THE EQUITY PREMIUM: WHY IS IT A PUZZLE?

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Almost two decades ago, Edward Prescott and I challenged the profession with a poser: the equity premium, (the return earned by a risky security in excess of that earned by a relatively riskfree T-bill), was an order of magnitude greater than could be rationalized in the context of the standard neo-classical paradigms of financial economics. This regularity, dubbed 'the equity premium puzzle' has spawned a plethora of research efforts to explain it away. In this paper, I take a retrospective look at the puzzle, and critically evaluate the various attempts to solve it.

1. Empirical Facts

Historical data provides us with a wealth of evidence documenting that for over a century stock returns have been considerably higher than those for T-bills. The average annual real return, (that is to say, the inflation adjusted return) on the U.S. stock market over the last hundred and ten years has been about 7.9 percent. Over the same period, the return on a relatively riskless security was a paltry 1.0 percent. The difference between these two returns, 6.9 percent, is termed the “equity premium”. This statistical difference is even more pronounced over the post war period, with the premium of stock returns over bonds being almost 8 percent.

Furthermore, this pattern of excess returns to equity holdings is not unique to U.S. capital markets. Equity returns compared to the return to debt holdings in other countries also exhibit this historical regularity. The annual return on the British stock market was 5.7 percent over the post war period, an impressive 4.6 percent premium over the average bond return of 1.1 percent. Similar statistical differentials are documented for France, Germany and Japan. Together USA, UK, Japan, Germany and France account for more than 85% of the capitalized global equity value.

Jeremy Siegel (1998) has analyzed data on U.S. stock and bond returns going back to 1802 and found a similar though somewhat smaller premium in place for the past two hundred years. The tables below summarize the foregoing data.

<table>
<thead>
<tr>
<th>Time period</th>
<th>% real return on a market index</th>
<th>% real return on a relatively riskless security</th>
<th>% risk premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>1802-1998</td>
<td>7.0</td>
<td>2.9</td>
<td>4.1</td>
</tr>
<tr>
<td>1889-2000</td>
<td>7.9</td>
<td>1.0</td>
<td>6.9</td>
</tr>
<tr>
<td>1926-2000</td>
<td>8.7</td>
<td>0.7</td>
<td>8.0</td>
</tr>
<tr>
<td>1947-2000</td>
<td>8.4</td>
<td>0.6</td>
<td>7.8</td>
</tr>
</tbody>
</table>

Source: 1802-1998 from Siegel (1998), 1889-200 from Mehra & Prescott (1985). Data updated by the authors. The rest are the authors’ estimates.
The dramatic investment implications of this differential rate of return can be seen from the following table, which maps the capital appreciation of $1 invested in different assets from 1802 - 1997 and from 1926-2000.

<table>
<thead>
<tr>
<th>Country</th>
<th>Time period</th>
<th>% real return on a market index Mean</th>
<th>% real return on a relatively riskless security Mean</th>
<th>% risk premium Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.K</td>
<td>1947-1999</td>
<td>5.7</td>
<td>1.1</td>
<td>4.6</td>
</tr>
<tr>
<td>Japan</td>
<td>1970-1999</td>
<td>4.7</td>
<td>1.4</td>
<td>3.3</td>
</tr>
<tr>
<td>Germany</td>
<td>1978-1997</td>
<td>9.8</td>
<td>3.2</td>
<td>6.6</td>
</tr>
<tr>
<td>France</td>
<td>1973-1998</td>
<td>9.0</td>
<td>2.7</td>
<td>6.3</td>
</tr>
</tbody>
</table>

Source: U.K from Siegel (1998), the rest are from Campbell (2001)

The dramatic investment implications of this differential rate of return can be seen from the following table, which maps the capital appreciation of $1 invested in different assets from 1802 - 1997 and from 1926-2000.

<table>
<thead>
<tr>
<th>Investment period</th>
<th>Terminal value of $1 invested in Stocks</th>
<th>T-bills</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Real</td>
<td>Nominal</td>
</tr>
<tr>
<td>1802-1997</td>
<td>$558,945</td>
<td>$7,470,000</td>
</tr>
<tr>
<td>1926-2000</td>
<td>$266.47</td>
<td>$2,586.52</td>
</tr>
</tbody>
</table>

Source: Ibbotson (2001) and Siegel (1998)

As the table illustrates, $1 invested in a diversified stock index yields an ending wealth of $558,945 versus a value of $276, *in real terms*, for $1 invested in a portfolio of T-bills for the period 1802-1997. While for the 75-year period 1926-2000 the corresponding values are $266.47 and $1.71. We assume that all payments to the underlying asset, such as dividend payments to stock and interest payments to bonds are reinvested and that there are no taxes paid.

This long-term perspective underscores the remarkable wealth building potential of the equity premium. It should come as no surprise therefore, that the equity premium is of central importance in portfolio allocation decisions, estimates of the cost of capital and is front and center in the current debate about the advantages of investing Social Security funds in the stock market.

### 2. A Premium for Bearing Risk?

Why have stocks been such an attractive investment relative to bonds? Why has the rate of return on stocks been higher than on relatively risk free assets? One intuitive answer is that since stocks are ‘riskier’ than bonds, investors require a larger premium for bearing this additional risk; and indeed, the standard deviation of the returns to stocks (about 20% per annum historically) is larger
than that of the returns to T-bills (about 4% per annum), so, obviously they are considerably more risky than bills! But are they?

The figures below illustrate the variability of the annual real rate of return on the S&P 500 index and a relatively risk free security over the period 1889-2000.

Source: Mehra and Prescott (1985). Data updated by the authors.
To enhance and deepen our understanding of the risk-return trade-off in the pricing of financial assets, we make a detour into modern asset pricing theory and look at why different assets yield different rates of return. The deux ex machina of this theory is that assets are priced such that, ex-ante, the loss in marginal utility incurred by sacrificing current consumption and buying an asset at a certain price is equal to the expected gain in marginal utility contingent on the anticipated increase in consumption when the asset pays off in the future.

The operative emphasis here is the *incremental loss or gain* of well being due to consumption and should be differentiated from incremental consumption. This is because the *same* amount of consumption may result in different degrees of well-being at different times. (A five-course dinner after a heavy lunch yields considerably less satisfaction than a similar dinner when one is hungry!)

As a consequence, assets that pay off when times are good and consumption levels are high, i.e. when the incremental value of additional consumption is low, are less desirable than those that pay off an equivalent amount when times are bad and additional consumption is both desirable and more highly valued.

