Solutions to problem set 1.

Some ground rules for problem sets and exams:

Write your homework answers on the sheets supplied. If necessary, you can get new sheets on the class web site.

All homework and examination items are answered with a limitation on space. Limitations on the homework items are typical of those on examination items. Ordinarily, one-half page is allowed. Sometimes a word limit is also set. Write the best answer that you can in the space available. Writing that is illegible or unreasonably tiny is penalized.

In many cases you will know more than can fit in the space. You must decide which parts are most important to write down. In preparing homeworks or examinations, it is usually best not to leave blank space. Instead, explain the answer a bit more deeply.

You should outline the answer for yourself before writing it out for us, whether on homeworks or examinations. It also helps to connect the text to the diagrams and equations, which you do by labeling points in the diagrams (for instance by A, B, C, ...) or labeling equations (for instance, *, **, ***, ...) and then referring to the labels at the proper point in the text. For instance, you might write, "From the initial equilibrium of supply and demand, point A in the figure, the increase in demand leads to a new equilibrium at point B." You should practice this technique in the homeworks and in your preparation for examinations.

1. (Like 1.1 of the text) A lunch menu contains exactly two items: meat and fish. The meat can be either good or poor, and so can the fish. The diner has a modest appetite and chooses one dish or the other. List the exclusive, exhaustive events \( \theta_i \) and the decisions \( d_j \).

\[
\begin{align*}
\theta_1 & \quad \text{meat and fish are good} \\
\theta_2 & \quad \text{meat is good, fish is bad} \\
\theta_3 & \quad \text{meat is bad, fish is good} \\
\theta_4 & \quad \text{meat and fish are bad} \\
\end{align*}
\]

\[
\begin{align*}
d_1 & \quad \text{choose meat} \\
d_2 & \quad \text{choose fish} \\
\end{align*}
\]

On further thought, the diner might also go elsewhere for lunch. Moreover, he or she might think about dinner, which could be again, either meat or fish. Make a new listing of events (there are eight) and decisions (three), reflecting the new considerations.

\[
\begin{align*}
\theta_1 & \quad \text{Dinner is fish. At lunch meat and fish are good.} \\
\theta_2 & \quad \text{Dinner is fish. At lunch meat is good, fish is bad.} \\
\theta_3 & \quad \text{Dinner is fish. At lunch meat is bad, fish is good.} \\
\theta_4 & \quad \text{Dinner is fish. At lunch meat and fish are bad.} \\
\theta_5 & \quad \text{Dinner is meat. At lunch meat and fish are good.} \\
\theta_6 & \quad \text{Dinner is meat. At lunch meat is good, fish is bad.} \\
\theta_7 & \quad \text{Dinner is meat. At lunch meat is bad, fish is good.} \\
\theta_8 & \quad \text{Dinner is meat. At lunch meat and fish are bad.} \\
\end{align*}
\]

\[
\begin{align*}
d_1 & \quad \text{choose meat} \\
d_2 & \quad \text{choose fish} \\
d_3 & \quad \text{eat lunch elsewhere} \\
\end{align*}
\]

Suppose the diner never under any circumstances chooses fish for lunch. The event space can be reduced to essentially four events: What are they?

\[
\begin{align*}
\theta_1 & \quad \text{Dinner is fish. At lunch, meat is good.} \\
\theta_2 & \quad \text{Dinner is fish. At lunch, meat is bad.} \\
\theta_3 & \quad \text{Dinner is meat. At lunch, meat is good.} \\
\theta_4 & \quad \text{Dinner is meat. At lunch, meat is bad.} \\
\end{align*}
\]
2. Consider the let’s-make-a-deal game discussed in class. What are the events (three of them)?

\[ \theta_1 \quad \text{The prize is behind curtain one.} \]
\[ \theta_2 \quad \text{The prize is behind curtain two.} \]
\[ \theta_3 \quad \text{The prize is behind curtain three.} \]

What are the decisions (six of them)?

\[ d_1 \quad \text{Choose curtain one, stay} \]
\[ d_2 \quad \text{Choose curtain one, switch.} \]
\[ d_3 \quad \text{Choose curtain two, stay} \]
\[ d_4 \quad \text{Choose curtain two, switch.} \]
\[ d_5 \quad \text{Choose curtain three, stay} \]
\[ d_6 \quad \text{Choose curtain three, switch.} \]

Explain fairly carefully why the contestant is best advised to switch.

