Some ground rules for home works and exams:

Write your homework answers on the sheets supplied. If necessary, you can get new sheets on the class web site.

All homework and examination items are answered with a limitation on space and a word count. Limitations on the homework items are typical of those on examination items. Write the best answer that you can in the space available. Writing that is clear and readable will be rewarded. As much as possible write in sentences and paragraphs. In many cases you will know more than can fit within the space and word allowance. You decide which parts are most important to write down.

You should outline the answer for yourself before writing it out for us. It also helps to connect the text to the diagrams and equations, which you do by labeling points in the diagrams (for instance by A, B, C, ...) or labeling equations (for instance, *, **, ***,...) and then referring to the labels at the proper point in the text. For instance, you might write, "From the initial equilibrium of supply and demand, point A in the figure, the increase in demand leads to a new equilibrium at point B." You should practice this technique.

SOLUTIONS ECONOMICS 172A
Problem Set #1, Due Wednesday, January 17, 2007

1. Consider the hide-and-seek game with three doors. If the prize is in door one it is one dollar; if it is in door two it is two dollars; and if it is in door three it is three dollars.

(a) The seeker's problem is to maximize $V$ subject to $V \leq \min(x_1, 2x_2, 3x_3)$. In thirty words explain why this problem makes sense.
Answer: The seeker is maximizing the worst case outcome. She assumes that the hider knows her strategy and plays against it in a way that minimizes her expected reward. (28 words).

(b) In seventy-five words or less, explain why in the solution we have $x_1 = 2x_2 = 3x_3$.
Answer: $x_1$, $2x_2$, and $3x_3$ are the expected payoffs to the seeker when the hider plays, respectively doors one, two, or three. Suppose they are not equal. Suppose for instance $x_1 < 2x_2 \leq 3x_3$. The hider knows this and plays door one. More generally, if the expected values from the doors are not equal, the hider will play the door that is least favorable to the seeker. (59 words)

(c) In thirty words and a few equations, explain why the solution is $x_1^* = 6/11$, $x_2^* = 3/11$, $x_3^* = 2/11$.
Answer: The probabilities sum to unity, therefore $x_1 + x_2 + x_3 = 1$. From the information in part b, rewrite as
\[ x_1 + \frac{1}{2}x_1 + \frac{1}{3}x_1 = 1 \] (1)
Now combine terms and solve getting $x_1^* = 6/11$. Using part b again yields $x_2^* = 3/11$ and $x_3^* = 2/11$ (24 words)

(d) Write the problem as a problem in linear programming problem.
Answer:
\[ \text{Maximize } V \] (2)
subject to $V - x_1 \leq 0$ (3)
$V - 2x_2 \leq 0$ (4)
$V - 3x_3 \leq 0$ (5)
$x_1 + x_2 + x_3 \leq 1$ (6)
(a) Given the linear program (incidentally, the solution is $x_1^* = 10$, $x_2^* = 0$, $x_3^* = 20$ and the value of the program is 30).

$$\begin{align*}
\text{Maximize} & \quad x_1 + x_2 + x_3 \\
\text{subject to} & \quad x_1 + x_2 \leq 10 \\
& \quad x_2 + x_3 \leq 20
\end{align*}$$

Write the dual of the program.

Answer:

$$\begin{align*}
\text{Minimize} & \quad 10y_1 + 20y_2 \\
\text{subject to} & \quad y_1 \geq 1 \\
& \quad y_1 + y_2 \geq 1 \\
& \quad y_2 \geq 1
\end{align*}$$

(b) Using the diagrammatic approach, show that the solution in the dual is $y_1^* = 1$ and $y_2^* = 1$, and the value of the dual is 30.

The optimum feasible solution is at A.

The value of the dual is $10 \times 1 + 20 \times 1 = 30$.

SOLUTIONS ECONOMICS 172A
Problem Set #2, Due Wednesday, January 24, 2007

1. (a) Consider a standard linear programming problem in which the primal is a maximization of revenue given technology and resources. The dual is, of course, a minimization. In 50 words or less, explain the sense in which the dual is a valuation and the dual variables are shadow prices.