Let us illustrate this principle in the context of the standard, popular paradigm, the Capital Asset Pricing Model (CAPM). The model postulates a linear relationship between an asset’s ‘beta’, a
measure of systematic risk, and expected return. Thus, high beta stocks yield a high-expected rate of return. That is so because in the CAPM, good times and bad times are captured by the return on the market. The performance of the market as captured by a broad based index acts as a surrogate indicator for the relevant state of the economy. A high beta security tends to pay off more when the market return is high, that is, when times are good and consumption is plentiful; as discussed earlier, such a security provides less incremental utility than a security that pays off when consumption is low, is less valuable and consequently sells for less. Thus assets that pay off in states of low marginal utility will sell for a lower price than similar asset that pay off in states of high marginal utility. Since rates of return are inversely proportional to asset prices, the latter class of assets will, on average, give a lower rate of return than the former.

Another perspective on asset pricing emphasizes that economic agents prefer to smooth patterns of consumption over time. Assets that pay off a relatively larger amount at times when consumption is already high, “destabilize” these patterns of consumption, whereas assets that pay off when consumption levels are low “smooth” out consumption. Naturally, the latter are more valuable and thus require a lower rate of return to induce investors to hold these assets. (Insurance policies are a classic example of assets that smooth consumption. Individuals willingly purchase and hold them, in spite of their very low rates of return.)

To return to the original question: are stocks that much more riskier than bills so as to justify a 7% differential in their rates of return?

What came as a surprise to many economists and researchers in finance was the conclusion of a research paper that we wrote in 1979. Stocks and bonds pay off in approximately the same states of nature or economic scenarios and hence, as argued earlier, they should command approximately the same rate of return. In fact, using standard theory to estimate risk-adjusted returns, we found that stocks on average should command, at most, a 1% return premium over bills. Since, for as long as we had reliable data, (about a hundred years), the mean premium on stocks over bills was considerably and consistently higher, we realized that we had a puzzle on our hands. It took us six more years to convince a skeptical profession and for our paper “ The Equity Premium: A Puzzle” to be published. (Mehra and Prescott (1985)).
To illustrate the puzzle we consider a frictionless economy that has a single representative 'stand-in' household. This unit orders its preferences over random consumption paths by

\[ E_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\} , \quad 0 < \beta < 1 \]

where \( c_t \) is the per capita consumption, \( \beta \) is the subjective time discount factor, \( E_0 \{ \} \) is the expectation operator conditional upon information available at time zero, (which denotes the present time) and \( U \) is an increasing, continuously differentiable concave utility function. We further restrict the utility function to be of the constant relative risk aversion class (CRRA)

\[ U(c, \alpha) = \frac{c^{1-\alpha}}{1-\alpha}, \quad 0 < \alpha < \infty \]

where the parameter \( \alpha \) measures the curvature of the utility function. When \( \alpha = 1 \), the utility function is defined to be logarithmic, which is the limit of the above representation as \( \alpha \) approaches 1. The feature that makes this the “preference function of choice” in much of the literature of Growth and Real Business Cycle Theory is that it is scale invariant. Although the level of aggregate variables such as capital stock have increased over time, the resulting equilibrium return process is stationary.

A second attractive feature is that it is one of only two preference functions that allows for aggregation and a “stand in” representative agent formulation that is independent of the initial distribution of endowments. One disadvantage of this representation is that it links risk preferences with time preferences. With CRRA preferences, agents who like to smooth consumption across various states of nature also prefer to smooth consumption over time, that is, they dislike growth. Specifically, the coefficient of relative risk aversion is the reciprocal of the elasticity of intertemporal substitution. There is no fundamental economic reason why this must be so. We will revisit this issue in Section 3 where we examine preference structures that do not impose this restriction.\(^1\)

We assume there is one productive unit which produces output \( y_t \) in period \( t \), which is the period dividend. There is one equity share with price \( p_t \) (denominated in consumption units) that is competitively traded; it is a claim to the stochastic process \( \{ y_t \} \).

Consider the intertemporal choice problem of a typical investor at time \( t \). He equates the loss in utility associated with buying one additional unit of equity to the discounted expected utility of the

\(^1\) Epstein and Zin (1991) and Weil (1989)
resulting additional consumption next period. To carry over one additional unit of equity, \( p_t \) units of the consumption good must be sacrificed and the resulting loss in utility is \( p_t U'(c_t) \). By selling this additional unit of equity next period, \( p_{t+1} + y_{t+1} \) additional units of the consumption good can be consumed and \( \beta E_t \{ (p_{t+1} + y_{t+1}) U'(c_{t+1}) \} \) is the expected value of the incremental utility next period. At an optimum these quantities must be equal. Hence the fundamental relation that prices assets is

\[
p_t U'(c_t) = \beta E_t \{ (p_{t+1} + y_{t+1}) U'(c_{t+1}) \}. \tag{1}
\]

Versions of this expression can be found in Rubinstein (1976), Lucas (1978), Breeden (1979) and Prescott and Mehra (1980) among others. Excellent textbook treatments of asset pricing can be found in Cochrane (2001) and Danthine and Donaldson (2001).

We use (1) to price both stocks and riskless one period bonds.

For equity we have

\[
1 = \beta E_t \left\{ \frac{U'(c_{t+1})}{U'(c_t)} R_{e,t+1} \right\} \tag{2}
\]

where

\[
R_{e,t+1} = \frac{p_{t+1} + y_{t+1}}{p_t}
\]

and for the riskless one period bonds the relevant expression is

\[
1 = \beta E_t \left\{ \frac{U'(c_{t+1})}{U'(c_t)} \right\} R_{f,t+1} \tag{3}
\]

Where the gross rate of return on the riskless asset is, by definition,

\[
R_{f,t+1} = \frac{1}{q_t}
\]

with \( q_t \) being the price of the bond. We can rewrite (1) as

\[
1 = \beta E_t \{ M_{t+1} R_{e,t+1} \} \tag{4}
\]
where \( M_{t+1} = \frac{U'(c_{t+1})}{U'(c_t)} \). Since \( U(c) \) is assumed to be increasing, \( M_{t+1} \) is a strictly positive stochastic discount factor. This guarantees that the economy will be arbitrage free and the law of one price holds.

A little algebra shows that

\[
E_t(R_{e,t+1}) = R_{f,t+1} + \text{Cov}_t \left\{ \frac{-U'(c_{t+1}), R_{x,t+1}}{E_t(U'(c_{t+1}))} \right\}
\]

The equity premium \( E_t(R_{e,t+1}) - R_{f,t+1} \) can thus be easily computed. Expected asset returns equal the risk free rate plus a premium for bearing risk which depends on the covariance of the asset returns with the marginal utility of consumption. Assets that co vary positively with consumption, that is, they payoff in states when consumption is high and marginal utility is low, command a high premium since these assets “destabilize” consumption.

The question we need to address is the following: is the magnitude of the covariance between the marginal utility of consumption large enough to justify the observed 6% equity premium in US equity markets?

