Recursive Competitive Equilibrium
Real Business Cycles

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Today

- FWT and SWT
- Business Cycles

First Welfare Theorem

Theorem

The recursive competitive equilibrium is optimal.

Proof.

$V(K,z) \geq v(K,K,z)$ is obvious. To show that $V(K,z) \leq v(K,K,z)$, define $\tilde{V}(k,z)$ to be the value of not trading at markets at all and having technology $zF(k,1)$. It follows that $v(K,K,z) \geq \tilde{V}(K,z)$. But since F is CRS, $\tilde{V}(K,z) = V(K,z)$ and so $v(K,K,z) \geq V(K,z)$. $\square$

Second Welfare Theorem

Theorem

Suppose that $C(K,z)$*,* $K'(K,z)$ *are the optimal policy functions in the recursive Pareto optimum. Then there are prices* $w(K,z)$*,*$r(K,z)$*, value function* $v(k,K,z)$ *and the optimal policy functions* $c(k,K,z)$ *and* $h(k,K,z)$ *such that* v, c, k, K', w, r *constitute the recursive competitive equilibrium and* $v(K,K,z) = V(K,z)$*,*$c(K,K,z) = C(K,z), h(K,K,z) = K'(K,z)$*.*

Proof.

Guess the candidate pricing functions $w(K,z) = zF_n(K,1)$ and $r(K,z) = zF_k(K,1)$.
Let $\mathcal{P}$ be a set of value functions $v(k,K,z)$ that satisfy:

P1. are continuous and concave in k

P2. satisfy $V(K,z) = v(K,K,z)$

P3. has a zero derivative w.r.t. K at $k = K$.

□

Second Welfare Theorem

Proof.

Define an operator T by

$$\begin{aligned} Tv(k,K,z) &= \max_{c\geq 0, k'\geq 0} \{U(c) + \beta \sum_{z'\in Z} v(k', K'(K,z), z')\pi(z'|z)\} \\ s.t. \quad c + k' &\leq [r(K,z) + 1 - \delta]k + w(K,z) \end{aligned} \tag{1}$$

The result will be proven if we show that $T : \mathcal{P} \rightarrow \mathcal{P}$, hence $v^* = Tv^* \in \mathcal{P}$. Thus, T must preserve Properties 1-3 and $\mathcal{P}$ be closed. Two useful preliminary results:

1. If $k = K$ then, since the production function is CRS,

$$[r(K,z) + 1 - \delta]K + w(K,z) = zF(K,1) + (1-\delta)K$$

2. P3 implies that

$$v_k(K,K,z) = V_K(k,z)$$

Second Welfare Theorem

Proof.

T preserves P1: Can be shown using standard arguments. □

Second Welfare Theorem

Proof.

T preserves P2: Take the first order conditions w.r.t. c for $k = K$:

$$\begin{aligned} & U'([r(K,z)+1-\delta]K + w(K,z) - k') \\ = \; & U'(zF(K,1)+(1-\delta)K - k') \\ = \; & \beta \sum_{z' \in Z} v_k(k', K'(K,z), z')\pi(z'|z) \end{aligned}$$

If $k' = K'(K,z)$ then the first order condition becomes

$$\begin{aligned} & U'(zF(K,1)+(1-\delta)K - K'(K,z)) \\ = \; & \beta \sum_{z' \in Z} v_k(K'(K,z), K'(K,z), z')\pi(z'|z) \\ = \; & \beta \sum_{z' \in Z} V_K(K'(K,z), z')\pi(z'|z), \end{aligned}$$

which holds by definition of $K'(K,z)$. Thus
$h(K,K,z) = K'(K,z)$. Implies P2: $Tv(K,K,z) = V(K,z)$. □

Second Welfare Theorem

Proof.

T preserves P3: The envelope condition implies

$$\begin{aligned} & Tv_k(K, K, z) \\ = \; & U'[zF(K,1) + (1-\delta)K - h(K, K, z)][zF_K(K,1) + 1 - \delta] \\ = \; & U'[zF(K,1) + (1-\delta)K - K'(K, z)][zF_K(K,1) + 1 - \delta] \\ = \; & V_K(K, z). \end{aligned}$$

One can also show that $\mathcal{P}$ is closed.
Hence $v^* \in \mathcal{P}$, $v^*(K, K, z) = V(K, z)$ and
$h(K, K, z) = K'(K, z)$. □

