A Competitive Theory of Mismatch

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September 21, 2011

Abstract

This paper studies a general equilibrium island model of mismatch and examines the stationary properties of the equilibrium. Aggregate demand is uncertain and market participants cannot instantaneously adjust to changes in demand. Changes in technological and demand conditions trigger capital and labor reallocations that increase aggregate unemployment and vacancies. Mismatch increases due to higher dispersion of unemployment and vacancies across locations, and because the number of unemployed workers and vacant jobs in different locations increases. The theory provides an explanation for the shifts in the Beveridge curve consistent with microeconomic evidence on labor market dispersion.

Keywords: unemployment, vacancies, mismatch, competitive equilibrium

JEL classification: E20; E24; D52.

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*I would like to thank Daron Acemoglu, Björn Brügemann, Juan Carlos Cordoba, Aspen Gorry, Jang-Ting Guo, William Hawkins, Marek Kapicka, Tee Kilelthong, Philipp Kircher, Finn Kydland, David Lagakos, Ricardo Lagos, Benjamin Lester, Jianjun Miao, Dale Mortensen, Borghen Narajabad, Richard Rogerson, Peter Rupert, Raaj Sah, Rob Shimer, Richard Suen, Ted Temzelides, Ludo Visschers, Eric Young, Neil Wallace, Etienne Wasmer, Pierre-Olivier Weill, and Randy Wright. George Akerlof, Cheng-Zhong Qin, and Bill Zame commented on an earlier draft of this paper. I would also like to thank seminar participants at Northwestern University, UCSB, UC Santa Cruz, Rice University, USC, UC Riverside, the conference “Business Cycles: Theoretical and Empirical Advances,” the Midwest Macroeconomics meetings, the NBER summer meeting, the summer meeting of the Econometric Society, and the SED meetings for their comments and suggestions. Kang Cao provided valuable research assistance.
1 Introduction

Many labor markets exhibit a persistent mismatch between available workers and available jobs. Mismatch is likely to arise due to the inability of market participants to fully anticipate and instantaneously adjust to changes in the labor market. Workers acquire skills or decide where to reside without knowing the needs of the labor market, and firms design jobs with specific qualifications in mind, not necessarily those required by the local labor market. Thus, workers may find themselves unemployed because the qualifications they offered no longer meet existing demands. Workers cannot easily relocate or learn new skills, and jobs cannot be easily redesigned to accommodate the characteristics of unemployed workers. These resource reallocations are costly, uncertain, and typically involve long spells of unemployment.

This paper studies a general equilibrium model of mismatch and examines the stationary properties of the equilibrium. The basic structure of the model is that of island economies described by Lucas and Prescott [24]. In their paper, however, production depends on the labor input only and there is no notion of jobs or unfilled vacancies. I introduce vacancies, capital mobility and accumulation, and a notion of unemployment that does not require worker mobility. The model relies on three simple assumptions: (i) capital and labor are exchanged in competitive but segmented markets, i.e., “islands”; (ii) aggregate demand is uncertain; and, (iii) economic decisions are made before the resolution of demand uncertainty and in the absence of state-contingent contracts.

The role of these assumptions is the following. Assumption (i) implies that in order to find employment (or to find workers), workers (firms) must be physically present in a given sub-market. Assumption (ii) implies that, at each point in time, only some sub-markets will have a positive demand (and consequently positive production). Finally, Assumption (iii) implies that workers and firms make decisions based on imperfect information about the state of aggregate demand. That is, agents cannot wait until the full resolution of uncertainty before they decide to visit any particular sub-market. The workers who visit sub-markets with no final demand will remain unemployed and the capital in these sub-markets will remain vacant.
In addition to search unemployment, the model yields a new notion of unemployment I call *waiting unemployment*. The idea is that when workers visit a sub-market, say after a period searching for work, they are uncertain about demand in that sub-market. Workers will then wait for jobs that may or may not become available. Daily labor markets provide an ideal illustration of this phenomenon. In daily labor markets, workers typically know the locations where jobs may be found and the wage they will get paid if hired. Workers, however, enter the labor market before demand is known. Thus, workers will become unemployed when the realization of demand is low.\(^1\)

In equilibrium, the mean and the dispersion of unemployment and vacancies differ across sub-markets because changes in aggregate demand are sectorally unbalanced. I use this microeconomic model to study the individual and aggregate effects of changes in technological and market conditions. Shifts in the mean and the dispersion of unemployment and vacancies are a central subject in this paper. In discussing mismatch, Petrongolo and Pissarides ([30], pp. 399-400) note that mismatch “measures the degree of heterogeneity in the labor market across a number of dimensions” and that a “rise in mismatch [implies] a shift in the aggregate matching function.” The framework proposed here examines the aggregate labor market but with a microeconomic structure focused on understanding labor market heterogeneity.