Answer: The dual answers the question: what is the least price the firm should accept for all or part of its resources. The prices are "shadow" because one expects that they are different from the market prices. (36 words)

(b) Consider a standard linear programming problem in which the primal is a minimization of the cost of producing a given vector of outputs, given the technology and costs of processes. The dual is, of course, a maximization. In 50 words of less, explain the sense in which the dual is a valuation and the dual variables are shadow prices.

Answer: The dual determines the greatest price the firm should pay to purchase some or all of the required outputs, instead of making them itself. The prices are also
marginal costs of producing more of the required outputs. The prices are "shadow" because they are unequal to market prices. (48 words)

2. In the following linear programing problem, the solution is \( x^*_1 = 1, x^*_2 = 1, x^*_3 = \frac{1}{2}, x^*_4 = 0 \)

Maximize \( 2x_1 + 4x_2 + x_3 + x_4 \)  

subject to \( x_1 + 3x_2 + x_4 \leq 4 \)  
\( 2x_1 + x_2 + \leq 3 \)  
\( x_2 + 4x_3 + x_4 \leq 3 \)  
\( x_i \geq 0 \ for \ i = 1, 2, 3, 4 \)  

(a) Write the dual.  
**Answer:**

Minimize \( 4y_1 + 3y_2 + 3y_3 \)  

subject to \( y_1 + 2y_2 \geq 2 \)  
\( 3y_1 + y_2 + y_3 \geq 4 \)  
\( 4y_3 \geq 1 \)  
\( y_1 + y_3 \geq 1 \)  
\( y_i \geq 0 \ for \ i = 1, 2, 3 \)  

(b) Show that the solution of the dual is \( y^*_1 = \frac{11}{10}, y^*_2 = \frac{9}{20}, y^*_3 = \frac{1}{4} \). Explain each step briefly.  
**Answer:** Apply the complementary slackness theorem. Because \( x^*_1 > 0, x^*_2 > 0, \) and \( x^*_3 > 0 \), the first three constraints in the dual (lines 16, 17, and 18) are satisfied exactly. Then from equation (18), \( y^*_3 = \frac{1}{4} \). Now from (16) and (17) one must solve

\[
\begin{align*}
y_1 + 2y_2 &= 2 \\
3y_1 + y_2 + \frac{1}{4} &= 4
\end{align*}
\]

for which the solutions are \( y^*_1 = \frac{11}{10}, y^*_2 = \frac{9}{20} \). As a check, the value of the dual is \( 13/2 \), which is the same as the value in the primal.

3. Consider the following linear program. Solve it by writing the dual and solving the dual by diagrammatic means. Give brief explanations. Write in sentences, please. No bullets. Key your explanation to the equations you use. Indicate how you checked your results.

Minimize \( 5x_1 + 8x_2 + 8x_3 \)  

subject to \( x_1 + 2x_2 + x_3 \geq 3 \)  
\( x_1 + x_2 + 2x_3 \geq 4 \)  
\( x_i \geq 0 \ for \ i = 1, 2, 3 \)  

**Answer:** The dual is

Maximize \( 3y_1 + 4y_2 \)  

subject to \( y_1 + y_2 \leq 5 \)  
\( 2y_1 + y_2 \leq 8 \)  
\( y_1 + 2y_2 \leq 8 \)  
\( y_i \geq 0 \ for \ i = 1, 2, 3 \)  

The diagram is

3
The feasible set is 0ABCD. Solution is at B.

The three constraints are represented by AB, BC, and CD. The objective function has slope equal to $-3/4$. The solution occurs at the point (B) where the slope of the objective is between the slopes ($-1/2$ and $-1$) of the adjacent edges of the constraint set.

Find the solution by solving

\[
\begin{align*}
y_1 + y_2 &= 5 \\ y_1 + 2y_2 &= 8
\end{align*}
\]  

That leads to $y_2^* = 3$, and $y_1^* = 2$.

Now for the primal. Because the solutions in the dual are both positive, both constraints in the primal are satisfied exactly. Because the second constraint in the dual (equation 29) is not satisfied exactly, the corresponding primal activity is $x_2^* = 0$. Thus solve

\[
\begin{align*}
x_1 + x_3 &= 3 \\ x_1 + 2x_3 &= 4
\end{align*}
\]  

leading to $x_3^* = 1$ and $x_1^* = 2$.