To address this issue we make some additional assumptions. While they are not necessary and were not, in fact, part of our original paper, we include them to facilitate exposition and because they result in closed form solutions. These assumptions are:

(a) the growth rate of consumption \( x_{t+1} \equiv \frac{c_{t+1}}{c_t} \) is i.i.d.

(b) the growth rate of dividends \( z_{t+1} \equiv \frac{y_{t+1}}{y_t} \) is i.i.d.

(c) \((x_t, z_t)\) are jointly log normally distributed.

A consequence of these assumptions is that the gross return on equity \( R_{e,j} \) (defined above) is i.i.d and that \((x_t, R_{e,j})\) are jointly lognormal.

Substituting \( U'(c_t) = c_t^{-\alpha} \) in the fundamental pricing relation

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2 See the Appendix.
3 Abel (1988) uses a similar framework.
4 See equation A1 in the Appendix
\[ p_t = \beta E_t \left\{ (p_{t+1} + y_{t+1})U'(\varepsilon_{t+1}) / U'(\varepsilon_t) \right\} \]

we get

\[ p_t = \beta E_t \left\{ (p_{t+1} + y_{t+1})x_s^{\alpha} \right\} \]  \hspace{1cm} (6)

It is easily shown that\(^5\)

\[ E_t \{ R_{x,t+1} \} = \frac{E_t \{ z_{t+1} \}}{\beta E_t \{ x_{t+1}^{\alpha} \}} \]  \hspace{1cm} (7)

Analogously, the gross return on the riskless asset can be written as

\[ R_{f,t+1} = \frac{1}{\beta} \frac{1}{E_t \{ x_{t+1}^{\alpha} \}} \]  \hspace{1cm} (8)

Since we have assumed the growth rate of consumption and dividends to be log normally distributed,

\[ E_t \{ R_{e,t+1} \} = e^{\mu_{x,+1} / \hat{a} - \sigma^2 / 2 \hat{a}^2 \sigma^2} \]  

and

\[ \ln E_t \{ R_{e,t+1} \} = -\ln \beta + \alpha \mu_x - 1 / \hat{a} \sigma^2 + \alpha \sigma_{x,z} \]  \hspace{1cm} (9)

where \( \mu_x = E(\ln x) \), \( \sigma^2 = Var(\ln x) \), \( \sigma_{x,z} = \text{Cov}(\ln x, \ln z) \) and \( \ln x \) is the \textit{continuously compounded} growth rate of consumption. The other terms involving \( z \) and \( R_e \) are defined analogously.

Furthermore, since the growth rate of consumption is i.i.d the conditional and unconditional expectations are the same.

Similarly

\[ R_f = \frac{1}{\beta e^{-\alpha \mu_x + 1/2 \alpha^2 \sigma^2}} \]

and

\[ \ln R_f = -\ln \beta + \alpha \mu_x - 1 / 2 \alpha^2 \sigma^2 \]  \hspace{1cm} (10)

\[ \therefore \ln E(R_e) - \ln R_f = \alpha \sigma_{x,z} \]  \hspace{1cm} (11)

In this model it also follows that

\[ \ln E(R_e) - \ln R_f = \alpha \sigma_{x,R_e} \]  \hspace{1cm} (12)

where

\[ \sigma_{x,R_e} = \text{Cov}(\ln x, 1 \ln R_e) \]

The (log) equity premium in this model is the product of the coefficient of risk aversion and the covariance of the (continuously compounded) growth rate of consumption with the (continuously compounded) return on equity or the growth rate of dividends. If we impose the equilibrium condition that \( x = z \), a consequence of which is that the restriction that the return on equity is perfectly correlated to the growth rate of consumption, we get

\[ \ln E(R_e) - \ln R_f = \alpha \sigma^2 \]

\(^5\)See the Appendix for derivations of equations 7-10.
and the equity premium then is the product of the coefficient of relative risk aversion and the variance of the growth rate of consumption. As we see below, this variance is 0.001369 so unless the coefficient of risk aversion $\alpha$ is large, a high equity premium is impossible. The growth rate of consumption just does not vary enough!

In Mehra & Prescott (1985) we report the following sample statistics for the U.S. economy over the period 1889-1978:

- risk free rate, $R_f = 1.008$
- mean return on equity, $E(R_e) = 1.0698$
- mean growth rate of consumption, $E(x) = 1.0183$
- standard deviation of the growth rate of consumption, $\sigma(x) = 0.0357$
- mean equity premium, $E(R_e) - R_f = 0.0618$

In our calibration, we are guided by the tenet that model parameters should meet the criteria of cross-model verification. Not only must they be consistent with the observations under consideration but they should not be *grossly inconsistent* with other observations in growth theory, business cycle theory, labor market behavior and so on. There is a wealth of evidence from various studies that the coefficient of risk aversion $\alpha$ is a small number, certainly less than 10. A number of these studies are documented in Mehra and Prescott (1985). We can then pose a question: if we set the risk aversion coefficient $\alpha$ to be 10 and $\beta$ to be 0.99 what are the expected rates of return and the risk premium using the parameterization above?

Using the expressions derived earlier we have

$$\ln R_f = -\ln \beta + \alpha \mu_x - 1/2 \alpha^2 \sigma_x^2 = 0.12383$$

or

$$R_f = 1.1318$$

that is, a risk free rate of 13.18%!

Since

$$\ln E(R_e) = \ln R_f + \alpha \sigma_x^2$$

$$= 0.1361$$

we have

$$E(R_e) = 1.1458$$

or a return on equity of 14.58%. This implies an equity risk premium of 1.40%, far lower than the 6.18% historically observed equity premium. In this calculation we have been very liberal in
choosing the values for $\alpha$ and $\beta$. Most studies indicate a value for $\alpha$ that is close to 3. If we pick a lower value for $\beta$, the riskfree rate will be even higher and the premium lower. So the 1.4% value represents the maximum equity risk premium that can be obtained in this class of models given the constraints on $\alpha$ and $\beta$. Since the observed equity premium is over 6%, we have a puzzle on our hands that risk considerations alone cannot account for.

Philippe Weil (1989) has dubbed the high risk free rate obtained above, “the risk free rate puzzle.” The short term real rate in the US averages less than one percent, while the high value of $\alpha$ required to generate the observed equity premium results in an unacceptably high risk free rate. The late Fischer Black\(^6\) proposed that $\alpha = 55$ would solve the puzzle. Indeed it can be shown that the US experience from 1889-1978, reported above, can be reconciled with $\alpha = 48.4$ and $\beta = 0.55$.