The quantitative significance of changes in the structure of labor markets is reflected in the high volatility of the trend component of unemployment and vacancies, and in the shifts in the Beveridge curve. As an illustration, Figure 1 displays the U.S. time series of unemployment and vacancies and their trends obtained using the Hodrick-Prescott filter. The trend components are as volatile as the cyclical components and the correlation between trend components, 0.85, is precisely the opposite of the correlation between cyclical components, −0.90. This strong positive correlation between medium-run trends

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\(^1\)Many occupational, educational, and locational decisions are made with imperfect information about the state of the labor market. In rural-urban migration models such as the Harris-Todaro model, workers move to urban markets before knowing if they will be employed. This uncertainty is not captured by the price uncertainty of search models or by a bilateral matching process. Moreover, typical labor markets display a wide range of waiting activity enough to distinguish it conceptually from search; see, e.g., Jones and Riddell [19].
suggests shifts in the aggregate Beveridge curve.\textsuperscript{2} The 1950s and 1960s featured low trend unemployment and vacancies. Most of the 1970s and 1980s featured high economic volatility and high trend unemployment and vacancies. The 1990s and the early 2000s featured reduced economic volatility and low trend unemployment and vacancies. Economic volatility appears to have increased in recent times. The Beveridge curve also appears to have shifted recently; see, e.g., Daly et al. [14].

The theory provides an explanation for the changes in medium-run trends based on capital and labor reallocations. In the model, resource reallocations are accompanied by an increase in aggregate unemployment and vacancies. A decline in aggregate demand

\textsuperscript{2}Abraham [2], Blanchard and Diamond [7], Bleakley and Fuhrer [9], Medoff [25], and Nickell et al. [28] are some of the studies that have examined empirically the shifts in the Beveridge curve.
(e.g., higher demand uncertainty) or changes in productivity induce workers to move across sub-markets. Search unemployment takes place as workers leave declining sub-markets, as in Lucas and Prescott [24]. Workers who do not move experience waiting unemployment. Aggregate vacancies shift in the same direction as aggregate unemployment because capital and labor are complements in production.

Reallocations also increase labor market dispersion. A key fact about the relationship between aggregate and disaggregate labor markets is that periods of high aggregate unemployment or vacancies are periods of high cross-sectional dispersion in these variables (see Section 4). In the model, labor market dispersion increases because the movements of capital and labor are uneven across sub-markets. Mismatch increases because the dispersion of unemployment and vacancies increases, and because the number of unemployed workers and vacant jobs in different locations increases.

Related literature. The idea that mismatch plays a relevant role in explaining unemployment goes back, at least, to discussions regarding the increase in European unemployment rates since the mid-1970s; see Jackman et al. [18] and Schioppa [33]. These early approaches provided many empirical measures of mismatch, but their theoretical bases are ambiguous. The model in this paper displays more plausible properties and it relies on an explicit microeconomic structure.3

This paper also complements recent models of mismatch, e.g., Lagos [21], Shimer [34], and Mortensen [27]. Mismatch in these models arises because some sub-markets have more workers than jobs, others more jobs than workers, and these excesses cannot be used locally. Such models have yielded many insights. Shimer [34], for instance, quantitatively accounts for movements along a stable Beveridge curve. These models, however, have focused exclusively on short-run fluctuations in the aggregate labor market. This paper, instead, emphasizes the medium run and labor market dispersion. Moreover, this literature relies on shortages due to a Leontief technology. The results in this paper do not require a Leontief function, but I use this function as a special case.

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3 As pointed out by Sherwin Rosen in a comment to their paper, a puzzling feature in Jackman et al. [18] is that a rise in unemployment lowers their measure of mismatch. I explain their puzzling results. Rosen also remarked that Jackman et al. [18] defined mismatch “without reference to equilibrium or disequilibrium.” Other earlier measures of mismatch suffered similar shortcomings; see Schioppa [33].
The focus on labor market dispersion is clearly related to Lilien [23], although he only examined cyclical unemployment; see Abraham and Katz [1] and Petrongolo and Pissarides [30] for critical discussions of Lilien [23]. The idea that unused resources may result from demand uncertainty is discussed in Prescott [32]. The market decentralization in this paper follows Eden [16]. In contrast to these papers, this paper is aimed directly at examining worker-job mismatch.

The rest of the paper is as follows. Section 2 examines an economy without worker mobility. Section 3 introduces worker search and capital accumulation. Section 3 also discusses a numerical exploration of the general model under directed search, as in Lucas and Prescott [24], and under random search, as in Alvarez and Veracierto [4]. Section 4 discusses the empirical relevance of the model and Section 5 presents some possible extensions. Section 6 concludes. All technical proofs are in the Appendix.

2 The basic model

Environment. Time is discrete, indexed by \( t \in \{0, 1, \ldots, \infty\} \). There is a continuum of locations indexed by \( x \in [0, 1] \). Each point represents a potential sub-market, not only or necessarily a geographic location (i.e., skill, occupation, industry, or any combination of these categories).