As a check, the value in the primal is $5 \times 2 + 8 \times 1 = 18$ and the value in the dual is $3 \times 2 + 4 \times 3 = 18$.

**SOLUTIONS**

**ECONOMICS 172A**

Midterm exam, Wednesday, January 31, 2007

The midterm is closed book and closed note. No calculators are allowed. Write your answers in the available space and observe any restrictions on the number of words. Clear, legible writing will be rewarded.

1. (20 minutes)

   (a) In no more than fifty words, state the complementary slackness theorem (perhaps in your own words).

   Answer: The feasible solutions $x_1^*, x_2^*, \ldots, x_n^*$ and $y_1^*, y_2^*, \ldots, y_n^*$ of the standard primal and dual problems are optimum solutions if and only if

   \[
   y_j^* = 0 \text{ whenever } a_{j1}x_1^* + a_{j2}x_2^* + \ldots + a_{jn}x_n^* < b_j
   \]  

   and

   \[
   x_i^* = 0 \text{ whenever } a_{1i}y_1^* + a_{2i}y_2^* + \ldots + a_{mi}y_m^* > c_i
   \]  

   (21 words)
(b) Write out the four distinct statements that are useful in applying the theorem, and explain in no more than fifty words their relations to each other.

i. Positive slack in a primal constraint implies that the corresponding dual variable is zero.
ii. A positive dual variable implies that the corresponding constraint in the primal is satisfied exactly.
iii. Positive slack in a dual constraint implies that the corresponding primal variable is zero.
iv. A positive primal variable implies that the corresponding constraint in the dual is satisfied exactly.
The relations stem from the logical fact that \( (p \implies q) \) if and only if \( (\neg q \implies \neg p) \). This relation exists between statements i and ii, and between statements iii and iv. (34 words)

2. (30 minutes)

(a) Solve the following linear programming problem using the diagrammatic method. (Hint: the solution is \( x_1^* = 5 \) and \( x_2^* = 7 \).) Explain your steps carefully.

\[
\begin{align*}
\text{Maximize} & \quad 3x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + x_2 \leq 12 \\
& \quad 3x_1 \leq 15 \\
& \quad x_2 \leq 9 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Answer: The objective has a slope of \(-\frac{3}{2}\) and is represented by line L in the diagram. The three constraints give rise to the feasible region bounded by \(0ABCD\). The slope of the constraint in the segment \(BC\) is \(-1\), which is flatter than the slope of the objective. Therefore the optimum occurs at point \(C\). Solve the two constraint boundaries

\[
\begin{align*}
3x_1 &= 15 \\
x_1 + x_2 &= 12
\end{align*}
\]

The result is \( x_1^* = 5 \) and \( x_2^* = 7 \). The value in the optimum is \(3 \times 5 + 2 \times 7 = 29\).
(b) Write the dual. In no more than fifty words interpret the dual and the dual variables.

\[ \text{Minimize } 12y_1 + 15y_2 + 9y_3 \]  

subject to \[ y_1 + 3y_2 \geq 3 \]  
\[ y_1 + y_3 \geq 2 \]  
\[ y_1, y_2, y_3 \geq 0 \]  

The dual seeks the least price at which the firm or its resources might be sold. The \( y_i \)'s are called shadow prices because they differ from market prices. Each shadow price is the least the firm would accept for the corresponding resource. (42 words)

(c) Solve the dual and use the solution to check the proposed solution to the primal. Explain your solution in no more than 100 words.

Answer: Apply the complementary slackness theorem. Because both activities are positive in the solution to the primal, both constraints in the dual are satisfied exactly. Therefore

\[ y_1 + 3y_2 = 3 \]  
\[ y_1 + y_3 = 2 \]  

Because the third constraint in the primal is not satisfied exactly, the corresponding dual variable, \( y_3^* \) is equal to zero. From the two equations immediately above, it follows that \( y_1^* = 2 \) and \( y_2^* = 1/3 \). Then the value in the dual is \( 12*2 + (1/3)*15 = 29 \). Because the proposed solutions are feasible in the primal and dual respectively, the optimality theorem assures that they are also optimum. The values satisfy the requirements of the complementary slackness theorem, a further confirmation of optimality. (97 words)