To see this, observe that since $\sigma_x^2 = \ln \left[ 1 + \frac{\text{var}(x)}{[E(x)]^2} \right] = 0.00123$

and $\mu_x = \ln E(x) - 1/2 \sigma_x^2 = 0.01752$

this implies $\alpha = \frac{\ln E(R) - \ln R_F}{\sigma_x^2}$

$= 48.44$

since $\ln \beta = -\ln R_F + \alpha \mu_x - 1/2 \alpha^2 \sigma_x^2$

$= -0.60041$

this implies $\beta = 0.55$.

Besides postulating an unacceptably high $\alpha$, another problem is that this is a “knife edge” solution. No other set of parameters will work and a small change in $\alpha$ will lead to an unacceptable risk free rate. An alternate approach is to experiment with negative time preferences; however there seems to be no empirical evidence that agents do have such preferences.

\(^6\) Private communication 1981.
The relation \( \ln R_f = -\ln \beta + \alpha \mu - 1/2 \alpha^2 \sigma^2 \) shows why an extremely high \( \alpha \) can be consistent with a “low” risk free rate. The last term dominates when \( \alpha \) is very large; however, then a small changes in the growth rate of consumption will have a large impact on interest rates. This is inconsistent with a cross-country comparison of real risk free rates and their observed variability. For example, throughout the eighties South Korea had a much higher growth rate than the US but real rates were not appreciably higher. Nor does the risk-free rate vary considerably over time, as would be expected if \( \alpha \) was large.

An alternative perspective on the puzzle is provided by Hansen and Jagannathan (1991). The fundamental pricing equation can be written as

\[
E_t(R_{e,t+1}) = R_{f,t+1} + \frac{\text{Cov}_t\left[ M_{t+1}, R_{e,t+1} \right]}{E_t(M_{t+1})}
\]

This expression also holds unconditionally, so that

\[
E(R_{e,t+1}) = R_{f,t+1} + \sigma(M_{t+1}) \sigma(R_{e,t+1}) \rho_{R,M} / E_t(M_{t+1})
\]

or

\[
E(R_{e,t+1}) - R_{f,t+1} / \sigma(R_{e,t+1}) = \sigma(M_{t+1}) \rho_{R,M} / E_t(M_{t+1})
\]

and since

\[-1 \leq \rho_{R,M} \leq 1\]

\[
\left| E(R_{e,t+1}) - R_{f,t+1} / \sigma(R_{e,t+1}) \right| \leq \sigma(M_{t+1}) / E(M_{t+1})
\]

This inequality is referred to as the Hansen-Jagannathan lower bound on the pricing kernel.

For the US economy, the Sharpe Ratio \( E(R_{e,t+1}) - R_{f,t+1} / \sigma(R_{e,t+1}) \) can be calculated to be 0.37. Since \( E(M_{t+1}) \) is the expected price of a one period risk free bond its value must be close to one. In fact, for the parameterization discussed earlier, \( E(M_{t+1}) = 0.96 \) when \( \alpha = 2 \). This implies that the lower bound on the standard deviation for the pricing kernel must be close to 0.3 if the Hansen-Jagannathan bound is to be satisfied. However, when this is calculated in the Mehra-Prescott framework, we obtain an estimate for \( \sigma(M_{t+1}) = 0.002 \) which is off by more than an order of magnitude.

We would like to emphasize that the equity premium puzzle is a \textit{quantitative} puzzle; standard theory is consistent with our notion of risk that, on average, stocks should return more than bonds. The puzzle arises from the fact that the quantitative predictions of the theory are an order of

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7 Kandel and Stambaugh (1991) have suggested this approach.
magnitude different from what has been historically documented. The puzzle cannot be dismissed lightly, since much of our economic intuition is based on the very class of models that fall short so dramatically when confronted with financial data. It underscores the failure of paradigms central to financial and economic modeling to capture the characteristic that appears to make stocks comparatively so risky. Hence the viability of using this class of models for any quantitative assessment, say, for instance, to gauge the welfare implications of alternative stabilization policies, is thrown open to question.

For this reason, over the last 15 years or so, attempts to resolve the puzzle have become a major research impetus in finance and economics. Several generalizations of key features of the Mehra and Prescott (1985) model have been proposed to better reconcile observations with theory. These include alternative assumptions on preferences, modified probability distributions to admit rare but disastrous events, survival bias, incomplete markets, and market imperfections; none have fully resolved the anomalies. In section 3 below we examine some of the research efforts to resolve the puzzle.


a) Modifying the conventional time and state separable utility function.

The analysis above shows that the isoelastic preferences used in Mehra and Prescott (1985) can only be made consistent with the observed equity premium if the coefficient of relative risk aversion is implausibly large. One restriction imposed by this class of preferences is that the coefficient of risk aversion is rigidly linked to the elasticity of intertemporal substitution. One is the reciprocal of the other. What this implies is that if an individual is averse to variation of consumption across different states at a particular point of time then he will be averse to consumption variation over time. There is no a priori reason that this must be so. Since, on average, consumption is growing

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9 Rietz (1988)
10 Brown, Goetzmann and Ross (1995)
11 For example, Constantinides and Duffie (1996), Heaton and Lucas (1997), Mankiw (1986) and Storesletten, Telmer and Yaron (1999).
13 The reader is also referred to the excellent surveys by Narayana Kocherlakota (1996), John Cochrane (1997) and by John Campbell (1999).
over time, the agents in the Mehra and Prescott (1985) setup have little incentive to save. The demand for bonds is low and the riskfree rate as a consequence is counterfactually high. Epstein and Zin (1991) have presented a class of preferences that they term “Generalized Expected Utility” (GEU) which allows independent parameterization for the coefficient of risk aversion and the elasticity of intertemporal substitution.

In this class of preferences, utility is recursively defined by
\[
U_t = \{c_t^{-\rho} + \beta [E_t U_{t+1}]^{(1-\rho)/(1+\alpha)}\}^{1/(1-\rho)}
\]
The usual isoelastic preferences follow as special case when \(\alpha = \rho\). The major advantage of this class of models is that a high coefficient of risk aversion, \(\alpha\), does not necessarily imply that agents will want to smooth consumption over time. This modification has the potential to resolve the risk free rate puzzle. However, the main difficulty in testing this alternative preference structure stems from the fact that the counterparts of equations (1) and (2) using GEU depend on the unobserved utility at time \(t+1\), and this makes calibration tricky. One would need to make specific assumptions on the consumption process to obtain first order conditions in terms of observables. Although Epstein and Zin (1991) claim that their framework offers a solution to the equity premium puzzle, we feel that the proxies that they use for the unobservables overstate their claim. This framework does, however, ameliorate the risk free rate puzzle.

b) Habit Formation.

A second approach to modifying preferences was initiated by Constantinides (1990), by incorporating habit formation. This formulation assumes that utility is affected not only by current consumption but also by past consumption. It captures the notion that utility is a decreasing function of past consumption and marginal utility is an increasing function of past consumption.