There is a measure one of workers in the economy. At the beginning of period \( t \), workers are attached to some location \( x \). Let \( l_t(x) \) be the number of workers in \( x \) and let \( \mu_t \) represent the distribution of workers across locations during period \( t \), e.g., \( \mu_t([0, 1]) \equiv \int_0^1 l_t(x)dx = 1 \), by the previous convention. In the basic model, workers cannot move. The next section allows worker movements which represent search.

There is a representative firm in each location. This firm has access to a production function \( f(k_t(x), l_t(x)) \), which combines capital, \( k_t(x) \), and labor, \( l_t(x) \), to produce output. \( f(k_t(x), l_t(x)) \) is increasing in \( k_t(x) \) and \( l_t(x) \), has diminishing returns to scale, satisfies

\[
\frac{\partial^2 f}{\partial k_t \partial l_t} < 0, \quad \frac{\partial^2 f}{\partial k_t^2} > 0, \quad \frac{\partial^2 f}{\partial l_t^2} > 0, \quad \frac{\partial^2 f}{\partial k_t \partial l_t} < \frac{\partial^2 f}{\partial k_t^2} \frac{\partial^2 f}{\partial l_t^2}
\]

In Prescott [32], sellers of hotel rooms set prices before they know how many buyers will arrive. (In Butters [10], sellers send price offers to potential costumers.) When the realization of demand is low, rooms will remain vacant. Most existing studies focus on monopolistic pricing under demand uncertainty; see, e.g., Bryant [11], Carlton [12], Dana [15], and Peters and Winter [29].
Inada conditions, and $f_{kt}(k_t(x), l_t(x)) > 0$. At the beginning of period $t$, a location-specific productivity shock, $z_t(x)$, is realized. Productivity shocks are persistent but independent across locations. The transition function for $z_t(x)$ is Markovian and has a unique stationary distribution.

Aggregate demand is uncertain. There is a representative household made up of a large number of individual members distributed along $[0, 1]$. The preferences of the different individual members differ and each member obtains utility from her own consumption of goods. I assume, for simplicity, that utility is linear in consumption. There is a continuum of possible demand states indexed by $\omega$. Consumption in location $x$ and state $\omega$ is $c_t(x, \omega) \equiv c_t(x)n_t(x, \omega)$, where $c_t(x)$ denotes the consumption per member in $x$ and $n_t(x, \omega)$ is the fraction of individual members that will arrive at the market in $x$. $c_t(x)$ is deterministic but $n_t(x, \omega)$ is random. In particular, $n_t(x, \omega) \in \{0, 1\}$ is driven by a preference shock to the marginal utility of consumption: $n_t(x, \omega) \equiv \min\{x - X_t(\omega), 0\}/\{x - X_t(\omega)\}$. This implies that consumers will only visit locations $x \leq X_t(\omega)$, e.g., $n_t(x, \omega) = 1$ if $x \leq X_t(\omega)$, and $n_t(x, \omega) = 0$, otherwise. The random variable $X_t(\omega) \in [0, 1]$ thus determines the state of aggregate demand.

Aggregate consumption is $C_t(\omega) = \int_0^1 c_t(x, \omega)dx$, where

$$c_t(x, \omega) = \begin{cases} c_t(x) & \text{if } x \leq X_t(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

(1)

The objective of the representative household is to maximize $\mathbb{E}_\omega[C_t(\omega)]$, where the expectation is over possible realizations of the state $\omega$.\footnote{Individual demands may fluctuate stochastically as in Lucas and Prescott [24]. This alternative source of uncertainty can be included in (1), but it does not contribute to the main points raised here. The paper only requires that some locations face a positive probability of not producing. Other possible assumptions such as technological obsolescence, an uncertain regulatory environment, or an uncertain labor supply would, after appropriate modifications, yield equivalent results. The assumptions used here highlight the role of changes in aggregate demand in generating output and unemployment.}

If demand is positive in $x$, capital and labor are hired instantaneously and firms produce to clear the goods market. To be hired, however, capital and labor must be present in a given location. Capital is assigned before $X_t(\omega)$ is known. (In the general
model, worker movements will also take place before \( X_t(\omega) \) is known.) This sequence of events implies that there will be no excess capacity but that the capital and labor available in locations \( x > X_t(\omega) \) will remain idle. It is crucial that capital and labor cannot be reassigned after \( X_t(\omega) \) is realized. A sequential resolution of uncertainty in which some sub-markets transact before others provides a rationalization of this assumption. I discuss this point below.

Let \( y_t(x, \omega) \) denote the market-clearing level of output in location \( x \),

\[
y_t(x, \omega) = \begin{cases} 
  z_t(x) f(k_t(x), l_t(x)) & \text{if } x \leq X_t(\omega) \\
  0 & \text{otherwise.}
\end{cases}
\]  

(2)

The structure of demand (1) implies that firms face a Bernoulli random variable: with probability \( q_t(x) \equiv \Pr(\{\omega : x \leq X_t(\omega)\}) \) production in \( x \) takes place and with probability \( 1 - q_t(x) \equiv \Pr(\{\omega : X_t(\omega) > x\}) \) no production occurs. These are the same probabilities capital and labor have of being hired. These probabilities and the shocks \( z_t(x) \) are common knowledge among workers and firms.