(d)

SOLUTIONS
ECONOMICS 172A

Problem set #3, due Wednesday, February 7, 2007

Compose your answers to the problem set in excel. Because skill with excel is part of this exercise, we will reward compliance with the following instructions. Shrink column widths etcetera so that the answers to problems one and two are printed on one side of one sheet of 8.5 × 11 paper. Your name should be printed in the upper right hand corner. The answer to problem three should be printed on (part of) one side of one sheet of 8.5 × 11 paper. Again, your name should be in the upper right-hand corner. Please don’t ask us to decipher ugly, sprawling answers.

We do not need to see your formulas. We trust you. Just print out the results. If you work in a group, which is recommended, do not simply copy the group solution and turn it in. After the group settles on solutions, each member should independently recreate the solutions and formats and turn in his or her unique product.

1. This is an exercise in using excel and solver. Go to the class web site and find the excel spread sheet that demonstrates solver. Start with the second template given in the introductory page. You may recognize that it is the Wyndor-Glass problem. Modify the template and use the modified version to solve both the primal and dual of the following problem. Comment on any difficulties you had in modifying the template.

\[ \text{Maximize } 3x_1 + 4x_2 + 2x_3 + 5x_4 \]  

subject to \[ x_1 + x_2 \leq 8 \]  
\[ 2x_3 + 2x_4 \leq 12 \]  
\[ 3x_1 + 2x_2 + x_3 + 2x_4 \leq 18 \]  
\[ x_i \geq 0 \]  

6
2. This is another exercise in using excel and solver. Start with the template that solved problem one above. Modify the template and use the modified version to solve both the primal and dual of the following problem. In fifty words or less, what is interesting about the solutions?

\[ \text{Maximize } 3x_1 + 4x_2 + 2x_3 + 5x_4 \]

subject to
\[
\begin{align*}
  x_1 + x_2 & \leq 8 \\
  2x_3 + 2x_4 & \leq 12 \\
  3x_1 + 2x_2 + 2x_3 + 2x_4 & \leq 32 \\
  x_1 + 2x_2 + x_3 + 3x_4 & \leq 20 \\
  3x_1 + 2x_2 + x_3 + 2x_4 & \leq 18 \\
  x_i & \geq 0
\end{align*}
\]

3. This is another exercise in using excel and solver. Start with the template that solved problem two above. Modify the template and use the modified version to solve both the primal and dual of the following assignment problem. There are three workers and three tasks. One worker must take on each task. The table below gives the time cost of each task if done by each worker. The objective is to assign workers in such a way as to minimize total cost. In a few words, define the notation you use for the problem. Solve primal and dual. Is the solution unique?

<table>
<thead>
<tr>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worker 1</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>Worker 2</td>
<td>14</td>
<td>19</td>
</tr>
<tr>
<td>Worker 3</td>
<td>13</td>
<td>13</td>
</tr>
</tbody>
</table>

ECONOMICS 172A
Problem set #4, due Wednesday, February 14, 2007
Solutions are in excel

We recommend that you work in groups. After the group settles on a solution, each member should go apart and independently recreate the solution and format and turn in his or her unique product. Please, for your own benefit and in fairness to others, do not merely copy the group solution and turn it in.
Neatly printed, well-organized answers will be rewarded. Do not turn in this sheet.

1. (Like problem 3.1-10 in Hillier and Lieberman) A manufacturing firm has excess production capacity. It is considering devoting the excess capacity to one or more of three products, A, B, and C. The potential bottlenecks in producing the products are inputs of the processes milling, turning, and grinding. The available capacities of milling, turning, and grinding are 500, 350 and 150, respectively. The hours required per unit of the three products are given in the table.

<table>
<thead>
<tr>
<th>Machine</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milling</td>
<td>9</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Turning</td>
<td>5</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Grinding</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

The sales department reports that the potential sales of products A and B are far beyond any feasible production rate, but the market for product C is only 20 units. Net revenues per unit of products A, B, and C are $50, $20, and $25 respectively. The objective is to maximize net revenue from the available capacities.