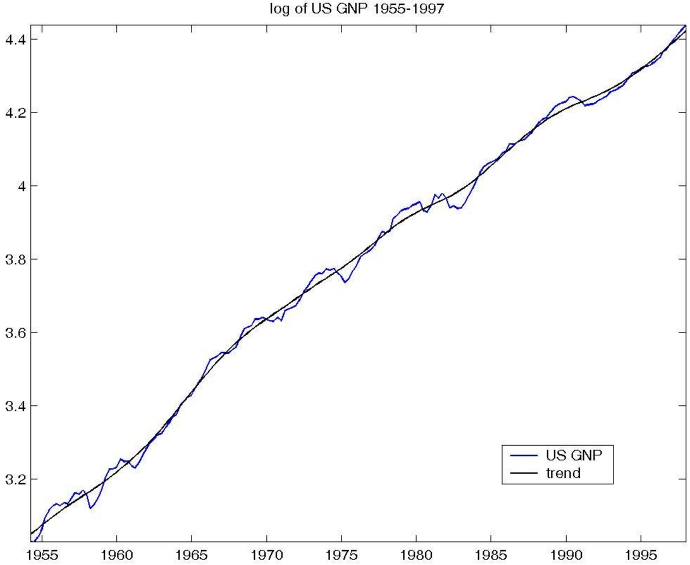
Business Cycles

Definition (Lucas)

Business Cycles are deviations of agregate output from a trend

- Properties:
  - Recurrent, not periodic
  - Two to eight years
  - Comovement among variables
- We will use the stochastic growth model to study business cycles
- Read Cooley-Prescott (1995), "Economic Growth and Business Cycles" (on the web)

US GNP


log of US GNP 1955-1997
4.4
4.2
4
3.8
3.6
3.4
3.2
1955
1960
1965
1970
1975
1980
1985
1990
1995
US GNP
trend

Trend vs. Business Cycles

- How to extract the trend?
- Hodrick-Prescott filter: let
    - $\{y_t\}_{t=1}^T$ be a time series
    - $\{z_t\}_{t=1}^T$ be the trend component
    - $\{y_t - z_t\}_{t=1}^T$ be the business cycle component.
    - $\{z_t\}_{t=0}^T$ solves

$$\arg\min_{\{z_t\}_{t=0}^T} \sum_{t=1}^T (y_t - z_t)^2 + \lambda \sum_{t=1}^T [(z_{t+1} - z_t) - (z_t - z_{t-1})]^2$$

- λ determines smoothness of the trend
    - for quarterly data, take $\lambda = 1600$
    - for annual data, $\lambda = 400$

Business Cycle Facts

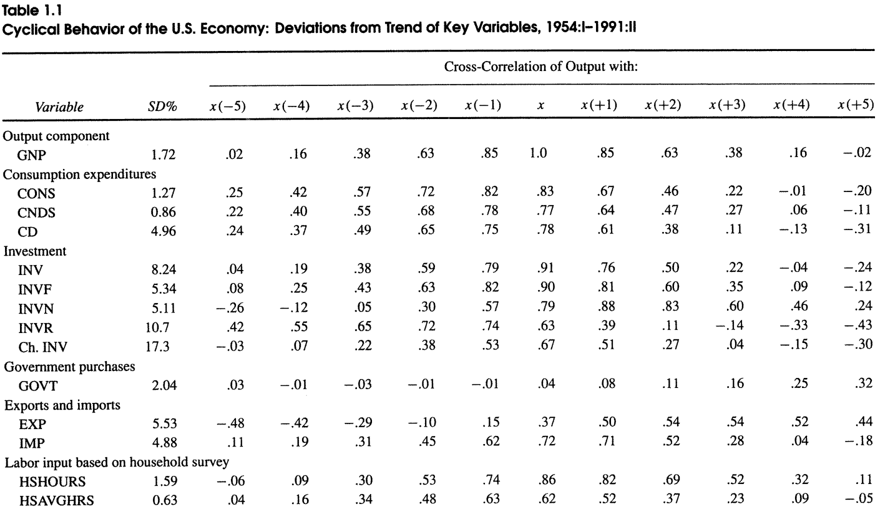


Table 1.1
Cyclical Behavior of the U.S. Economy: Deviations from Trend of Key Variables, 1954:I–1991:II