Utility is defined as
\[
U(c) = E \sum_{s=0}^{\infty} \beta^s \frac{(c_{t+s}-\lambda c_{t+s-1})^{1-\alpha}}{1-\alpha}, \lambda > 0
\]
This preference ordering makes the agent extremely averse to consumption risk even when the risk aversion is small. For small changes in consumption, changes in marginal utility can be large. So while this approach cannot solve the equity premium puzzle without invoking extreme aversion to consumption risk, it can address the risk free rate puzzle since the induced aversion to consumption risk increases the demand for bonds, thereby reducing the riskfree rate.
A modification of the Constantinides (1990) approach is to define utility of consumption relative to average per capita consumption. Abel (1990) terms this “Keeping up with the Joneses.” The idea is that one’s utility depends not on the absolute level of consumption but on how one is doing relative to others. The effect is that, once again, an individual can become extremely sensitive and averse to consumption variation. Equity may have a negative rate of return and this can result in personal consumption falling relative to others. Equity, thus becomes an undesirable asset relative to bonds. Since average per capita consumption is rising over time, the induced demand for bonds with this modification helps in mitigating the risk free rate puzzle.

An alternate approach expounded by Campbell and Cochrane (1999) incorporates the possibility of recession, that is to say, a major economic downturn as a state variable. In this model the risk aversion of investors rises dramatically when the chances of a recession become larger and thus the model can generate a high equity premium. Since risk aversion increases precisely when consumption is low, it generates a precautionary demand for bonds that helps lower the risk free rate. This model is consistent with both consumption and asset market data. However, it is an open question as to whether investors actually display the huge, time varying, counter cyclical variations in risk aversion postulated in this model.

To summarize, models with habit formation, relative or subsistence consumption have had success in addressing the risk free rate puzzle but only limited success with resolving the equity premium puzzle since in these models effective risk aversion & prudence become impossibly large.

4. Idiosyncratic and uninsurable income risk.

In infinite horizon models, agents when faced with uninsurable income shocks, dynamically self insure; agents simply stock up on bonds when times are good and deplete them in bad times, thereby effectively smoothing consumption. Hence the difference in the equity premium in incomplete markets and complete markets is small. (Heaton and Lucas (1996,1997))

The analysis changes when a shock is permanent. A model incorporating heterogeneity, proposed by Constantinides and Duffie (1996), captures the notion that consumers are subject to idiosyncratic income shocks that cannot be insured away. Simply put, consumers face the risk of job loss, or other major personal disasters that cannot be hedged away or insured against. Equities and related pro-cyclical investments exhibit the undesirable feature that they drop in
value when the probability of job-loss increases, as it does, for instance, in recessions. In economic downturns, consumers thus need an extra incentive to hold equities and other similar investment instruments; the equity premium can then be rationalized as the added inducement needed to make equities palatable to investors. This model can generate a high risk premium but it remains to be seen if the required degree of consumption variation can be generated in an economy populated with agents displaying a relatively low level of risk aversion.

5. Models Incorporating a Disaster State and Survivorship Bias.

Reitz (1988) has proposed a solution to the puzzle that incorporates a very small probability of a very large drop in consumption. He finds that in such a scenario the riskfree rate is much lower than the return on an equity security. This model requires a one in hundred chance of a 25% decline in consumption to reconcile the equity premium with a risk aversion parameter of 10. Such a scenario has not been observed in the US for the last hundred years, for which we have economic data. Nevertheless, one can evaluate the implications of the model. One implication is that the real interest rate and the probability of the occurrence of the extreme event move inversely. For example, the perceived probability of a recurrence of a depression were probably very high just after World War II and subsequently declined over time. If real interest rates rose significantly as the war years receded, that evidence would support the Reitz hypothesis. Similarly, if the low probability event precipitating the large decline in consumption were a nuclear war, the perceived probability of such an event has surely varied over the last hundred years. It must have been low before 1945, the first and only year the atom bomb was used; and it must have been higher before the Cuban Missile Crisis than after it. If real interest rates moved as predicted, that would support Rietz’s disaster scenario. But they did not.

Another attempt at resolving the puzzle proposed by Brown et al (1995) focuses on survival bias.

The central thesis here is that the ex-post measured returns reflect the premium, in the US, on a stock market that has successfully weathered the vicissitudes of fluctuating financial fortunes. Many other exchanges were unsuccessful and hence the ex-ante equity premium was low. Since it was not known a priori which exchanges would survive, for this explanation to work, stock and bond markets must be differentially impacted by a financial crises. Governments have expropriated much of the real value of nominal debt by the mechanism of unanticipated
inflation. Five historical instances come readily to mind: During the post war period of German hyperinflation, holders of bonds denominated in Reich marks lost virtually all of the value invested in those assets. During the 1920s Poincare’s administration in France, bondholders lost nearly 90% of the value invested in nominal debt. And in the 1980s, Mexican holders of dollar denominated debt lost a sizable fraction of its value when the Mexican government, in a period of rapid inflation, converted the debt to pesos and limited the rate at which these funds could be withdrawn. Czarist bonds in Russia and Chinese debt holdings (subsequent to the fall of the Nationalists), suffered a similar fate under the communist regimes.

The above examples demonstrate that in times of financial crises, bonds are as likely to lose value as stocks. Although a survival bias may impact on the levels of both the return on equity and debt there is no evidence to support the assertion that these crises impact differentially on the returns to stocks and bonds, hence the equity premium is not impacted. In every instance where, due to political upheavals, etc., trading in equity has been suspended, governments have either reneged on their debt obligations or expropriated much of the real value of nominal debt by the mechanisms of unanticipated inflation.


In models with borrowing constraints and transaction costs the effect is to force investors to hold an inventory of bonds (precautionary demand) to smooth consumption. Hence in infinite horizon models with borrowing constraints, agents come close to equalizing their marginal rates of substitution with little effect on the equity premium. Some recent attempts to resolve the puzzle incorporating both borrowing constraints and consumer heterogeneity appear promising. One approach, which departs from the representative agent model, has been proposed in Constantinides, Donaldson and Mehra (2001).

The novelty of their paper lies in incorporating a life-cycle feature to study asset pricing. The idea is appealingly simple. As discussed earlier, the attractiveness of equity as an asset depends on the correlation between consumption and equity income. If equity pays off in states of high marginal utility of consumption, it will command a higher price, (and consequently a lower rate of return), than if its payoff is in states where marginal utility is low. Since the

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14 This is true unless the supply of bonds is unrealistically low. See Aiyagari and Gertler (1991).
marginal utility of consumption varies inversely with consumption, equity will command a high rate of return if it pays off in states when consumption is high, and vice versa.\textsuperscript{15}

A key insight of their paper is that as the correlation of equity income with consumption changes over the life cycle of an individual, so does the attractiveness of equity as an asset. Consumption can be decomposed into the sum of wages and equity income. A young person looking forward in his life has uncertain future wage \textit{and} equity income; furthermore, the correlation of equity income with consumption will not be particularly high, as long as stock and wage income are not highly correlated. This is empirically the case, as documented by Davis and Willen (2000). Equity will thus be a hedge against fluctuations in wages and a “desirable” asset to hold as far as the young are concerned.