(a) Formulate as a linear program, preferably using (a modification of) the templates used in class and available on the class web site.
(b) Briefly identify the \( x, y, c, \) and \( b \) vectors.
(c) Solve primal and dual using Solver.
(d) Print the program and your other answers on one side of one sheet of 8.5 × 11 paper, in portrait configuration. Print your name (using excel, of course) in the upper right hand corner. (Hint from the answer key in the text: the optimum productions are 26.19, 54.76, 20 and the net revenue is \( V = 2904.76 \).)

2. (Like problem 3.4-14 in Hillier and Lieberman) The problem is to supply staffing for a university statistics lab fourteen hours a day, five days a week. There should be one and only one staffer in the lab at any time. Six undergraduate and graduate students are available. Their daily availability is given in the table. For instance, student one is available for 6 hours on Mondays and not at all on Tuesdays.

<table>
<thead>
<tr>
<th>Student</th>
<th>wage rate</th>
<th>M</th>
<th>T</th>
<th>W</th>
<th>R</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>6</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>10.1</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>9.9</td>
<td>4</td>
<td>8</td>
<td>4</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>9.8</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>10.8</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>11.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

There are minimum hours needed in order for each staffer to maintain competency. Students 1 through 4 are undergraduates and must have at least 8 hours per week. Students 5 and 6 are graduate students and must have a minimum of 7 hours per week. The objective is to assign hours that minimize total staffing cost.

(a) Formulate as a linear program, preferably using (a modification of) the templates used in class and available on the class web site.

(b) Briefly identify the \( x, y, c, \) and \( b \) vectors.

(c) Solve primal and dual using Solver.

(d) Print the solution and answers on one side of one sheet of 8.5 × 11 paper, in portrait configuration. Again print your name in the upper right hand corner. (Hint: Instead of dealing with thirty variables, ignore those that are constrained to be zero. Then there remain 18 variables, a more manageable number.)

ECONOMICS 172A

Problem set #5, due Wednesday, February 21, 2007

We recommend that you work in groups. After the group settles on a solution, each member should go apart and independently recreate the solution and format and turn in his or her unique product. Please, for your own benefit and in fairness to others, do not merely copy the group solution and turn it in. Neatly printed, well-organized answers will be rewarded. Do not turn in this sheet.

1. (Like the problem in class in the twelfth lecture, but be warned, the numbers are slightly different). This is the transport problem of a distribution firm. There are two supply sources \( S_1 \) and \( S_2 \), and three retail points \( R_1, R_2, \) and \( R_3 \). The transport costs are given in the table. The requirements are given in the top row of the table, and the supply at \( S_1 \) is given in the left column. At \( S_2 \) the firm can buy as much as needed at a price of $60 per unit. The objective of the firm is to minimize the total cost of acquiring and distributing the required product to the three retail points.

<table>
<thead>
<tr>
<th>Transport cost matrix requirements</th>
<th>100</th>
<th>500</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supply</td>
<td>( R_1 )</td>
<td>( R_2 )</td>
<td>( R_3 )</td>
</tr>
<tr>
<td>700</td>
<td>( S_1 )</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>as much as needed at $60 per unit</td>
<td>( S_2 )</td>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

Other firms are available to supply product at \( R_1 \) at some price. The question is this: how much should our firm be willing to pay to buy a unit of the product at \( R_1 \)? You might solve this problem in various ways, but the requirement here is to solve it through linear programming. Answer in excel neatly printed in portrait configuration on one side of one sheet of 8.5 x 11 paper with your name in the upper right-hand corner. Explain carefully.
2. (Like 3.4 - 8 in Hillier and Lieberman, with minor modifications). Web Mercantile sells seasonal products. As it builds inventory in the slow months, it leases storage containers as needed. Requirements for storage containers in the next five months are

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Containers needed</td>
<td>30</td>
<td>20</td>
<td>40</td>
<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>

The firm can lease containers month-to-month for a cost of $65 per container per month. In addition the firm can enter into leases of longer term that start and terminate at any of the five months. Monthly costs of leasing for one or more months are as follows:

<table>
<thead>
<tr>
<th>Leasing period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost per container</td>
<td>65</td>
<td>100</td>
<td>135</td>
<td>160</td>
<td>190</td>
</tr>
</tbody>
</table>

The objective is to satisfy the storage requirements at minimum cost. Answer the following in excel neatly printed in portrait configuration on one side of one sheet of 8.5 x 11 paper with your name in the upper right-hand corner.