I assume that \( X_t(\omega) \) is independently drawn each period. I retain the index \( t \) in \( q_t(x) \) to discuss the case of persistence in demand later on. Notice that \( q_t(x) \) differs across locations: \( q_t(x') \leq q_t(x) \) for \( x' \geq x \). This order means that demand is “more uncertain” for some firms. As expected, the volatility of output will be higher for firms with a more uncertain demand. Differences in \( q_t(x) \) and \( z_t(x) \) will also support wage dispersion in equilibrium. The special case when \( q_t(x) \) is not indexed by \( x \) does not particularly simplify the analysis. As a drawback, this case assigns all heterogeneity across locations to productivity differences.\(^6\)

Finally, the total number of jobs is given by the economy’s total capital, \( K_t \). The resource constraint for capital is

\[
\int_0^1 k_t(x)dx \leq K_t, 
\]  

(3)

\(^6\)Discussing Lillien’s [23] analysis of unemployment, Abraham and Katz [1] argued that variations in the cross-sectional distribution of unemployment could be driven by heterogeneous responses to aggregate demand fluctuations. The function \( q_t(x) \) implies a sectorally unbalanced response.
A planner problem. To maximize $E_{\omega}[C_{t}(\omega)]$, the (constrained) social planner must assign capital $k_{t}(x)$ across locations knowing $l_{t}(x)$, $z_{t}(x)$, and $q_{t}(x)$, but with no information about $X_{t}(\omega)$. Let $F_{t}(K_{t},\mu_{t})$ be the value function of the planner problem. This value function is the maximized expected aggregate output,

$$F_{t}(K_{t},\mu_{t}) \equiv \max_{k_{t}(x)} \int_{0}^{1} E_{\omega}[y_{t}(x,\omega)]dx,$$

subject to (3) and the distribution of workers.7

**Proposition 1** There exists a unique Pareto optimal capital assignment, and it is such that capital is assigned to locations with a positive probability of production.

**Proof.** The proof is trivial. Let $k_{t}^{*}(x)$ be the optimal capital assignment. Then,

$$q_{t}(x)z_{t}(x)f_{k}(k_{t}^{*}(x),l_{t}(x)) = r_{t},$$

with $r_{t}$ as the Lagrange multiplier of (3), i.e., the opportunity cost of capital. By Inada conditions, $q_{t}(x) > 0$ implies $k_{t}^{*}(x) > 0$. ■

Capital is assigned trading off its opportunity cost, $r_{t}$, and the production probability, $q_{t}(x)$. A positive assignment of capital implies that the the expected marginal product of labor in $x$ will be positive. It also implies that higher realizations of demand, i.e., a high value of $X_{t}(\omega)$, generate more capital and labor to be used and hence higher aggregate output. These comparative statics, however, are not the focus of the paper. My focus is on the long-run stationary allocation.

Recall (2) and let $y_{t}(x)$ be the mean of $y_{t}(x,\omega)$. Then, $y_{t}(x) \equiv E_{\omega}[y_{t}(x,\omega)] = q_{t}(x)z_{t}(x)f_{k}(k_{t}^{*}(x),l_{t}(x))$. The variance of the output produced in $x$ is $Var_{\omega}[y_{t}(x,\omega)] \equiv E_{\omega}[(y_{t}(x,\omega) - y_{t}(x))^{2}] = q_{t}(x)[1 - q_{t}(x)]\{z_{t}(x)f_{k}(k_{t}^{*}(x),l_{t}(x))\}^{2}$, which follows due to the Bernoulli all-or-nothing nature of demand.

Mean output and its variance are increasing in $z_{t}(x)$, $q_{t}(x)$, and $l_{t}(x)$; see, e.g., (5).

The response in the variance of output to changes in $q_{t}(x)$ is, in general, ambiguous: a

7To avoid notational clutter, I omitted the terms $q_{t}(x)$ and $z_{t}(x)$. Thus, $F_{t}(K_{t},\mu_{t})$ is a short-hand notation for $F(K_{t},\mu_{t};\{q_{t}(x),z_{t}(x)\}_{x\in[0,1]})$. Similar notation applies to the rest of the paper.
decline in \( q_t(x) \) increases the variance directly but the decline in \( k_t^*(x) \) lowers it. To resolve the ambiguity, I assume that \( q_t(x) \geq 1/2 \) for all \( x \). At very fine levels of disaggregation, it is possible to find firms with low chances of selling their output. The fraction of idle resources in the aggregate, however, appears to be small, e.g., \( q_t(x) \) is likely to be close to one. I also assume that the response in \( k_t^*(x) \) to changes in \( q_t(x) \) is “small.” To satisfy this assumption, I assume throughout that \(-f_{kk}(k,l)k/f_k(k,l) \geq q\) for \( q > 0 \). Conventional concave functions satisfy this property. (An example that fails to do so is the linear technology.) Under these assumptions, output dispersion in \( x \) is decreasing in \( q_t(x) \).