The same asset (equity) has a very different characteristic for the middle aged. Their wage uncertainty has largely been resolved. Their future retirement wage income is either zero or deterministic and the innovations (fluctuations) in their consumption occur from fluctuations in equity income. At this stage of the life cycle, equity income is highly correlated with consumption. Consumption is high when equity income is high, and equity is no longer a hedge against fluctuations in consumption; hence, for this group, it requires a higher rate of return.

The characteristics of equity as an asset therefore change, depending on who the predominant holder of the equity is. Life cycle considerations thus become crucial for asset pricing. If equity is a “desirable” asset for the marginal investor in the economy, then the observed equity premium will be low, relative to an economy where the marginal investor finds it unattractive to hold equity. The \textit{deus ex machina} is the \textit{stage} in the life cycle of the marginal investor.

The authors argue that the young, who should be holding equity in an economy without frictions and with complete contracting, are effectively shut out of this market because of borrowing constraints. They are characterized by low wages; ideally they would like to smooth lifetime consumption by borrowing against future wage income (consuming a part of the loan and investing the rest in higher return equity). However, they are prevented from doing so because human capital alone does not collateralize major loans in modern economies for reasons of moral hazard and adverse selection.

\textsuperscript{15} This is precisely the reason as explained earlier why high-beta stocks in the simple CAPM framework have a high rate of return. In that model, the return on the market is a proxy for consumption. High-beta stocks pay off when the market return is high, i.e. when marginal utility is low, hence their price is (relatively) low and their rate of return high.
In the presence of borrowing constraints, equity is thus exclusively priced by the middle-aged investors since the young are effectively excluded from the equity markets and we observe a high equity premium. If the borrowing constraint is relaxed, the young will borrow to purchase equity, thereby raising the bond yield. The increase in the bond yield induces the middle-aged to shift their portfolio holdings from equity to bonds. The increase in the demand for equity by the young and the decrease in the demand for equity by the middle-aged work in opposite directions. On balance, the effect is to increase both the equity and the bond return while simultaneously shrinking the equity premium. Furthermore, the relaxation of the borrowing constraint reduces the net demand for bonds and the risk free rate puzzle re-emerges.

In order to systematically illustrate these ideas, the authors construct an overlapping-generations (OLG) exchange economy in which consumers live for three periods. In the first period, a period of human capital acquisition, the consumer receives a relatively low endowment income. In the second period, the consumer is employed and receives wage income subject to large uncertainty. In the third period, the consumer retires and consumes the assets accumulated in the second period. The authors explore the implications of a borrowing constraint by deriving and contrasting the stationary equilibria in two versions of the economy. In the borrowing-constrained version, the young are prohibited from borrowing and from selling equity short. The borrowing-unconstrained economy differs from the borrowing-constrained one only in that the borrowing constraint and the short-sale constraint are absent.

The model introduces two forms of market incompleteness. First, consumers of one generation are prohibited from trading claims against their future wage income with consumers of another generation. Second, consumers of one generation are prohibited from trading bonds and equity with consumers of an unborn generation. The model suppresses a third and potentially important form of market incompleteness that arises from the inability of an age cohort of consumers to insure, via pooling, the risks of their persistent, heteroscedastic idiosyncratic income shocks. Specifically, the authors model each generation of consumers with a representative consumer. This assumption is justified only if there exists a complete set of claims through which heterogeneous consumers within a generation can pool their idiosyncratic income shocks. Absent a complete set of such claims, consumer heterogeneity in the form of uninsurable, persistent and heteroscedastic idiosyncratic income shocks, with counter-cyclical

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16 Being homogeneous within their generation, consumers have no incentive to trade claims with consumers of their own generation.
conditional variance, has the potential to resolve the empirical difficulties encountered by representative-consumer models. Nevertheless, consumer heterogeneity within a generation is downplayed in the model in order to isolate and explore the implications of heterogeneity across generations in a parsimonious paradigm.

7. Liquidity Premium

Bansal and Coleman (1996) develop a monetary model that offers an explanation of the equity premium. In their model, assets other than money play a key feature by facilitating transactions. This affects the rate of return they offer in equilibrium.

To motivate the importance of considering the role of a variety of assets in facilitating transactions, they argue that on the margin, the transaction service return of money relative to interest bearing checking accounts should be the interest rate paid on these accounts. They estimate this to be 6% based on the rate offered on NOW accounts for the period they analyze. Since this is a substantial number they suggest that other money-like assets may also implicitly include a transaction service component to their return. Insofar as T-bills and equity have a different service component built into their returns, this may offer an explanation for the observed equity premium. In fact, if this service component differential was about 5% there would be no equity premium puzzle. However, this approach can be challenged on three accounts. First, the bulk of T-bills holdings are concentrated in institutions that do not use them as compensatory balances for checking accounts and thus it is difficult to reconcile their having a significant transaction service component. Secondly, the returns on NOW and other interest bearing accounts have varied over time, being higher post 1980 than in earlier periods. In fact, for most of the twentieth century, checking accounts were not interest bearing. However, contrary to the implications of this model, the equity premium has not diminished in the post 1980 period when presumably the implied transaction service component was the greatest. Finally, this model implies that there should be a significant yield differential between T-Bills and long term government bonds that presumably do not have a significant transaction service component. However, this has not been the case.

17 See Mankiw (1986) and Constantinides and Duffie (1996).
8. Taxes

In Mehra and Prescott (1985) we did not take into account changes in tax and regulatory policies or institutions affecting asset markets. McGrattan and Prescott (2001) have proposed an explanation for the equity premium based on changes in tax rates. It is important to note that theirs is not a risk-based explanation. They can account for an equity premium but not as an equity risk premium.

They find that, at least over the postwar period, the equity premium is not puzzling. They argue that the large reduction in individual income tax rates and the increased opportunity to shelter income from taxation lead to a doubling of equity prices between 1960 and 2000. This increase in equity prices in turn, resulted in much higher ex-post returns on equity than on debt.

Their argument can be illustrated in a simple one sector (a corporate sector) model that includes only taxes on corporate distributions and taxes on corporate profits as shown below. In their paper, they extend the model to include sufficient details of the U.S. economy - especially in relation to the tax code - to allow them to calibrate the model. They match up the model with data from the National Income and Product Accounts (NIPA) and the Statistics of Income (SOI) of the Internal Revenue Service.