(a) Formulate as a linear programming problem. Explain the $x$ vector carefully. Identify the $b$, $c$, and $y$ vectors.

(b) Solve the primal and dual linear programming problems using solver in excel and, preferably, the templates developed in class.

(c) Interpret the dual variables with particular attention to those that are zero.

3. On one side of one sheet of 8.5 x 11 paper with your name in the upper right-hand corner, solve the following systems of equations using Gaussian elimination. Display all of the tableaux! Circle the pivot points. Show which vectors are in the basis at each step. Check the answers by matrix-vector multiplication.

(a)

\[
\begin{align*}
    x_1 + x_3 &= 10 \quad (66) \\
    x_2 + x_3 &= 3 \quad (67) \\
    x_1 + x_2 + x_3 &= 11 \quad (68)
\end{align*}
\]

(b)

\[
\begin{align*}
    x_1 + 2x_3 &= -2 \quad (69) \\
    x_1 + x_2 + x_3 &= 2 \quad (70) \\
    2x_1 + 3x_2 - x_3 &= 1 \quad (71)
\end{align*}
\]

(c)

ECONOMICS 172A

SOLUTIONS

Midterm exam, Wednesday, February 28, 2007

The midterm is closed book and closed note. No calculators are allowed. Write your answers in the available space and observe any restrictions on the number of words. Clear, legible writing will be rewarded.

1. (15 points) In the tables below there is a simplex-method solution to the Wyndor-Glas problem, which is a maximization.

\[
\begin{align*}
    V & (x_1) & (x_2) & (s_1) & (s_2) & (s_3) & (b) & \text{line} \\
    V & 1 & -3 & -5 & 0 & 0 & 0 & 0 & 1 \\
    u_1 & 0 & 1 & 0 & 1 & 0 & 0 & 4 & 2 \\
    u_2 & 0 & 0 & 2** & 0 & 1 & 0 & 12 & 3 \\
    u_3 & 0 & 3 & 2 & 0 & 0 & 1 & 18 & 4 \\
\end{align*}
\]

\[
\begin{align*}
    V & (x_1) & (x_2) & (s_1) & (s_2) & (s_3) & (b) & \text{line} \\
    V & 1 & -3 & 0 & 0 & 5/2 & 0 & 30 & 5 \\
    u_1 & 0 & 1 & 0 & 1 & 0 & 0 & 4 & 6 \\
    (x_2) & 0 & 0 & 1 & 0 & 1/2 & 0 & 6 & 7 \\
    u_3 & 0 & 3** & 0 & 0 & -1 & 1 & 6 & 8 \\
\end{align*}
\]
(a) At line 3, how did we choose the pivot element indicated by **? (50 words maximum)

**Answer:** The column is chosen because of the minus sign in line one, which reads $V = 3x_1 + 5x_2$. The row is chosen to retain feasibility. Choose the least ratio, either $12/2$ or $18/2$ (or $4/0$). The $12/2$ is chosen. Thus the $(x_2)$ vector is entering the basis and the $(s_2)$ is leaving it. (44 words)

(b) At lines 5, 7, 8, 11, and 12 fill in the blanks.

(c) At lines 9 through 12, how do you know that the optimum has been reached? (50 words maximum)

**Answer:** There are no further negative entries in line one, which now reads $V = 36 - \frac{3}{2}s_2 - s_3$. (12 words) Alternatively, use the complementary slackness theorem to confirm the optimum.

2. (15 points) In the diet problem, you are minimizing the cost of the barely adequate diet. You set up the required linear program, which is shown below. The primal runs fine using solver, yielding the numbers seen below (for instance, 241.7142 for the optimized value, using optimum $X^*_C = 1.1428$ kilograms of corn. etc.). You know that the primal is right, but when you try to run the dual, solver finds a maximum that is unequal to the minimum found in the primal. You restore the trial values of the primal to $(0.1,0.1,0.1)$, yielding the situation in the table.

