Real Business Cycles

Marek Kapicka, Econ 204b

May 13, 2009

Calibration of the model

- **Calibration**: a procedure to find the parameters and functional forms of the model
- Three steps
  1. Restrict the **utility and production functions** to parametric classes that are consistent with the long run growth facts ✓
  2. Constructs **measurements of the U.S. economy** that are consistent with the model ✓
  3. Restrict the **parameters** of the model so that the model matches certain long-run facts about the U.S. economy

3.2. Matching some long-run observations

Facts

- The following facts are approximately true about the U.S. economy (1954-1992)
  1. Annual rates of growth of GDP per person is 1.56%
  2. Annual rate of growth of population is 1.2%
  3. The capital share is constant over time and is 0.4
  4. Investment/capital ratio is trendless and is 7.6% on average
  5. Time spent on market activities is approximately constant over time and constitutes 31% of total disposable time
  6. Capital/output ratio is trendless and is 3.32 on average

$$6 \text{ facts} \;\Leftrightarrow\; 6 \text{ parameters } \mu, \eta, \gamma, \beta, \delta, \alpha$$

3.2. Matching some long-run observations

How to match these facts?

- Compute a nonstochastic version of the model (i.e. $z = 1$ in all periods) and choose the parameters $\mu, \eta, \gamma, \beta, \delta, \alpha$ to match these facts in **steady state**

$$\hat{V}(k) = \max_{c,n,k'} \{(1-\mu)\ln c + \mu \ln(1-n) + \beta(1+\eta)\hat{V}(k')\}$$

s.t.

$$c + (1+\eta)(1+\gamma)k' = k^{\alpha} n^{1-\alpha} + (1-\delta)k$$

- First order conditions

$$\begin{aligned} \frac{1-\mu}{c} &= \lambda \\ \beta(1+\eta)\hat{V}_k(k') &= \lambda(1+\eta)(1+\gamma) \\ \frac{\mu}{1-n} &= \lambda(1-\alpha)\left(\frac{n}{k}\right)^{-\alpha} \end{aligned}$$

- Envelope condition

$$\hat{V}_k(k) = \lambda\left(\alpha\left(\frac{n}{k}\right)^{1-\alpha} + 1 - \delta\right)$$

3.2. Matching some long-run observations

How to match these facts?

- In steady state,
  - c, k are constant over time
  - population grows at rate η
  - output per person grows at rate γ

1. Use fact 1 to set

$$\boxed{\gamma = 0.0156 \text{ (annual)}}$$

2. Use fact 2 to set

$$\boxed{\eta = 0.012 \text{ (annual)}}$$

3. Use fact 3 to set

$$\boxed{\alpha = 0.4}$$

3.2. Matching some long-run observations

How to match these facts?

4. From the law of motion for capital

$$\delta = \frac{i}{k} + 1 - (1+\eta)(1+\gamma)$$

Use fact 4 ($\frac{i}{k} = 0.076$) and this equation to set

$$\boxed{\delta = 0.048 \text{ (annual)}}$$

5. Combine the first order conditions in c, n to get

$$\frac{n}{1-n} = (1-\alpha)\frac{1-\mu}{\mu}\frac{y}{c}$$

Use fact 5 ($n = 0.31$) and facts 4,6 ($\frac{y}{c} = 1.33$) to set

$$\boxed{\mu = 0.64}$$

3.2. Matching some long-run observations

How to match these facts?