Consider an infinitely-lived representative agent economy with household preferences defined over consumption and leisure. Each household chooses sequences of consumption and leisure to maximize

\[ \max \sum_{t=0}^{\infty} \beta^t U(c_t, l_t), \quad 0 < \beta < 1 \]

subject to their budget constraint,

\[ \sum_{t=0}^{\infty} p_t \{ c_t + V_t(s_{t+1} - s_t) \} = \sum_{t=0}^{\infty} p_t \{ (1 - \tau_{pers})(d_t s_t + w_t n_t) + \kappa_t \} \]

where \( c_t \) is per-capita consumption at time \( t \) and \( l_t \) is the fraction of productive time allocated to non-market activities such as leisure. The fraction of time allocated by households to market activities is denoted by \( n = 1 - l_t \). The budget constraint represents the condition that the present discounted
value of expenditures must be less than or equal to the present discounted value of after-tax income. Expenditures of the household are consumption and purchases of stocks, $V_t(s_{t+1} - s_t)$, where $s_t$ is the number of shares held in period $t$ and $V_t$ is the price per share. Income for the households is received from three sources: dividends ($d_t$), wages ($w_t$), and government transfers ($\kappa_t$). Households pay personal taxes ($\tau_{pers}$) on dividend and wage income.

Firms own capital and hire labor to produce output with a constant-returns-to-scale production technology,

$$y_t = f(k_{m,t}, k_{u,t}, z_t, n_t)$$

This specification assumes that firms use both tangible assets ($k_m$), and intangible assets $k_u$ to produce output $y$. Tangible assets include structures, equipment, inventories and land. Intangible assets are the result of on-the-job training, R&D, organization building, and firm-specific learning by doing. In addition to capital, labor services $n$ are required. The parameter $z_t$ is the level of technology in period $t$, which is assumed to grow at rate $\gamma$. Firms choose capital and labor to maximize the present value of dividends. In this framework it is shown that the value of corporate equity is

$$V_t = (1 - \tau_{pers})[k_{m,t+1} + (1 - \tau_{corp})k_{u,t+1}]$$

From this it is clear that a large drop in the personal tax rate with little change in the corporate tax rate will indeed raise the value of equity. However, to show that the market value of U.S. corporate equity relative to GNP, $V/y$, rises with a decline in the personal tax rate with little change in the corporate tax rate, it must be the case that capital-output ratios do not change with changes in $\tau_{pers}$, only with changes in $\tau_{corp}$. This is shown in the paper.

In the U.S., corporate income tax rates have changed little since 1960 while personal income tax rates have declined significantly. In particular, personal tax rates in the period prior to the tax cuts initiated by the Kennedy administration were considerably higher than in the period after the Tax Reform Act of 1986. McGrattan and Prescott (2001) argue that this reduction in personal tax rates was largely unforeseen and that the large unanticipated increase in equity prices had a significant effect on equity returns.

As an illustration, McGrattan and Prescott (2001) propose the following hypothetical
example. Suppose that the tax rate on dividends falls from 50 percent to 0 percent in forty years. This fall in the tax rate implies that equity prices will grow by approximately 1.8 percent per year higher than the growth in the GNP. Growth in per-capita GNP over the postwar period has been 2 percent per year, while the growth in population was roughly 1.5 percent in the early part of the postwar period and fell to about 1 percent per year towards the end of the postwar period. This implies an average growth in equity prices of around 4.8 to 5.3 percent per year. To compute a return, a dividend yield must be added. The rise in equity prices implies a fall in the dividend yield since the dividend-output ratio stays roughly constant. If, on average, the dividend yield is in the range of 3 to 4 percent, then equity returns will be in the range of 7.8 to 9.3 percent, a figure which is of the same magnitude as documented in Table 2 for the post war period.

Additionally, if it is assumed that households had a liquidity motive for holding debt, bond returns would be low and the resulting equity premium large. The low yield on debt is further reinforced by the additional demand for debt induced by constraints on individuals to hold their retirement assets in debt securities. Indeed, in the first half of the postwar period, pension funds assets were almost entirely invested in debt securities because of institutional restrictions on pension fund managers. In 1974, the Employee Retirement Income Security Act (ERISA) was passed. This act made pension plan fiduciaries personally liable for imprudence or misconduct. Prior to that, fiduciary breaches typically resulted in a loss of tax qualification for the pension fund. Such penalties were likely to be avoidable if pension fund managers held debt assets of various maturities, so as to avoid large fund losses and to facilitate the timing of the distribution of benefits.

A potential calibration problem with the McGrattan-Prescott model arises because of the difficulty in identifying the marginal investor. While the marginal tax rate has dramatically declined in the post war period, the same cannot be said about the rates that apply to the marginal investor, without identifying the marginal investor. It is possible that the marginal tax rates and the rates that apply to the marginal investors may be quite different and of course it is the latter rate that is relevant for pricing. To estimate the tax rate that applies to the marginal investor, McGrattan and Prescott compute a weighted average rate, averaged across income groups, for each year in the 1947—1996 time period. However, the accuracy of this estimate is open to question.

One implication of their model is that it predicts that eventually, as the dividend yield falls and tax rates level out, ceteris paribus, equity returns will decline. The real before-tax return on equity is the sum of three returns: the dividend yield, the anticipated capital gain, and the unanticipated
capital gain. The dividend yield has been high, (over 3 percent), for much of the postwar period because high tax rates have implied a low price of equity. Recently it has declined and is presently just over one percent. The anticipated capital gain is the growth rate of productive assets, which is roughly three percent. This growth rate has not changed. The unanticipated capital gain is the growth in the price of equity due to unanticipated changes in tax rates. This growth rate has changed, falling from a range of 1.5 to 2 percent, to 0 percent. Adding these rates, we would expect an 8 percent (3+3+2) real, before-tax stock return in the early postwar period and, barring any further unexpected changes in tax rates, a 4 percent (1+3+0) return in the future.

9. No Premium?

There is also an alternative point of view, held by a group of academicians and professionals who claim that at present there is no equity premium and, by implication, no equity premium puzzle. To address these claims we need to differentiate between two different interpretations of the term “equity premium”. One is the ex-post or realized equity premium. This is the actual, historically observed difference between the return on the market, as captured by a stock index, and the risk free rate, as proxied by the return on government bills. This is what Edward Prescott and I addressed in the 1985 paper. However, there is a related concept -- the ex-ante equity premium. This is a forward-looking measure of the premium, that is, the equity premium that is expected to prevail in the future or the conditional equity premium given the current state of the economy. To elaborate, after a bull market, when stock valuations are high relative to fundamentals the ex-ante equity premium is likely to be low. However, it is precisely in these times, when the market has risen sharply, that the ex-post, or the realized premium is high. Conversely, after a major downward correction, the ex-ante (expected) premium is likely to be high while the realized premium will be low. This should not come as a surprise since returns to stock have been documented to be mean reverting.