**Vacancies.** The capital assigned to locations with no production will remain vacant. Let \( v_t(x, \omega) \) denote job vacancies,

\[
v_t(x, \omega) = \begin{cases} 
  k_t^*(x) & \text{if } x > X_t(\omega) \\
  0 & \text{otherwise.}
\end{cases}
\]  

(6)

Let \( v_t(x) \) be the mean of \( v_t(x, \omega) \),

\[
v_t(x) \equiv \mathbb{E}_\omega[v_t(x, \omega)] = [1 - q_t(x)] k_t^*(x).
\]  

(7)

The first term in the right-hand-side of (7) is the probability that no production takes place, and the second is the assigned capital. The dispersion of vacancies is

\[
Var_\omega[v_t(x, \omega)] \equiv \mathbb{E}_\omega[(v_t(x, \omega) - v_t(x))^2] = q_t(x) [1 - q_t(x)] \{k_t^*(x)\}^2.
\]  

(8)

The mean and variance of vacancies are ambiguous with respect to \( q_t(x) \). As the Appendix shows, a bounded elasticity in the marginal product of capital resolves those ambiguities:

**Proposition 2** The mean number of job vacancies, \( v_t(x) \), and the dispersion of vacancies are increasing in the productivity shock, \( z_t(x) \), and the number of workers, \( l_t(x) \); and decreasing in the production probability, \( q_t(x) \).

These comparative statics follow from (5), (7), and (8), and are very intuitive.
**Waiting unemployment.** Let \( u_t(x, \omega) \) be the number of unemployed workers in \( x \) given \( \omega \). As in (6), \( u_t(x, \omega) = l_t(x) \) if \( x > X_t(\omega) \), and \( u_t(x, \omega) = 0 \), otherwise. The mean number of unemployed workers in \( x \), \( u_t(x) \), is

\[
    u_t(x) \equiv \mathbb{E}_\omega[u_t(x, \omega)] = [1 - q_t(x)]l_t(x).
\]

The interpretation of \( u_t(x) \) is analogous to (7). Employment is \( e_t(x, \omega) \equiv l_t(x) - u_t(x, \omega) \), and its mean is \( \mathbb{E}_\omega[e_t(x, \omega)] = q_t(x)l_t(x) \). The dispersion of unemployment, which equals the dispersion of employment, is

\[
    Var_\omega[u_t(x, \omega)] \equiv \mathbb{E}_\omega[(u_t(x, \omega) - u_t(x))^2] = q_t(x)[1 - q_t(x)]\{l_t(x)\}^2.
\]

The mean number of unemployed workers and the variance of unemployment are both increasing in \( q_t(x) \).

**Job “matches.”** I treat capital in producing locations as being *matched* to employed workers. Let \( m_t(x) \) denote the mean number of job matches,

\[
    m_t(x) \equiv \mathbb{E}_\omega[k^*_t(x) - v_t(x, \omega)] = q_t(x)k^*_t(x).
\]

The mean number of job matches is increasing in the productivity shock, the number of workers, and the production probability. These predictions follow directly from (5) and (10). Notice that changes in \( q_t(x) \) do not generate an ambiguous response for \( m_t(x) \), and that the dispersion of job matches is the same as the dispersion of vacancies, (8).

**Discussion.** It is not surprising that some capital and labor will remain idle when decisions are made with imperfect information about the state of aggregate demand. The significance of this result is that the model yields differences in mean unemployment and vacancies across locations, and differences in their dispersion. These differences are consistent with well-known observations.\(^8\) The model also implies that average unemployment (and vacancies) and their dispersion should covary positively.

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\(^{8}\) Jackman et al. ([18], pp. 45) remarked that “as everybody knows, unemployment rates differ widely between occupations and between regions, as well as across age, race and (sometimes) sex groups.” Sherwin Rosen, in his discussion of mismatch, listed persistent differences in unemployment across skills groups. Further, “employment and output variability are always greater in construction and in durable goods manufactures than in services and non-durables manufacturing, and urban-rural and North-South regional differences can persist for generations”; see Jackman et al. ([18], pp. 102).
To distinguish conceptually the unemployment that arises due to demand uncertainty from search unemployment, I have used the notion of *waiting unemployment*. Waiting unemployment captures the idea that workers enter the labor market before demand is known. Since workers cannot immediately go to an alternative location if demand does not arrive, workers have to wait out a spell of unemployment.