6. Use first order condition in k', envelope condition and resource constraint to get

$$\frac{1+\gamma}{\beta} + \delta - 1 = \theta \frac{y}{k}$$

Use fact 6 ($\frac{k}{y} = 3.32$) to get

$$\boxed{\beta = 0.947 \text{ (annual)}}$$

3.3. Using Solow Residual to determine the stochastic process for z

- From the production function

$$\begin{aligned} Y_t &= z_t K_t^{\alpha} N_t^{1-\alpha} \\ \ln z_t &= \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln N_t \end{aligned}$$

- Compute the time series for $\ln z_t$ and estimate its autocorrelation ($\rightarrow \rho$) and variance ($\rightarrow \nu$).
- Results

$\rho = 0.95$
$\nu = 0.007$

4. How to solve the RBC model in 0.31 second

- Rewrite the model:

$$
\begin{aligned}
V(k,\omega) &= \max_{n,i}\{(1-\mu)\ln(e^{\omega}k^{\alpha}n^{1-\alpha}-i)+\mu\ln(1-n) \\
&\quad +\beta(1+\eta)E[V(k',\omega')|\omega]\}
\end{aligned}
$$

  s.t.

$$
\begin{aligned}
k' &= \frac{1-\delta}{(1+\eta)(1+\gamma)}k+\frac{1}{(1+\gamma)(1+\eta)}i \\
\omega' &= \rho\omega+\nu\varepsilon \qquad \varepsilon\sim N(0,1)
\end{aligned}
$$

  where $\omega=\ln z$.

  - The constraints are linear functions of $k', k, i, \omega', \omega, \varepsilon$
  - The objective function is a nonlinear function of ω, k, n, i

- **Linear-Quadratic Approximation:** Approximate the objective function by a second order Taylor expansion around the nonstochastic steady state

4.1. Linear-Quadratic Approximation

1. Form the linear-quadratic approximation of the period return function
2. Rearrange the terms
3. Guess that the approximated value function is quadratic in the states
    3.1 Certainty equivalence
4. Derive the Riccati equation that solves for the value function

4.1. Linear-Quadratic Approximation

1. Form the LQ approximation

- Let $y = \begin{bmatrix} \omega & k & n & i \end{bmatrix}^T$. Define

$$r(y) = (1-\mu)\ln(e^{\omega}k^{\alpha}n^{1-\alpha} - i) + \mu\ln(1-n)$$

- The second order Taylor approximation of $r(y)$ around the steady state $\overline{y} = \begin{bmatrix} 0 & \overline{k} & \overline{n} & \overline{i} \end{bmatrix}^T$ is given by a function $\tilde{r}(y)$

$$\tilde{r}(y) = r(\overline{y}) + \overline{J}^T(y-\overline{y}) + \frac{1}{2}(y-\overline{y})^T\overline{H}(y-\overline{y})$$

where

$$\overline{J} = \begin{bmatrix} r_{\varepsilon}(\overline{y}) & r_k(\overline{y}) & r_n(\overline{y}) & r_i(\overline{y}) \end{bmatrix}^T$$

is the Jacobian and

$$\overline{H} = \begin{bmatrix} r_{\varepsilon\varepsilon}(\overline{y}) & r_{\varepsilon k}(\overline{y}) & r_{\varepsilon n}(\overline{y}) & r_{\varepsilon i}(\overline{y}) \\ r_{\varepsilon k}(\overline{y}) & r_{kk}(\overline{y}) & r_{kn}(\overline{y}) & r_{ki}(\overline{y}) \\ r_{\varepsilon n}(\overline{y}) & r_{kn}(\overline{y}) & r_{nn}(\overline{y}) & r_{ni}(\overline{y}) \\ r_{\varepsilon i}(\overline{y}) & r_{ki}(\overline{y}) & r_{ni}(\overline{y}) & r_{ii}(\overline{y}) \end{bmatrix}$$

is the Hessian.

4.1. Linear-Quadratic Approximation

2a Rearrange the terms

- Rewrite the function as follows:

$$\tilde{r}(y) = \begin{bmatrix} 1 & y^T \end{bmatrix} Q \begin{bmatrix} 1 \\ y \end{bmatrix}$$

where the symmetric matrix $Q = \begin{bmatrix} Q_{11} & Q_{1y} \\ Q_{y1} & Q_{yy} \end{bmatrix}$ is given by

$$\begin{aligned} Q_{11} &= r(\overline{y}) - \overline{J}^T \overline{y} + \frac{1}{2} \overline{y}^T \overline{H} \overline{y} \\ Q_{yy} &= \frac{1}{2} \overline{H} \\ Q_{1y} &= \frac{1}{2} (J^T - \overline{y}^T \overline{H}) \\ Q_{y1} &= Q_{1y}^T . \end{aligned}$$