Which of these interpretations of the equity premium is relevant for an investment advisor? Clearly this depends on the planning horizon. The equity premium that we documented in our 1985 paper is for very long investment horizons. It has little to do with what the premium is going to be over the next couple of years. The ex-post equity premium is the realization of a stochastic process over a certain period and it has varied considerably over time. Furthermore, the variation depends on the time horizon over which it is measured.
There have even been periods when it has been negative, as illustrated in the figures below.

Source: Ibbotson 2001

Market watchers and other professionals who are interested in short term investment planning will wish to project the conditional equity premium over their planning horizon. This is by no means a simple task. Even if the conditional equity premium given current market conditions is small, and there appears to be general consensus that it is, this in itself does not imply that either the historical premium was too high or that the equity premium has diminished.

The data used to document the equity premium over the past hundred years is as good an economic data set as we have and a hundred years is long series when it comes to economic data. Before we dismiss the premium, not only do we need to understand the observed phenomena but we also need a plausible explanation as to why the future is likely to be any different from the past. In the absence of this, and based on what we currently know, we can make the following claim: over the long horizon, the equity premium is likely to be similar to what it has been in the past and that returns to investment in equity will continue to substantially dominate returns to investment in T-bills for investors with a long planning horizon.
References


Appendix A

Some facts about the log normal distribution.

a) If \( \ln z \sim N(\mu_z, \sigma_z^2) \) then \( a \ln z \sim N(a \mu_z, a^2 \sigma_z^2) \)

b) \( E(z^a) = E[\exp[a \ln z]] = \exp[a \mu_z + 1/2 a^2 \sigma_z^2] \)

c) \( a \ln z + b \ln x \sim N(a \mu_z + b \mu_x, a^2 \sigma_z^2 + b^2 \sigma_x^2 + 2 ab \rho \sigma_z \sigma_x) \)
   where \( \rho = \text{corr}(\ln x, \ln z) \)

d) \( E[z^a x^b] = \exp[a \mu_z + b \mu_x + 1/2 (a^2 \sigma_z^2 + b^2 \sigma_x^2)] \)

e) if \( x = z \) then
   \( E[z^a x^b] = E[x^{a+b}] = \exp[(a+b) \mu_x + 1/2 (a+b)^2 \sigma_x^2] \)

f) \( \text{Var}(x) = E[x^2] - (E(x))^2 = \exp[2\mu_x + 1/2 \cdot 4 \sigma_x^2] - \exp[2\mu_x + 1/2 \cdot 2] = \exp[2\mu_x + 1/2 \cdot 2 \sigma_x^2] \cdot [\exp[1/2 \cdot 2 \sigma_x^2] - 1] = (E(x))^2 \cdot (\exp 1/2 \cdot 2 \sigma_x^2 - 1) \)

g) \( \exp(\sigma_x^2) = 1 + \frac{\text{var}(x)}{(E(x))^2} \)

h) \( \ln E(x) = \mu_x + 1/2 \sigma_x^2 \)
   \( \mu_x = \ln E(x) - 1/2 \sigma_x^2 \)
Derivation of equations (5).

Expanding (4) results in
\[
\beta E_t \left\{ \frac{U'(c_{t+1})}{U'(c_t)} \right\} E_t(R_{t,t+1}) + \beta \text{Cov}_t \left\{ \frac{U'(c_{t+1}), R_{t,t+1}}{U'(c_t)} \right\} = 1
\]

Substituting for \( R_{f,t+1} \) from equation 3 below
\[
1 = \beta E_t \left\{ \frac{U'(c_{t+1})}{U'(c_t)} \right\} R_{f,t+1}
\]
results in
\[
E_t(R_{e,t+1}) = R_{f,t+1} + \text{Cov}_t \left\{ \frac{-U'(c_{t+1}), R_{t,t+1}}{E_t(U'(c_{t+1}))} \right\}
\]
This is equation 5 in the paper.

Derivation of equations (7)- (10) in the text.

As \( p_t \) is homogeneous of degree one in \( y \) we can represent it as
\[
p_t = wy_t
\]
Substituting for \( U'(c_t) = c_t^{-\alpha} \) and \( p_t \) in the fundamental pricing relation
\[
p_t = \beta E_t \left\{ \left( p_{t+1} + y_{t+1} \right) U'(c_{t+1}) / U'(c_t) \right\}
\]
results in
\[
w y_t = \beta E_t \left\{ \left( w y_{t+1} + y_{t+1} \right) x_{t+1}^{-\alpha} \right\}
\]
hence
\[
w = \beta E_t \left\{ (w+1) z_{t+1} x_{t+1}^{-\alpha} \right\}
\]
or
\[
w = \frac{\beta E_t \left\{ z_{t+1} x_{t+1}^{-\alpha} \right\}}{1 - \beta E_t \left\{ z_{t+1} x_{t+1}^{-\alpha} \right\}}
\]
By definition, \( R_{e,t+1} \) the gross rate of return on equity is
\[
R_{e,t+1} = \frac{p_{t+1} + y_{t+1}}{p_t}
\]
Substituting for \( p_t \) we get
\[
R_{e,t+1} = \frac{(w+1) \cdot y_{t+1}}{w} = \frac{w+1}{w} \cdot z_{t+1} \quad (A1)
\]
or
\[
E_t\{R_{e,t+1}\} = \frac{w+1}{w} \cdot E_t\{z_{t+1}\}
\]
since 

\[ \frac{w + 1}{w} = \frac{1}{\beta E_t(z_{t+1}x_{t+1}^a)} \]

we have 

\[ E_t(R_{t+1}) = \frac{E_t(z_{t+1})}{\beta E_t(z_{t+1}x_{t+1}^a)} \]

which is equation (7) in the text.

Analogously, the gross return on the riskless asset can be written as 

\[ R_{f,t+1} = \frac{1}{\beta} \frac{1}{E_t(x_{t+1}^a)} \]

Using the lognormal properties b) and d) above we get 

\[ E_t(R_{t+1}) = \frac{e^{\mu_{t+1} / \sigma^2}}{\beta e^{\mu_t - \alpha \mu + 1 / 2 \sigma^2 + \alpha \sigma^2 - \lambda \alpha \sigma z_t}} \]

and 

\[ R_f = \frac{1}{\beta e^{-\alpha \mu + 1 / 2 \alpha \sigma^2}} \]

taking logs on both sides results in 

\[ \ln E_t(R_{t+1}) = -\ln \beta + \alpha \mu - 1 / \alpha^2 \sigma_x^2 + \alpha \sigma z_t \]

and \[ \ln R_f = -\ln \beta + \alpha \mu - 1 / 2 \alpha^2 \sigma_z^2 \]