Jobs are divisible capital. This notion of a job is typical in business cycle models (e.g., Cooley [13]), although it is slightly different from that used in existing models of mismatch. Lagos [21], Shimer [34], and Mortensen [27] used a Leontief technology \( f(k, l) = \min[k, l] \). If \( f(k, l) = \min[k, l] \), the planner solution is such that \( k^*_t(x) = l_t(x) \) given aggregate capital availability. In this case, capital may be unused due to a shortage of workers, i.e., (6) becomes \( v_t(x, \omega) = \max\{k^*_t(x) - l_t(x), 0\} \) if \( x > X_t(\omega) \) and \( v_t(x, \omega) = 0 \) otherwise. If \( K_t > 1 \), capital is in excess supply. On the other hand, if \( K_t < 1 \), there will be an excess supply of labor in the least productive locations, i.e., workers in locations with the lowest values of \( q_t(x)z_t(x) \) will be in excess supply.\(^9\)

The Leontief assumption provides a complementary reason for the existence of unused capital and labor. Unemployment in the Leontief case is due to a shortage of jobs. Unemployment here arises because there is no demand for the commodities certain workers help produce. The difficulty here is not necessarily about job shortages in the local labor market. In both cases, however, firms and workers will report unfilled vacancies and unemployment when capital and labor are unused.

Shortages and the absence of factor substitution may be reasonable approximations in the short term. Over a longer time period, allowing for factor substitution seems natural. The Leontief assumption may also be restrictive empirically. If \( K_t = 1 \) this assumption implies \( u_t(x) = v_t(x) \), see (7) and (9). Differences between local unemployment and vacancies have in fact been previously used to measure matching inefficiencies; see, e.g.,

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\(^9\)Shimer [34] and Mortensen [27] considered a framework where capital and labor are assigned randomly. There are no economic decisions regarding \( k^*_t(x) \) or \( l_t(x) \) in these papers. Factors are not assigned randomly in Lagos [21], but he studied an inefficient allocation (see Section 5). The empirical distinction between posted, filled, and unfilled job vacancies is not present in the theory. As in these previous models, some locations have an “excess” of capital. This excess does not correspond to the traditional notion of capital utilization, often viewed as an endogenous depreciation rate.
Schioppa ([33], pp. 11). In here, not all sub-markets should be equally “tight.”

**Aggregation.** The previous microeconomic structure can be easily aggregated. Aggregate vacancies integrate (6), \( \bar{V}_t(\omega) = \int_0^1 v_t(x, \omega)dx \). Let \( V_t(K_t, \mu_t) \) be mean aggregate vacancies,

\[
V_t(K_t, \mu_t) \equiv \int_0^1 \mathbb{E}_\omega[v_t(x, \omega)]dx = \int_0^1 v_t(x)dx. \tag{11}
\]

The dispersion of aggregate vacancies will equal the aggregate over (8) if covariance terms were zero. The Appendix shows that this approximation is valid when \( q_t(x) \) is close to 1. In that case,

\[
Var_\omega[\bar{V}_t(\omega)] \simeq \int_0^1 Var_\omega[v_t(x, \omega)]dx = \int_0^1 q_t(x)[1 - q_t(x)]\{k^*_t(x)\}^2dx. \tag{12}
\]

Let \( U_t(K_t, \mu_t) \) be mean aggregate unemployment, \( U_t(K_t, \mu_t) \equiv \int_0^1 \mathbb{E}_\omega[u_t(x, \omega)]dx \). The dispersion of aggregate unemployment can be obtained as in (12). Mean aggregate job matches integrate (10),

\[
M_t(K_t, \mu_t) = \int_0^1 m_t(x)dx. \tag{13}
\]

The degree of returns to scale in this “aggregate matching function” depends primarily on the degree of returns to scale in the production function. The next proposition provides a simple illustration:

**Proposition 3** Suppose that \( f(k, l) \) has constant returns to scale. Then, \( M_t(K_t, \mu_t) \), has constant returns to scale with respect to the mean number of unemployed workers, \( U_t \), and vacant jobs, \( V_t \).

The proof is quite simple because unemployment is proportional to \( l_t(x) \) and vacancies and job matches are proportional to \( k^*_t(x) \). Notice that \( M_t(K_t, \mu_t) \) differs from the conventional reduced-form aggregate matching function on which search and matching models rely (e.g., Pissarides [31]). This literature focuses on the flow of aggregate job matches given the stock of unemployed workers and vacant jobs. \( M_t(K_t, \mu_t) \) measures the stock of aggregate job matches.
Mismatch. An immediate application is to compare mean individual and aggregate outcomes in environments with different levels of uncertainty. Let an increase in demand uncertainty be a first-order stochastic worsening in the distribution of $X_t(\omega)$, e.g., a decline in the production probabilities from $q_t(x)$ to $q'_t(x) < q_t(x)$ for all $x \in [0, 1]$. An increase in demand uncertainty represents a reduction in aggregate demand. (It can also be viewed as a decline in aggregate total factor productivity since $K_t$ and $\mu_t$ are constant.)

**Proposition 4** In response to an increase in demand uncertainty, mean individual and aggregate unemployment and vacancies increase, and their individual and aggregate dispersion increases.

An increase in demand uncertainty increases mean unemployment and vacancies “everywhere.” The increase in mean unemployment and vacancies implies that higher demand uncertainty “shifts” individual and aggregate Beveridge curves. These shifts are accompanied by higher dispersion in unemployment and vacancies.