<table>
<thead>
<tr>
<th>Slack resource</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>slack</th>
<th>costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td></td>
<td>71</td>
<td>84</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td></td>
<td>0.1</td>
<td>60</td>
<td>72</td>
</tr>
<tr>
<td>0.1</td>
<td></td>
<td>0.1</td>
<td></td>
<td>44</td>
<td>60</td>
</tr>
<tr>
<td>slack</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>costs</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X_C$</th>
<th>$X_T$</th>
<th>$X_A$</th>
<th>slack</th>
<th>resource</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1428</td>
<td>0</td>
<td>2.4285</td>
<td>0</td>
<td>200</td>
</tr>
<tr>
<td>0.1</td>
<td>90</td>
<td>20</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>30</td>
<td>80</td>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>10</td>
<td>20</td>
<td>60</td>
<td>7.1428</td>
</tr>
<tr>
<td>71</td>
<td>-60</td>
<td>-44</td>
<td>241.7142</td>
<td>53</td>
</tr>
</tbody>
</table>

(a) Now you must debug. What is wrong? How do you know? (Hint: the problem is not with solver. The bug is in the set-up. (100 words maximum, plus any equations you need)

**Answer:** The slacks in the second and third constraints of the dual are incorrect. Because the primal runs correctly, the primal slacks and objective must be correct. Check the dual objective and find $0.1 \times 200 + 0.1 \times 180 + 0.1 \times 150 = 53$ so that is okay. That leaves the slacks in the dual. The dual is a maximization. Using the trial values of the y’s, the actual slacks are $71 = 84 - 0.1 \times 130$, $60 = 72 - 0.1 \times 120$, and $44 = 60 - 0.1 \times 160$. The second and third slacks are off by a sign.

(b) Write out and explain the formulas needed to fix it.

**Answer:** Slack variables are always constructed to be nonnegative. Since the dual is a maximization, the slack should be the resource limit less the amount used in the solution. Specifically

\[ s_T = 72 - \text{sumproduct}(\langle 1,1,1 \rangle, \langle 20,80,20 \rangle) \]  
\[ s_A = 60 - \text{sumproduct}(\langle 1,1,1 \rangle, \langle 40,60,60 \rangle) \]

3. (20 points) A wholesaler has inventories at distribution centers $S_1$ and $S_2$ of 100 and 200 as shown in the diagram. It can purchase more units at the manufacturer $M$ at a cost of $40 per unit. The requirement is to supply 250 and 150 at retailers $R_1$ and $R_2$, as also illustrated. Distances are given in the diagram. The objective is to minimize cost of acquisition and shipment.
(a) Formulate as a linear program, explaining notation (i.e., the $b$, $c$, $x$, and $y$ vectors) carefully.

**Answer:** $b$ is the vector of requirements and supplies at the various nodes. $c$ is the vector of costs of acquisition and transport, that is $c = (40, 3, 2, 4, 6, 5, 7)$ perhaps not in that order. $x_M$ is the amount purchased at node $M$. The $x_{ij}$ are the quantities shipped from $i$ to $j$, where $M$ is one of the $i$'s. $y_i$'s are shadow prices at the five nodes.

Minimize $40x_M + 3x_{M1} + 2x_{M2} + 4x_{11} + 7x_{12} + 5x_{21} + 6x_{22}$

subject to $x_M - x_{M1} - x_{M2} \geq 0$

$x_{M1} - x_{11} - x_{12} \geq -100$

$x_{M2} - x_{21} - x_{22} \geq -200$

$x_{11} + x_{21} \geq 250$

$x_{12} + x_{22} \geq 150$

(b) Write the dual and comment on the typical dual constraints.

Maximize $-100y_s1 - 200y_s2 + 250y_r1 + 150y_r2$

subject to $y_M \leq 40$

$-y_M + y_{s1} \leq 3$

$-y_M + y_{s2} \leq 2$

$-y_{s1} + y_{r1} \leq 4$

$-y_{s1} + y_{r2} \leq 7$

$-y_{s2} + y_{r1} \leq 5$

$-y_{s2} + y_{r2} \leq 6$

Comment: The typical constraint says that the difference in shadow prices between any two adjacent nodes is never greater than the cost of shipping the commodity over the connecting route. If the price differential were ever greater, the firm could profit by shipping over that route.