		Cross-Correlation of Output with:										
Variable	*SD%*	$x(-5)$	$x(-4)$	$x(-3)$	$x(-2)$	$x(-1)$	x	$x(+1)$	$x(+2)$	$x(+3)$	$x(+4)$	$x(+5)$
Output component												
GNP	1.72	.02	.16	.38	.63	.85	1.0	.85	.63	.38	.16	−.02
Consumption expenditures												
CONS	1.27	.25	.42	.57	.72	.82	.83	.67	.46	.22	−.01	−.20
CNDS	0.86	.22	.40	.55	.68	.78	.77	.64	.47	.27	.06	−.11
CD	4.96	.24	.37	.49	.65	.75	.78	.61	.38	.11	−.13	−.31
Investment												
INV	8.24	.04	.19	.38	.59	.79	.91	.76	.50	.22	−.04	−.24
INVF	5.34	.08	.25	.43	.63	.82	.90	.81	.60	.35	.09	−.12
INVN	5.11	−.26	−.12	.05	.30	.57	.79	.88	.83	.60	.46	.24
INVR	10.7	.42	.55	.65	.72	.74	.63	.39	.11	−.14	−.33	−.43
Ch. INV	17.3	−.03	.07	.22	.38	.53	.67	.51	.27	.04	−.15	−.30
Government purchases												
GOVT	2.04	.03	−.01	−.03	−.01	−.01	.04	.08	.11	.16	.25	.32
Exports and imports												
EXP	5.53	−.48	−.42	−.29	−.10	.15	.37	.50	.54	.54	.52	.44
IMP	4.88	.11	.19	.31	.45	.62	.72	.71	.52	.28	.04	−.18
Labor input based on household survey												
HSHOURS	1.59	−.06	.09	.30	.53	.74	.86	.82	.69	.52	.32	.11
HSAVGHRS	0.63	.04	.16	.34	.48	.63	.62	.52	.37	.23	.09	−.05

Figure:

Business Cycle Facts

- **GDP Components**
  - Consumption of nondurables and services fluctuates much less than output and is procyclical
  - Investment fluctuates much more than output and is procyclical
  - Government expenditures are essentially uncorrelated with output
  - Both imports and exports are procyclical and fluctuate more than output

Business Cycle Facts

- **Factors of production**
  - Total hours worked fluctuate about as much as output and are highly procyclical
  - Most of the volatility in hours worked can be accounted for by volatility in employment eather than hours per worker
  - hours worked are uncorrelated with productivity
  - The capital stock fluctuates much less than output and is uncorrelated with output
- **Prices**
  - Average hourly compensation is essentially uncorrelated with output

The Real Business Cycle Model

- Essentially the stochastic growth model with some bells and whistles
    - Variable labor supply
    - Labor productivity growth γ
    - Population growth η
- Social Planner's problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t N_t U(\hat{c}_t, 1 - \hat{n}_t)$$

s.t.

$$\begin{aligned} N_t \hat{c}_t + N_{t+1} \hat{k}_{t+1} &= z_t f(N_t \hat{k}_t, A_t N_t \hat{n}_t) + (1-\delta) N_t \hat{k}_t \\ A_{t+1} &= (1+\gamma) A_t, \quad A_1 = 1 \\ N_{t+1} &= (1+\eta) N_t, \quad N_1 = 1 \end{aligned}$$

where $\hat{c}_t$ is per capita consumption, $\hat{n}_t$ is per capita labor supply and $\hat{k}_t$ is per capita capital stock.

Calibration of the model

- **Calibration**: a procedure to find the parameters and functional forms of the model
- Three steps
  1. Restrict the **utility and production functions** to parametric classes that are consistent with the long run growth facts
  2. Constructs **measurements of the U.S. economy** that are consistent with the model
  3. Restrict the **parameters** of the model so that the model matches certain long-run facts about the U.S. economy

1. Utility and Production Functions

- **Fact No. 1:** The labor share $1-\alpha$

  $$1-\alpha=\frac{w_tL_t}{Y_t}$$

  is approximately constant over time.
- **Implication:** The production function is Cobb-Douglas

  $$Y_t=z_tK_t^{\alpha}L_t^{1-\alpha}$$

1. Utility and Production Functions

- **Fact No. 2:** After WWII, per capita leisure is constant over time (despite changes in incomes and wages) is approximately constant over time.
- **Implication:** The utility function is CRRA

$$U(c, l) = \frac{(c^{1-\mu} l^{\mu})^{1-\sigma}}{1-\sigma}$$

1. Utility and Production Functions

Normalizing the model

- Let $n_t = \hat{n}_t$ and

  $$c_t = \frac{\hat{c}_t}{A_t} = \frac{\hat{c}_t}{(1+\gamma)^t}, \quad k_t = \frac{\hat{k}_t}{A_t} = \frac{\hat{k}_t}{(1+\gamma)^t}$$

  be consumption and capital stock per unit of effective labor.
- The model becomes

$$\max E_0 \sum_{t=0}^{\infty} \beta^t (1+\eta)^t \frac{\left[(1+\gamma)^{(1-\mu)t} c_t^{1-\mu} (1-n_t)^{\mu}\right]^{1-\sigma}}{1-\sigma}$$

s.t.