An increase in the mean and the dispersion of labor market outcomes has been typically viewed as an increase in mismatch; see, e.g., Medoff [25], Jackman et al. [18], Schioppa [33], and Petrongolo and Pissarides [30]. Higher demand uncertainty, however, is only one of many possible causes of a simultaneous increase in means and dispersion. In general, changes in productivity will also lead to resource reallocations that increase unemployment, vacancies, and labor market dispersion. Further, in the basic model, unemployed workers and vacant jobs are in the same locations. This feature is a consequence of the absence of worker mobility. In the general model, workers searching for jobs and vacant jobs will be in different places adding another dimension to mismatch. Finally, I ignored the effects of changes in demand uncertainty on $r_t$. The next section justifies this omission.

The model also has measurement implications. To measure mismatch, Jackman et al. ([18], pp. 70) proposed an index based on the variance of relative unemployment rates,\footnote{Petrongolo and Pissarides ([30], pp. 409) state that “in Britain the shifts in the regional Beveridge curves were of the same order of magnitude as the aggregate curve, casting doubt on the power of regional mismatch to explain the shift in the aggregate curve.” In here, individual and aggregate changes in unemployment and vacancies are of similar orders of magnitude and, hence, consistent with this finding.}
i.e., $\text{Var}_\omega[u_t(x, \omega)/u_t(x)]$. This index has been widely applied in the literature, but its properties are not well-established. In this paper, $\text{Var}_\omega[u_t(x, \omega)/u_t(x)] = q_t(x)l_t(x)/u_t(x)$. This index will not increase with a rise in demand uncertainty or in the unemployment rate. In fact, both will yield a decline in the relative variance. These observations are consistent with the puzzling finding of Jackman et al. [18] that measured mismatch fell while unemployment rates increased in Europe. Their puzzling results can be understood by taking into account that as $q_t(x)$ changes, the variance of unemployment will increase, but it will not increase as fast as the mean.

**The resolution of uncertainty.** Thus far, decisions are made before any information about demand is known. However, a partial revelation of information about demand conditions will yield the same assignment of capital.

Assume capital assignments to $x'$ can be conditioned upon demand conditions in $x$, with $x' > x$. Let $q_t(x'|x)$ be the conditional probability of production in location $x'$, given that production has taken place in $x$. The benefit of assigning capital to $x'$ is $q_t(x'|x)z_t(x')f_k(k_t(x'), l_t(x'))$, whereas the expected cost is $r_t/q_t(x)$. Bayes’ rule implies the benefit of production in $x$ provides no incentive for reallocations to $x'$. The intuition is simply that knowing demand conditions in $x$ provides no useful information about the state of aggregate demand, $X_t(\omega)$, which is the relevant source of uncertainty.\(^{11}\)

**A market assignment.** There are multiple ways to model competitive markets under demand uncertainty.\(^{12}\) I follow Eden [16] and consider competitive spot markets subject to *uncertain and sequential trade*.

Agents know prices in all potential markets. The market for capital opens at the beginning of each period, after $z_t(x)$ is known but before $X_t(\omega)$ is known. Workers supply labor inelastically. Once capital and labor are decided, firms and workers just wait for

\(^{11}\)Formally, $q_t(x'|x) = \text{Pr}\{\omega : x' \leq X_t(\omega) | x \leq X_t(\omega)\}$. Bayes’ rule yields $q_t(x'|x)q_t(x) = q_t(x|x')q_t(x') = q_t(x')$ since $q_t(x|x') = 1 \text{ as positive demand in } x' \text{ always implies positive demand in } x$. The first order condition is: $(q_t(x'|x)z_t(x')f_k(k_t(x'), l_t(x')) = r_t/q_t(x)$, which is (6). Landsburg and Eden [22] and Peters and Winter [29] provide related remarks.

\(^{12}\)In the Arrow-Debreu model, a market in contingent claims would open before $X_t(\omega)$ is known and claims would be redeemed after uncertainty is resolved. If decisions could be made contingent on $X_t(\omega)$, there would be no unused factors. An alternative is to assume that firms post wages before $X_t(\omega)$ is known. Wages would remain fixed after uncertainty is resolved; see Prescott [32]. With pre-determined prices, one needs to specify how rationing takes place as demand arrives; see, e.g., Dana [15].
the realization of demand. Demand arrives sequentially (but within a single period) and its arrival triggers the opening of competitive goods and labor markets. All transactions and production take place instantaneously. Markets then close. This process continues until uncertainty is resolved. The number of markets that transact is greater the higher the value of $X_t(\omega)$. Since markets do not transact simultaneously, idle capital cannot be reassigned to markets with positive demand because these markets already closed.