The master of ceremonies knows where the prize is and consequently his action reveals something to the contestant. If the contestant guessed wrong, which happens \( \frac{2}{3} \) of the time, the prize is behind the curtain that was neither chosen nor opened. If she guessed right, which happens \( \frac{1}{3} \) of the time, the prize is behind the chosen curtain. The policy of staying pays off \( \frac{1}{3} \) of the time, and the policy of switching pays off \( \frac{2}{3} \) of the time.

3. A salesperson is selecting a car to be used during the next three years. In that time, the price of gasoline may be high, medium or low, and the salesman’s job may require much or little travel by car. Moreover, the salesperson may choose to live near the office, to which he or she must go every day, or far from it. The available cars are the Hummer, the Ford Explorer, and the Nissan Sentra. What are the events and what are the decisions? Explain why you treated residential location as you did.

The key to this problem is to recognize which variables are choices and which are states. The price of gasoline and the amount of job travel are clearly not choices. That gives six exclusive, exhaustive states

<table>
<thead>
<tr>
<th>gasolene high</th>
<th>job travel much</th>
<th>job travel little</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_1 )</td>
<td>( \theta_2 )</td>
<td></td>
</tr>
<tr>
<td>gasoline middle</td>
<td>( \theta_3 )</td>
<td>( \theta_4 )</td>
</tr>
<tr>
<td>gasoline low</td>
<td>( \theta_5 )</td>
<td>( \theta_6 )</td>
</tr>
</tbody>
</table>

The decisions are home location and vehicle.

<table>
<thead>
<tr>
<th></th>
<th>live near the office</th>
<th>live far from the office</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hummer</td>
<td>( d_1 )</td>
<td>( d_2 )</td>
</tr>
<tr>
<td>Explorer</td>
<td>( d_3 )</td>
<td>( d_4 )</td>
</tr>
<tr>
<td>Sentra</td>
<td>( d_5 )</td>
<td>( d_6 )</td>
</tr>
</tbody>
</table>

Residential location is a choice because the problem says so. However, in life one’s home location might be dictated by other things such as spouse’s job, ability to make a down payment, or school district considerations. That would make home location a state variable. Then there would be twelve exhaustive, exclusive states and three decisions.
1. An item on an examination consists of a statement that is either true or false. The student must answer 0, .2, .5, .8, or 1 corresponding the probability that the statement is true. The item is scored using the quadratic penalty rule described in the text. What are the states and what are the decisions in this problem? For those states and decisions, write the payoff matrix.

There are two exclusive, exhaustive states: $\theta_1$ if the statement is true and $\theta_2$ if the statement is false. The exclusive exhaustive decisions are to answer either $d_1 = 0, d_2 = .2, d_3 = .5, d_4 = .8$, or $d_5 = 1$. The payoff matrix is

$$
\begin{array}{c|cccc}
\text{true} & 0 & .2 & .5 & .8 & 1 \\
\text{false} & -100 & -64 & -25 & -4 & 0
\end{array}
$$

The minus reminds us that these are penalty points.

2. (like 3.3) A part of a factory consists of a number of similar spinning machines attended by a single operator. When a machine malfunctions, the operator must restart it. At any moment the probability that exactly one machine is stopped and being restarted is .3. The probability that two are stopped is .2. The probability that three or more are stopped is .18. Show that the probability is .32 that all machines are running and the operator is idle. (Show means cite and apply the appropriate probability laws.)

The events \{none stopped\} \{one stopped\}, \{two stopped\} and \{three or more stopped\} are exclusive. Together they are exhaustive. By the addition law they sum to the probability of the universe, which is 1.

$$p(\text{none stopped}) + p(\text{one stopped}) + p(\text{two stopped}) + p(\text{three or more stopped}) = 1$$

Solve for the first probability, which is $p(\text{none stopped}) = .32$.

Given that the operator is busy restarting a machine, show that the probability is .558824 that at least one other machine is also stopped and therefore awaiting the operator’s attention. (Again, cite and apply the appropriate probability laws.)

The events \{two stopped\} and \{three or more stopped\} are exclusive and together exhaust the set \{two or more stopper\}. By the addition law, $p(\text{two or more stopped}) = .38$. The events \{one stopped\}, \{two stopped\} and \{three or more stopped\} are exclusive and collectively exhaust the event \{one or more stopped\}. By the addition law, $p(\text{one or more stopped}) = .68$.

