Economics 204b — Marek Kapicka

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Final Answers

1 Calibration (60 points)

Consider a representative agent economy whose preferences are given by

$$E\sum_{t=0}^{\infty}\beta^t[2\sqrt{c_t^{\gamma}(1-n_t)^{1-\gamma}}]$$

where $c_t \geq 0$ is consumption and $n_t \in [0,1]$ is labor supply. The aggregate resource constraint and the law of motion for capital are

$$\begin{aligned} c_t + x_t &\leq k_t^{\theta} n_t^{1-\theta} \\ k_{t+1} &= (1-\delta)k_t + x_t. \end{aligned}$$

where $k_t \geq 0$ is capital stock and x_t is investment.

1. Calibrate the model parameters $\gamma, \theta, \delta, \beta$ so that its competitive equilibrium matches steady state ratios $\frac{c}{y} = 0.8$, $\frac{k}{y} = 3$, labor income share is equal to 60% and people spend 1/3 of their time by working. (15 pts each parameter)

   The resource constraint can be written as

   $$c_t + k_{t+1} \leq k_t^{\theta} n_t^{1-\theta} + (1-\delta)k_t$$

   The first order conditions are:

   $$\begin{aligned} \frac{\gamma}{c_t}\sqrt{c_t^{\gamma}(1-n_t)^{1-\gamma}} &= \lambda_t \\ \frac{1-\gamma}{1-n_t}\sqrt{c_t^{\gamma}(1-n_t)^{1-\gamma}} &= \lambda_t(1-\theta)k_t^{\theta}n_t^{-\theta} \\ \lambda_t &= \beta\lambda_{t+1}(\theta k_t^{\theta-1}n_t^{1-\theta} + 1 - \delta) \end{aligned}$$

   where λ_t is Lagrange multiplier on the resource constraint. In steady state, those equations reduce to

   $$\begin{aligned} (1-\theta)\frac{y}{c}\frac{1-n}{n} &= \frac{1-\gamma}{\gamma} \qquad (1) \\ 1 &= \beta(\theta\frac{y}{k} + 1 - \delta). \end{aligned}$$

Since labor income share is 0.6, $\theta = 0.4$. Then equation (1) gives

$$\frac{1-\gamma}{\gamma} = 0.6 * 1.25 * \frac{2/3}{1/3} = 0.6 * 2.5 = 1.5.$$

Hence $\gamma = 0.4$. The law of motion for capital implies that,

$$\delta = \frac{x}{k} = \frac{y}{k}(1 - \frac{c}{y}) = \frac{1}{3} * 0.2 = \frac{1}{15}.$$

Finally,

$$\beta = \frac{1}{\theta \frac{y}{k} + 1 - \delta} = \frac{1}{\frac{0.4}{3} + 1 - \frac{1}{15}} = \frac{30}{32}.$$

2 Investment and Adjustment Costs (70 points)

Consider a problem of a firm that maximizes profits by choosing investment each period, but faces costs of adjusting its capital stock. The firm faces idiosyncratic productivity shock A that follows a Markov process and its production function is given by $\Pi(A, k)$ where k is its current capital stock. The cost of one unit of investment is p. In addition, a firm with capital stock k and investment x incurs adjustment costs $C(x, k)$ that increase in x and satisfy $C(0, k) = 0$. Net profits of the firm in a given period are therefore given by $\Pi(A, k) - px - C(x, k)$. Capital stock depreciates at rate δ.

1. Write down the Bellman Equation that solves the firm's dynamic profit maximization problem. Under what conditions is there going to be a unique solution to this equation in the space of bounded and continuous functions? (25 pts)

$$\begin{aligned} V(A, k) &= \max_x \{\Pi(A, k) - C(x, k) - px + \beta E(V(A', k')|A)\} \\ k' &= (1 - \delta)k + x. \end{aligned}$$

   There is going to be a unique solution if both Π and C are bounded and continuous, the stochastic process for A satisfies the Feller property, and $\beta < 1$.

2. Suppose now that $\Pi(A, k) = Ak$ and that the adjustment costs are convex and given by $C(x, k) = \frac{1}{2}(\frac{x}{k})^2 k$. Guess that the value function satisfies $V(A, k) = \phi(A)k$. Verify the guess. Derive a condition the function $\phi(A)$ must satisfy. (15 pts)