$$\begin{aligned} & (1+\eta)^t (1+\gamma)^t c_t + (1+\eta)^{t+1} (1+\gamma)^{t+1} k_{t+1} \\ = \; & z_t \left[(1+\eta)^t (1+\gamma)^t k_t\right]^{\alpha} \left[(1+\eta)^t (1+\gamma)^t n_t\right]^{1-\alpha} \\ & + (1-\delta)(1+\eta)^t (1+\gamma)^t k_t \end{aligned}$$

1. Utility and Production Functions

Normalizing the model

- Cancelling terms, we get

$$\max E_0 \sum_{t=0}^{\infty} \hat{\beta}^t \frac{[c_t^{1-\mu}(1-n_t)^{\mu}]^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_t + (1+\eta)(1+\gamma)k_{t+1} = z_t {k_t}^{\alpha} {n_t}^{1-\alpha} + (1-\delta)k_t$$

where $\hat{\beta} = \beta(1+\eta)(1+\gamma)^{(1-\mu)(1-\sigma)}$.

- !! Need $\hat{\beta} < 1$

2. Measurement of the U.S. Economy

- Problems:
  1. Some NIPA categories are not in the model
     - inventories
     - net export
     - government sector
  2. Some NIPA categories are wrongly attributed
     - Consumption of durable goods: NIPA: consumption. Model: investment
  3. Some items are not included in NIPA and need to be imputed
     - flow of services from durable goods
     - flow of services from government capital

2.1-2.2 Measurement of the U.S. Economy

	NIPA	Model
Personal Consumption Expenditures	C	
durable goods		I
nondurable goods		C
services		C
Gross Private Domestic Investment	I	I
Net Exports	NX	I
Government Purchases	G	
public consumption		C
public investment		I

2.3. Measurement of the U.S. Economy

- Three types of capital in the economy:
  - K : private capital
  - K_D : stock of durable goods
- To get a correct measurement of GDP, we need to impute incomes from K_D

  $$Y = wL + (r+\delta)K + (r+\delta_D)K_D$$

  where w is the wage rate, r is the interest rate (the same for all sectors) and δ, δ_D are depreciation rates
- We know:
  - $GNP^{NIPA} = wL + (r+\delta)K$
  - K, K_D
  - δK (NIPA)
- We do not know:
  - r, δ, δ_D

2.3. Measurement of the U.S. Economy

- Computations:

a. use the income side of NIPA to compute the private capital share α

b. use α to compute the income from private capital $Y^K = (r + \delta)K = \alpha GNP^{NIPA}$ and the interest rate $r = \frac{Y^K - \delta K}{K}$

c. use r to compute δ_D.

2.3. Measurement of the U.S. Economy

a. private capital share

$$\begin{aligned} GNP^{NIPA} &= wL + (r+\delta)K \\ &= wL + rK + \delta K \\ &= NNP^{NIPA} + \delta K \end{aligned}$$

Net National Product

1.	Compensation of Employees	Labor
2.	Corporate Profits	Private Capital
3.	Rental Income	Private Capital
4.	Net Interest	Private Capital
5.	Proprietor's Income	Both
6.	IBT	Both

2.3. Measurement of the U.S. Economy

a. private capital share

$$\begin{aligned} Y^K &= \alpha GNP^{NIPA} \\ &= (2) + (3) + (4) + \alpha[(5) + (6)] + \delta K \end{aligned}$$

hence

$$\alpha = \frac{(2) + (3) + (4) + \delta K}{GNP^{NIPA} - (5) - (6)}$$

2.3. Measurement of the U.S. Economy

b. interest rate

$$\begin{aligned} (r+\delta)K &= \alpha GNP^{NIPA} = Y^K \\ rK &= Y^K - \delta K \\ r &= \frac{Y^K - \delta K}{K} \end{aligned}$$

2.3. Measurement of the U.S. Economy

c. depreciation of durable goods and government capital

- From LOM for capital

$$\begin{aligned} K_{t+1,D} &= (1-\delta_D)K_{t,D} + I_{t,D} \\ \frac{K_{t+1,D}}{Y_{t+1,D}}\frac{Y_{t+1,D}}{Y_{t,D}} &= (1-\delta_D)\frac{K_{t,D}}{Y_{t,D}} + \frac{I_{t,D}}{Y_{t,D}} \end{aligned}$$

- Steady state): assume:
    - Y_D grows at the same rate as $Y (= \gamma)$
    - $\frac{K_{t,D}}{Y_{t,D}}$ is constant over time
-

$$\begin{aligned} (1+\gamma)\frac{K_D}{Y_D} &= (1-\delta_D)\frac{K_D}{Y_D} + \frac{I_D}{Y_D} \\ (\delta_D+\gamma)\frac{K_D}{Y_D} &= \frac{I_D}{Y_D} \\ \delta_D &= \frac{I_D}{K_D} - \gamma \end{aligned}$$