The intersection of these events is \{two or more stopped\} and by the conditional probability law (multiplication law) we have

$$p(\text{two or more stopped}) = p(\text{two or more stopped}|\text{one or more stopped}) \cdot p(\text{one or more stopped})$$

$$=.38 = p(\text{two or more stopped}|\text{one or more stopped}) \cdot .68$$

$$p(\text{two or more stopped}|\text{one or more stopped}) = \frac{.38}{.68}$$

from which the answer follows.

Incidentally, I also like to use the conditional probability law in the form: as long as $p(F) > 0$

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

In that case the left hand side is the desired number and the right hand side is just $\frac{.38}{.68}$. To me that is the most intuitive solution to this particular problem.
1. (Like 3.4, to which the answers are 19/128, 25/32, and 11/40.) Continuing with the example of the mountain pass on pages 31-32, 36, and 39-40 of the text.) Let $E$ denote the event of the pass being open and $A$ the event of an accident, and suppose that $p(E) = 5/6$, $p(A|E) = 1/18$, and $p(A|\overline{E}) = 1/6$. Let $B$ be the event of arriving late for the meeting and suppose

\[ p(B|E \text{ and } A) = 1/2 \quad \text{and} \quad p(B|E \text{ and } \overline{A}) = 1/6 \]

In answering the questions, briefly cite the rules of probability at the points you use them in the derivations.

(A) What is the probability of being late, given that the pass is open?

Before proceeding, recognize that the problem is one of aggregation, using the "Extension of the conversation Theorem" or more fundamental considerations. The problem is compound. To get the probability of being late, we aggregate over the events \{pass open\} and \{pass blocked\}. To get the probability of being late when the pass is open, we aggregate over the events \{accident\} and \{no accident\} and similarly if the pass is closed. As usual, the last part of the problem is done first, and that is the reason that (A) is asked first. To answer (A)

Use the exclusive union law (addition law) to write

\[ p(B|E) = p(B \text{ and } A|E) + p(B \text{ and } \overline{A}|E) \]

Use the conditionality law to amplify (or just invoke the extension of conversation law to get)

\[ p(B|E) = p(B|A \text{ and } E)p(A|E) + p(B|\overline{A} \text{ and } E)p(\overline{A}|E) \]
\[ = 1/2 * 1/18 + 1/6 * 17/18 \]
\[ = 5/27 \]

(B) Suppose further that

\[ p(B|E \text{ and } A) = 11/12 \quad \text{and} \quad p(B|E \text{ and } \overline{A}) = 5/6 \]

What is the probability of being late, given that the pass is blocked?

Use the exclusive union law (addition law) to write

\[ p(B|\overline{E}) = p(B \text{ and } A|\overline{E}) + p(B \text{ and } \overline{A}|\overline{E}) \]

Use the conditionality law to amplify (or just invoke the extension of conversation law to get)

\[ p(B|\overline{E}) = p(B|A \text{ and } \overline{E})p(A|\overline{E}) + p(B|\overline{A} \text{ and } \overline{E})p(\overline{A}|\overline{E}) \]
\[ = 11/12 * 6/6 + 5/6 * 5/6 \]
\[ = 61/72 \]

(C) What is the overall probability of being late?

Use the exclusive union law (addition law) to write

\[ p(B) = p(B \text{ and } E) + p(B \text{ and } \overline{E}) \]

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Use the conditionality law to amplify (or just invoke the extension of conversation law to get)

\[ p(B) = p(B|E)p(E) + p(B|\overline{E})p(\overline{E}) \]
\[ = 5/27 * 5/6 + 61/72 * 1/6 \]
\[ = \frac{383}{1296} \]
\[ = 0.29552 \]

\[ \text{(B) Suppose further that} \]
\[ p(B|E \text{ and } A) = 11/12 \quad \text{and} \quad p(B|E \text{ and } \overline{A}) = 5/6 \]

What is the probability of being late, given that the pass is blocked?

