1. Let \( r \) be equal to the last three digits of your registration number (PID, perm number), divided by 1000. That is, if the last three numbers are 678, \( r = .678 \). Consider the problem

Maximize \( 10 - (x_1 - 4)^2 - (x_2 - 6)^2 \)
subject to \( x_1 + x_2 = 5 - r \)

(a) Write the Lagrangian.

My \( r \) is .678 so the constraint becomes \( x_1 + x_2 = 4.322 \).

\[
\mathcal{L}(x, \lambda) = 10 - (x_1 - 4)^2 - (x_2 - 6)^2 - \lambda(x_1 + x_2 - 4.322)
\]

(b) Write the first order conditions.

\[
\frac{\partial \mathcal{L}(x, \lambda)}{\partial x_1} = -2(x_1 - 4) - \lambda = 0 \quad (\#)
\]
\[
\frac{\partial \mathcal{L}(x, \lambda)}{\partial x_2} = -2(x_2 - 6) - \lambda = 0 \quad (**)
\]
\[
\frac{\partial \mathcal{L}(x, \lambda)}{\partial \lambda} = x_1 + x_2 - 4.322 = 0 \quad (***)
\]

(c) Why are the first-order conditions sufficient? (35 words)

The objective function is concave. The constraint set is convex [or the functions in the constraint set are convex]. Under these and some regularity conditions, the first order conditions are sufficient. (31 words)

(d) Fill in the blanks for the optimum values and show your derivation briefly below.

\( x_1^* = \underline{1.161} \), \( x_2^* = \underline{3.161} \), \( \lambda^* = \underline{5.678} \)

From (*), \( 2x_1 = 8 - \lambda \) and from (**) \( 2x_2 = 12 - \lambda \). Adding the two equations gives

\[ 2(x_1 + x_2) = 20 - 2\lambda \]

From (***) \( x_1 + x_2 = 4.322 \), and substituting in the equation yields \( 8.644 = 20 - 2\lambda \). That solves to \( \lambda = 5.678 \). Substitute that value of \( \lambda \) in (*) and get \( x_1 = 1.161 \), and similarly from (**) \( x_2 = 3.161 \).

(e) What is the interpretation of \( \lambda^* \)? (50 words)

The \( \lambda^* \) is the shadow price of the resource. If the market price is less than that, the firm should buy more. If the market price is more, the firm should sell part of its holding. \( \lambda^* \) is the penalty rate that causes the resource constraint to be barely satisfied. (48 words)

2. Using the same value for \( r \) as in the first problem, look at the function

\[ .6 \ln(x_1 + 1) + .4 \ln(x_2 + 1) - \lambda(x_1 + 2x_2 - 7 - r) \]

(a) Taking \( \lambda \) as fixed, maximize over \( x_1 \) and \( x_2 \) and write out the solution functions \( x_1(\lambda) \), and \( x_2(\lambda) \).

Notice that the function is concave in \( x_1 \) and \( x_2 \).

\[
\frac{\partial f}{\partial x_1} = \frac{.6}{x_1 + 1} - \lambda = 0 \quad (((#))
\]

leads to

\[ x_1(\lambda) = \frac{.6}{\lambda} - 1 \]

and

\[
\frac{\partial f}{\partial x_1} = \frac{.4}{x_2 + 1} - 2\lambda = 0
\]

leads to

\[ x_2(\lambda) = \frac{.4}{2\lambda} - 1 \]
(b) Find the value of $\lambda$ that makes $x_1 + 2x_2 = 7 + r$. Use the numerical value of $r$ at this point.

My $r$ is .678. Write $x_1(\lambda) + 2x_2(\lambda) = 7.678$. Substitute from the previous part and get

$$\frac{.6}{\lambda} - 1 + 2\left(\frac{4}{2\lambda} - 1\right) = 7.678$$

That becomes

$$\frac{1}{\lambda} = 10.678 \text{ or } \lambda = \frac{1}{10.678}$$

$$\lambda = .09365$$

(c) Explain why this problem makes sense. (50 words)

The problem illustrates the Lagrange multiplier as a penalty rate. If the penalty rate is set too high, not all resources are used. If it is too low, the firm tries to use too much. (35 words)