Use the guess on the right-hand side. The first order condition in x is

$$\frac{x}{k} = \beta\rho(A) - p,$$

where $\rho(A) = E(\phi(A')|A)$. Substituting back in the Bellman Equation we get that

$$\begin{aligned} Ak - \frac{1}{2}(\frac{x}{k})^2 k - px + \beta\rho(A)k' &= Ak - \frac{1}{2}[\beta\rho(A) - p]^2 k - p[\beta\rho(A) - p]k + \beta\rho(A)[(1-\delta)k + \beta\rho \\ &= \{A - \frac{1}{2}[\beta\rho(A) - p]^2 - p[\beta\rho(A) - p] + \beta\rho(A)[1 - \delta + \beta\rho(A) - \end{aligned}$$

which verifies the guess since it is linear in k. The term in the brackets must equal to $\phi(A)$. Simplifying, $\phi(A)$ must satisfy

$$\phi(A) = A + \frac{1}{2}[\beta\rho(A) - p]^2 + \beta\rho(A)(1-\delta).$$

3. Assume that A is i.i.d. with $E(A) = 1$. Characterize $\phi(A)$ as sharply as possible. (10 pts)

   If the shocks are iid, $\rho(A) = \rho$ and so

   $$\phi(A) = A + \frac{1}{2}[\beta\rho - p]^2 + \beta\rho(1-\delta).$$

   Taking the expectation, we get that ρ satisfies

   $$\rho = 1 + \frac{1}{2}[\beta\rho - p]^2 + \beta\rho(1-\delta).$$

4. Empirically, firms' behavior is better described by adjustment costs that are not convex. To model nonconvexity, assume that the adjustment costs are equal to zero if no investment is taken, and to $\bar{C}k$ when investment is undertaken, regardless of the magnitude of the investment. Write down the Bellman Equation for the firm's problem. Describe in words what kind of investment behavior you would expect to emerge from such adjustment costs. (20 pts)

$$\begin{aligned} V(A,k) &= \max\{V_1(A,k), V_2(A,k)\} \\ V_1(A,k) &= Ak + \beta E(V(A',(1-\delta)k)|A) \\ V_2(A,k) &= \max_x\{Ak - Ck - px + \beta E(V(A',k')|A)\} \end{aligned}$$

One should expect that the investment will be undertaken only infrequently or only after large shocks. In particular, there is going to be an inaction region (set of shocks for each k) where investment is not going to be adjusted.

3 Search (70 points)

This problem modifies the Alvarez-Veracierto island search model we have seen in class by incorporating industry specific shocks. A worker on an island makes a decision about whether to stay on the island or whether to search for a better island. If the worker decides to leave the island, he can either search across all industries, or he can restrict his search to the industry he is currently working in. Searching takes one period, during which the worker earns unemployment benefits b. The workers are risk neutral, discount future at rate β, $0 < \beta < 1$, and maximize the present discounted value of earnings

Each island represents a firm. The state of an island is completely described by a pair (x, y, z) where x is the measure of workers on the island at the beginning of a period, y is current industry specific shock and z is current firm specific productivity shock. Both shocks are Markov, discrete and mutually uncorrelated. Their transition probabilities are given by $P(y, y')$ and $Q(z, z')$. The earnings of a worker are equal to the marginal product of labor $f(x, y, z)$.

Conditional on the decision whether to search within the industry or over all industries, the search is *undirected*. The measure of workers that have searched over all industries and arrive to each island at the end of the period is denoted by U. The measure of workers that have searched in an industry y and arrive to each firm in the same industry is denoted by $u(y)$. The stationary distribution of islands is denoted by $\lambda^*(x, y, z)$.

1. Let θ be the value of searching over all industries and $\phi(y)$ the value of searching in industry a. Formulate the Bellman equation of a worker on an island. (20 pts)

$$v(x,y,z) = \max\{\theta, \phi(y), f(x,y,z) + \beta \sum_{y'} \sum_{z'} v(x+U+u(y), y', z') P(y,y') Q(z,z')\}$$

2. Write down an equation that the value of searching θ must satisfy. (15 pts)

$$\theta = b + \beta \sum_{y} \sum_{z} \int_{x} v(x,y,z) \lambda^*(x,y,z) dx$$

3. Write down an equation that the value of searching $\phi(y)$ must satisfy. (15 pts)

$$\phi(y) = b + \beta \sum_{z} \int_{x} v(x,y,z) \hat{\lambda}^*(x,y,z) dx$$

where $\hat{\lambda}^*(x,y,z) = \frac{\lambda^*(x,y,z)}{\sum\limits_z \int\limits_x \lambda^*(x,y,z)}$ is the distribution of islands conditional on the industry shock y.

4. Describe the equilibrium employment $g(x,y,z)$ on an island (x,y,z).(20 pts)

   The employment is given by

$$g(x,y,z) = \min(x, g_1(z), g_2(z,y)).$$

   where $g_1(z,y)$ is given by an indifference between staying on an island and searching across all industries:

$$\theta = f[g_1(z,y), y, z), z] + \beta \sum_{y'} \sum_{z'} v(g_1(z,y) + U + u(y), y', z') P(y,y') Q(z,z'),$$

   and $g_2(z,y)$ is given by an indifference between staying on an island and searching in a given industry

$$\phi(y) = f[g_2(z,y), y, z), z] + \beta \sum_{y'} \sum_{z'} v(g_2(z,y) + U + u(y), y', z') P(y,y') Q(z,z')$$