Use the exclusive union law (addition law) to write

\[ p(B|\overline{E}) = p(B \text{ and } A|\overline{E}) + p(B \text{ and } \overline{A}|\overline{E}) \]

Use the conditionality law to amplify (or just invoke the extension of conversation law to get)

\[ p(B|\overline{E}) = p(B|A \text{ and } \overline{E})p(A|\overline{E}) + p(B|\overline{A} \text{ and } \overline{E})p(\overline{A}|\overline{E}) \]
\[ = 11/12 * 6/6 + 5/6 * 5/6 \]
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Use the conditionality law to amplify (or just invoke the extension of conversation law to get)

\[ p(B) = p(B|E)p(E) + p(B|\overline{E})p(\overline{E}) \]
\[ = 5/27 * 5/6 + 61/72 * 1/6 \]
\[ = \frac{383}{1296} \]
\[ = 0.29552 \]
That is the desired solution.

An alternative way to put the information together is illustrated below. The tree shows the compound nature of the problem very well, but the method of calculation is different. In this calculation, the probability of each late branch is worked out by multiplying probabilities along the branch. This is justified by the conditionality (multiplication) rule applied over and over. I hope that this method of solution helps some of you visualize the solution to the problem and that the rest think about it later.

Total probability of being late = \( \frac{383}{1296} \approx 0.295525 \)

2. You are choosing a career. You might become a drummer in an obscure rock band, a certified public accountant, or a Red Cross administrator. Your uncle Bill is a certified public accountant and quite rich. You feel that he will give you a sizable gift if you choose the career from the above list that he wants you to choose. He will only reward you if you choose correctly (some uncle!).

(A) What are the states (three of them) surrounding the decision?

- \( \theta_1 \) Uncle Bill wants me to be a drummer
- \( \theta_2 \) Uncle Bill wants me to be an accountant
- \( \theta_3 \) Uncle Bill wants me to be a Red Cross administrator

(B) You know that your uncle loves music, hates his job, and feels guilty about his lack of contributions to charitable organizations like the Red Cross. Nevertheless, you think that probability is \( \frac{1}{3} \) that he wants you to be a musician, \( \frac{1}{3} \) an accountant, and \( \frac{1}{3} \) a worker for the Red Cross. You ask your uncle to reveal one of the two careers that he does not approve, and he agrees to do that. He will choose at random between the careers he does not approve and he will reveal the one he chooses. That means, for instance, that

\[
p(\text{he says don't be a drummer} | \text{he wants you to be a CPA}) = 0.5
\]

\[
p(\text{he says don't be a RC worker} | \text{he wants you to be a CPA}) = 0.5
\]

and so on. What are the possible messages (three of them). Show that the probability is \( \frac{1}{3} \) that the message is "don't be a drummer?" Explain as fully as space permits, citing probability laws.

The possible messages are

- \( m_1 \) don't be a drummer
- \( m_2 \) don't be an accountant
- \( m_3 \) don't be a Red Cross administrator
The probability of a message is a problem of aggregation using the extension of conditionality (extension of the conversation) law.

\[
p(m_1) = p(m_1|\theta_1)p(\theta_1) + p(m_1|\theta_2)p(\theta_2) + p(m_1|\theta_3)p(\theta_3)
\]

Since Uncle Bill tells the truth, \( p(m_1|\theta_1) = 0 \). The other conditional probabilities are, by the assumption, equal to .5. The probabilities of the states are 1/3. Therefore

\[
p(m_1) = .5 \cdot 1/3 + .5 \cdot 1/3 = 1/3
\]

(C) Show that the conditional probabilities of the states associated with the message "don’t be a drummer" are 0, .5, and .5, explaining as fully as space permits and citing the probability laws used.

We are asked to find \( p(\theta_1|m_1) \), \( p(\theta_2|m_1) \), and \( p(\theta_3|m_1) \), using knowledge of \( p(m_1|\theta_1) \), \( p(m_1|\theta_2) \), and \( p(m_1|\theta_3) \). These are problems of conditionality and are most directly solved using Bayes’ law.

\[
p(\theta_1|m_1) = \frac{p(m_1|\theta_1)}{p(m_1)} p(\theta_1) = \frac{0}{1/3} \cdot 1/3 = 0
\]

\[
p(\theta_2|m_1) = \frac{p(m_1|\theta_2)}{p(m_1)} p(\theta_2) = \frac{1/2}{1/3} \cdot 1/3 = 1/2
\]

\[
p(\theta_3|m_1) = \frac{p(m_1|\theta_3)}{p(m_1)} p(\theta_3) = \frac{1/2}{1/3} \cdot 1/3 = 1/2
\]