Dynamic Programming Under Certainty

Marek Kapicka, Econ 204b

March 30 2009

1. Dynamic Programming under Certainty

- A method for solving dynamic models
- Suppresses the role of the time dimension and highlights the role of state variables
    - State variable: a variable that summarizes all past actions

1. Optimal Growth Problem

Setup

$$
\begin{aligned}
& \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
& s.t. \\
c_t + k_{t+1} \quad & \leq \quad F(k_t) + (1-\delta) k_t \\
c_t \quad & \geq \quad 0, k_{t+1} \geq 0 \\
& k_0 \text{ given}
\end{aligned}
$$

- The following is given:
  - β, $U(\cdot)$: describes preferences
  - δ, $F(\cdot)$ describes technology
  - k_0 : initial condition
- The social planner chooses *sequences* of consumption and capital $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize lifetime utility of the representative agent
  - The choice is made at time zero

1. Optimal Growth Problem

Simplification

- Simplify the setup by
  - eliminating consumption:

$$c_t = F(k_t) + (1-\delta)k_t - k_{t+1}$$

  - Defining

$$f(k_t) = F(k_t) + (1-\delta)k_t$$

- The social planner's **sequence problem (SP)** becomes

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

$$s.t.\ 0 \quad \leq \quad k_{t+1} \leq f(k_t) \qquad k_0 \text{ given}$$

1. Optimal Growth Problem

value function

- Define

$$
\begin{aligned}
v^*(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\
s.t.\ 0 &\leq k_{t+1} \leq f(k_t) \qquad k_0 \text{ given}
\end{aligned}
$$

- $v^*(k_0)$ is the value (maximized objective function) of the social planner's problem given k_0
- $v^*(\cdot)$ is the **value function** of the social planner (in a sequence problem)

2. Dynamic Programming

Main Idea

- In a sequence problem, everything is chosen at time 0
- That is not necessary: Only k_1 (and c_0) must be chosen at time 0, the rest can wait.
  - Since the problem has identical structure from time 1 on, the value of having k_1 at time 1 should be given by $v^*(k_1)$
- The social planner's problem sequence problem should satisfy

$$\begin{aligned} k_1^* &\in \underset{0\leq k_1 \leq f(k_0)}{\arg\max}\ U(f(k_0)-k_1)+\beta v^*(k_1) \\ v^*(k_0) &= \max_{0\leq k_1 \leq f(k_0)} U(f(k_0)-k_1)+\beta v^*(k_1) \qquad (1) \end{aligned}$$

- Note that nothing in these equations depends directly on time
  - time subscript can be dropped!

2. Dynamic Programming

Bellman Equation

- Do we need to solve the sequence problem first to get the value function v^*?
- No! We can consider the equation (1) as a primary object:

$$v(k) = \max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta v(y)$$

  - a **functional equation (FE)**: a function v is the solution (not a number as usual)
    - v is on both sides (evaluated at different values)
- k is a **state variable**: summarizes all the past actions
- $g(k)$ defined by

$$g(k) = \arg\max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta v(y)$$

is the **optimal policy function**

2. Dynamic Programming

Advantages of FE

1. Economic intuition: the problem is broken into "now" and "then"
2. Preferences over sequences replaced by preferences over current consumption and future capital stock
3. Computational algorithms available to efficiently solve FE
4. Analysis: easier to prove existence, uniqueness, monotonicity etc. of the solution to FE.

2. Dynamic Programming

Principle of Optimality

- However, before getting the benefits of (FE), we need to show that there is an equivalence between (FE) and (SP)
  1a. If v^* solves (SP) then v^* solves (FE)
  1b. If v solves (FE) (and additional conditions hold) then v also solves (SP)
  2a. If $\{k_{t+1}^*\}$ attains the maximum of (SP) then $k_{t+1}^* = g(k_t^*)$ all $t \geq 0$
  2b. If $g(k)$ attains the maximum of (FE) (and additional conditions hold) then $\{g(k_0).g(g(k_0)), ...\}$ attains the maximum of (SP)

2. Dynamic Programming

Potential Problems

- (FE) can have *more solutions* than just v^*
  - for instance, it but in general is not equal to v^*
- That's why we will need additional conditions to show that a solution to (FE) satisfies (SP)
- On the other hand, v^* will satisfy (FE)
- One way to show the equivalence will be to show that (FE) has a unique solution (in a certain class of functions). Then this solution will also satisfy (SP).

