SOLUTIONS
ECONOMICS 172B
Problem Set #5, Due Wednesday, May 16, 2007

1. (Like problems 3.1-10 and 12.1-1 in Hillier and Lieberman) A manufacturing firm has excess production capacity. It is considering devoting the excess capacity to one or more of three products, A, B, and C. The potential bottlenecks in producing the products are inputs of the processes milling, turning, and grinding. The available capacities of milling, turning, and grinding are 500, 350 and 150, respectively. The machine-hours required per unit of the three products are given in the table.

<table>
<thead>
<tr>
<th>Product</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milling machine</td>
<td>9</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Turning machine</td>
<td>5</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Grinding machine</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Variable costs per unit of products A, B, and C are $25, $10, and $15 respectively. The inverse demand curves for the products are $35 + 100x_A^{-1/3}$, $15 + 40x_B^{-1/4}$, and $20 + 50x_C^{-1/2}$. The objective is to maximize net revenue from the available capacities. Print the program and your other answers on one side of one sheet of 8.5 x 11 paper, in portrait configuration. Print your name (using excel, of course) in the upper right hand corner. (Hint: $x_A = 28.67205$, $p_A = 67.67235$, objective = 2548.079, $\lambda_G = 0$).

(a) Formulate as a convex program.

Maximize $x_A(10 + 100x_A^{-1/3}) + x_B(5 + 40x_B^{-1/4}) + x_C(5 + 50x_C^{-1/2})$
subject to $9x_A + 3x_B + 5x_C \leq 500$
$5x_A + 4x_B \leq 350$
$3x_A + 2x_C \leq 150$

(b) Briefly explain the objective.
Answer: Inverse demand gives the price at which the given quantity is sold. Subtract from that price the variable cost to get net revenue per unit sold.

(c) Solve using Solver in excel and find the values of the resources using the sensitivity analysis option. Excel sheet has the whole solution.

(d) Find the prices of the products. Explain very briefly
I entered the formulae for the prices in the spread sheet where usually the constant net revenues in a linear programming problem would go.

2. Solve by hand the following convex programming problem, writing your answers carefully in the available space. (Hint: The solutions are $x_1 = 3.350739$, $x_2 = 2.539621$, and $\lambda = 0.16219096$).

Maximize $2x_1^{1/2} + x_2^{1/2}$
subject to $x_1^3 + x_2^3 \leq 54$

(a) Write the Lagrangian. Explain why the first-order necessary conditions are sufficient for a global maximum.

$L(x, \lambda; b = 54) = 2x_1^{1/2} + x_2^{1/2} + \lambda(54 - x_1^3 - x_2^3)$

(b) Write the first-order necessary conditions.

$\frac{\partial L}{\partial x_1} = x_1^{-1/2} - \lambda 3x_1^2 = 0$ (*)
$\frac{\partial L}{\partial x_2} = \frac{1}{2}x_2^{-1/2} - \lambda 3x_2^2 = 0$ (**) 
$\frac{\partial L}{\partial \lambda} = 54 - x_1^3 - x_2^3 = 0$ (***)
(c) Solve the conditions for $x_1$ and $x_2$.

Answer: Solve the first two conditions for $x_2$ as a function of $x_1$. From (*) and (**)

$$2 \left( \frac{x_2}{x_1} \right)^{1/2} = \left( \frac{x_1}{x_2} \right)^2$$

$$\left( \frac{x_2}{x_1} \right)^{2.5} = \frac{1}{2}$$

$$\frac{x_2}{x_1} = \left( \frac{1}{2} \right)^{1/2.5}$$

$$x_2 = 0.75786x_1$$  (##)

Now substitute in the constraint:

$$x_1^3(1 + \left( \frac{1}{2} \right)^{3/2.5}) = 54$$

$$x_1 = \left( \frac{54}{1 + \left( \frac{1}{2} \right)^{3/2.5}} \right)^{1/3}$$

$$x_1 = 3.3508$$

$$x_2 = 2.5394$$

(d) Solve for the Lagrangian multiplier.

From (*)

$$\lambda = \frac{1}{3}x_1^{-5/2}$$

$$= .016218$$

Close enough given the accuracy of the software.

(e) Solve for the relative prices of $x_1$ and $x_2$.

$$\frac{p_1}{p_2} = \frac{f_1}{f_2} = \frac{\frac{\partial}{\partial x_1} (2x_1^{1/2} + x_2^{1/2})}{\frac{\partial}{\partial x_2} (2x_1^{1/2} + x_2^{1/2})}$$

$$= \frac{1}{2} \left( \frac{x_2}{x_1} \right)^{1/2}$$

$$= \frac{1}{2} \cdot 0.75786^{1/2}$$

$$= .43528$$

from (##).