4.1. Linear-Quadratic Approximation

2b. Rearrange the terms

- Partition the vector $\begin{bmatrix} 1 \\ y \end{bmatrix}$ into two parts:
  - vector of state variables $x = \begin{bmatrix} 1 \\ \omega \\ k \end{bmatrix}$
  - vector of control variables $u = \begin{bmatrix} n \\ i \end{bmatrix}$
- Partition the Q matrix correspondingly:

$$Q = \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix}$$

- The approximation of the objective function can now be written as

$$\tilde{r}(y) = x^T Q_{xx} x + u^T Q_{uu} u + 2u^T Q_{xu} x$$

4.1. Linear-Quadratic Approximation

3. Guess that the value function is quadratic

- The approximated objective function is quadratic in x, u :

$$\tilde{r}(y) = x^T Q_{xx} x + u^T Q_{uu} u + 2u^T Q_{xu} x$$

- The constraints are linear in x, u, ε :

$$x' = Ax + Bu + C\varepsilon$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \frac{1-\delta}{(1+\gamma)(1+\eta)} \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{(1+\gamma)(1+\eta)} \end{bmatrix}, C = \begin{bmatrix} 0 \\ \nu \\ 0 \end{bmatrix}$$

- Guess: the value function of the approximated problem is quadratic in x :

$$\tilde{V}(k, \varepsilon) = x^T P x$$

for some symmetric matrix P.

- If we find the matrix P, we solve the problem

4.1. Linear-Quadratic Approximation

3. Certainty equivalence

- The P matrix satisfies

  $$x^T Px = \max_u \{x^T Q_{xx} x + u^T Q_{uu} u + 2u^T Q_{xu} x + \hat{\beta} E(x'^T Px' | x)\}$$

  s.t.

  $$x' = Ax + Bu + C\varepsilon.$$

  where $\hat{\beta} = \beta(1 + \eta)$

  - The solution is linear in the states: For some matrix F,

    $$u = Fx.$$

- **Certainty Equivalence:** The solution is *independent* of the vector C
  - one can take $C = 0$ (no *perceived* uncertainty) and will get exactly the same solution

4.1. Linear-Quadratic Approximation

- Using the certainty equivalence, it is enough to solve the following problem:

$$x^T P x = \max_u \{x^T Q_{xx} x + u^T Q_{uu} u + 2u^T Q_{xu} x + \hat{\beta} x'^T P x'\}$$

s.t.

$$x' = Ax + Bu.$$

- Taking first order conditions, one gets that

$$u = -(Q_{uu} + \hat{\beta} B^T P B)^{-1} (\hat{\beta} B^T P A + Q_{xu}) x.$$

- Substituting into the objective function we get that P solves the **Riccatti equation**

$$\begin{aligned} P &= Q_{xx} + \hat{\beta} A^T P A \\ &\quad -(\hat{\beta} B^T P A + Q_{xu})^T (Q_{uu} + \hat{\beta} B^T P B)^{-1} (\hat{\beta} B^T P A + Q_{xu}) \end{aligned}$$

4.2. Linear-Quadratic Approximation

Numerical implementation

1. Start with some symmetric negative semi-definite matrix P_1
2. For a given matrix P_j, use the Riccati difference equation

$$\begin{aligned} P_{j+1} &= Q_{xx} + \hat{\beta} A^T P_j A \\ &\quad - (\hat{\beta} B^T P_j A + Q_{xu})^T (Q_{uu} + \hat{\beta} B^T P_j B)^{-1} (\hat{\beta} B^T P_j A + Q_{xu}) \end{aligned}$$

   to obtain a new matrix P_{j+1}
3. Iterate until $\| P_{j+1} - P_j \| \leq$ error tolerance
4. the iteration takes 0.31 second :-)