2.1. First example when (FE) has more solutions

- $v^* < \infty$
- However, $v = \inf$ solves the functional equation
- Hence, the $v = \inf$ is the "wrong" solution of (FE)

2.1. Second example when (FE) has more solutions

- $U(c) = c$,
- $f(k) = \frac{1}{\beta}k \rightarrow c_t = \frac{1}{\beta}k_t - k_{t+1}$
- Unlimited borrowing possible: k_{t+1} can be negative.
- (SP):

$$v^*(k_0) = \max_{\{k_{t+1} \leq \frac{1}{\beta}k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\frac{1}{\beta}k_t - k_{t+1}) = +\infty.$$

- (FE):

$$v(k) = \max_{y \leq \frac{1}{\beta}k} [\frac{1}{\beta}k - y + \beta v(y)]$$

  - Solution 1: $v(k) = +\infty$ (the right one!)
  - Solution 2: $v(k) = \frac{k}{\beta}$ (the wrong one!)

2.2a. A solution to (SP) satisfies (FE):

$$\begin{aligned}
v^*(k_0) &= \max_{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t)-k_{t+1}) \\
&= \max_{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty}} \left[U(f(k_0)-k_1)+\beta \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t)-k_{t+1})\right] \\
&= \max_{0\leq k_1\leq f(k_0)} [U(f(k_0)-k_1) \\
&\quad +\beta \max_{\{0\leq k_{t+1}\leq f(k_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t)-k_{t+1})] \\
&= \max_{0\leq k_1\leq f(k_0)} [U(f(k_0)-k_1)+\beta v^*(k_1)]
\end{aligned}$$

Theorem

v^* satisfies (FE)

2.2b. A solution to (FE) satisfies (SP):

$$
\begin{aligned}
v(k_0) &= \max_{0 \le k_1 \le f(k_0)} [U(f(k_0) - k_1) + \beta v(k_1)] \\
&= \max_{0 \le k_1 \le f(k_0)} \{U(f(k_0) - k_1) \\
&\quad + \beta \max_{0 \le k_2 \le f(k_1)} [U(f(k_1) - k_2) + \beta v(k_2)]\} \\
&= \max_{\{0 \le k_{t+1} \le f(k_t)\}_{t=0}^{1}} \{\sum_{t=0}^{1} \beta^t [U(f(k_t) - k_{t+1}) + \beta^2 v(k_2)\} \\
&\quad \cdots\cdot \\
&= \max_{\{0 \le k_{t+1} \le f(k_t)\}_{t=0}^{T}} \{\sum_{t=0}^{T} \beta^t [U(f(k_t) - k_{t+1}) + \beta^{T+1} v(k_{T+1})\}
\end{aligned}
$$

2.2b. A solution to (FE) satisfies (SP): harder

Theorem

If

$$\lim_{T\to\infty} \beta^{T+1} v(k_{T+1}) = 0 \tag{2}$$

for all $\{k_{T+1}\}$ *such that* $0 \leq k_{T+1} \leq f(k_T)$, *then* v *satisfies (SP)*

Proof.

If the condition is satisfied then the last term vanishes:

$$\begin{aligned} v(k_0) &= \lim_{T\to\infty} \max_{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{T}} \{\sum_{t=0}^{T} \beta^t [U(f(k_t) - k_{t+1}) + \beta^{T+1} v(k_{T+1})\} \\ &= \max_{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty}} \{\sum_{t=0}^{\infty} \beta^t [U(f(k_t) - k_{t+1}) \end{aligned}$$

□

2.1contd. The second Example revisited

- By Theorem 2, it must be true that the condition fails for some feasible sequence of capital
- Consider the following feasible sequence:

$$\begin{aligned} k_1 &= \frac{1}{\beta} k_0 \\ k_2 &= \frac{1}{\beta^2} k_0 \\ &\dots \end{aligned}$$

- Then

$$\begin{aligned} \lim_{T\to\infty} \beta^{T+1} v(k_{T+1}) &= \lim_{T\to\infty} \beta^{T+1} [\frac{1}{\beta^{T+1}} k_0] \\ &= k_0 \neq 0 \end{aligned}$$

- Hence Condition (2) is not satisfied for $v(k) = k$
- Note that Condition (2) is also not satisfied for the right solution!!
  - Condition (2) is sufficient but not necessary.