In a competitive market equilibrium, labor supply satisfies $l_t^x(x) = l_t(x)$ if $w_t(x) > 0$, where $w_t(x)$ is the wage rate in sub-market $x$. Firms demand capital and labor to maximize expected profits,

$$q_t(x)z_t(x)f(k_t^d(x), l_t^d(x)) - w_t(x)l_t^d(x) - r_t k_t^d(x),$$

where $r_t$ is the rental price of capital. Factor markets clear: (3) holds, and $l_t^d(x) = l_t^x(x)$, for all $x$. (The distribution of profits does not affect the workers’ incentives and therefore is not specified.) Equilibrium prices and quantities cannot be indexed by $X_t(\omega)$ because information about $X_t(\omega)$ becomes available later in time.

The competitive equilibrium coincides with the solution to the social planner problem. For example, the equilibrium rental price of capital coincides with $r_t$. Under uncertain and sequential trade, the equilibrium wage rate in $x$ is

$$w_t(x) = q_t(x)z_t(x)f_t(k_t^s(x), l_t(x)). \tag{14}$$

Wages are increasing in $z_t(x)$ and decreasing in $l_t(x)$. Demand uncertainty also yields equilibrium wage dispersion: sub-markets with a more uncertain demand exhibit higher unemployment rates and lower wages.\(^{13}\)

As in Prescott [32], the model does not feature the externalities typical of search models, although they can be accommodated (see Section 5). In this economy, the markets that transact always close, and agents are price-takers and fully informed about potential wages in the economy (e.g., there is no need to search). Under demand uncertainty, however, not all markets transact in any given period.

\(^{13}\)This implication is consistent with the fact that high-skilled workers typically have lower unemployment rates than low-skilled workers. Microeconometric earnings equations also show that workers in labor markets with higher unemployment rates earn lower wages; see, e.g., Blanchflower and Oswald [8].
3 Capital accumulation and worker search

So far, aggregate capital is constant and workers are not able to leave their location. This section allows for capital accumulation and worker search.

**Capital accumulation.** Capital accumulation decisions are made at the end of each period, after the realization of demand. Let $\omega^t = (\omega_t, \omega^{t-1})$ denote a history of demand states. Aggregate consumption in period $t$ is $C_t(\omega^t)$. The capital stock in period $t$ is $K_t(\omega^{t-1})$, and the distribution of workers at the beginning of period $t$ is $\mu_t(\omega^{t-1})$. Assume for now that the law of motion of $\mu_t(\omega^{t-1})$ is deterministic. (This will be the case under the conditions assumed below, but I will discuss the evolution of this distribution once I discuss worker search.)

Initial capital $K_0$ is given. Capital takes one period to enter production and it depreciates at a constant rate $\delta \in (0, 1)$. Unused capital also depreciates. Let $\rho \in (0, \infty)$ be the time discount rate. The objective of the representative household is to maximize the present discounted value of expected aggregate consumption,

$$
E_0 \sum_{\tau=0}^{\infty} (1 + \rho)^{-\tau} C_\tau(\omega^\tau),
$$

where the expectation is over possible histories of realizations of the states $\omega^\tau$. Since the allocation of capital across locations takes place as in the basic model, and since utility is linear, the objective of the representative household can be written simply as

$$
\max_{\{K_{t+1}(\omega^t)\}} E_0 \sum_{t=0}^{\infty} (1 + \rho)^{-t} \left\{ F_t(K_t(\omega^{t-1}), \mu_t(\omega^{t-1})) - [K_{t+1}(\omega^t) - (1 - \delta)K_t(\omega^{t-1})] \right\}, \quad (15)
$$

where $F_t(K_t(\omega^{t-1}), \mu_t(\omega^{t-1}))$ is the value function of the static problem, see, e.g., (4).

The maximizing choice of $K_{t+1}(\omega^t)$ is characterized by the first order condition

$$
E_t \left\{ \frac{\partial F_{t+1}(K_{t+1}(\omega^t), \mu_{t+1}(\omega^t))}{\partial K_{t+1}(\omega^t)} \right\} = \rho + \delta. \quad (16)
$$

The envelope theorem applied to the Lagrangian of the static problem, (4), yields

$$
\frac{\partial F_{t+1}(K_{t+1}(\omega^t), \mu_{t+1}(\omega^t))}{\partial K_{t+1}(\omega^t)} = r_{t+1}(\omega^t).$$

Thus, for a given realization of demand in period $t$, the optimal capital accumulation plan equates the expected value of the opportunity cost of capital in $t + 1$, net of depreciation, to the discount rate, e.g.,
\[ \mathbb{E}_t[r_{t+1}(\omega^t)] = \rho + \delta. \]
The expectation term in (16) is relevant only if demand in period \( t \) yields useful information about demand conditions in \( t + 1 \). Under the assumption that \( X_t(\omega) \) is independently drawn, \( q_{t+1}(x) = q_t(x) = q(x) \). Thus, the opportunity cost of capital in \( t + 1 \) will equal \( \rho + \delta \) regardless of the demand conditions in \( t \).

The aggregate state of the economy is \( (K_t(\omega^{t-1}), \mu_t(\omega^{t-1})) \). In equilibrium, factor prices depend on the aggregate state. This dependence means that workers, when making search decisions, would need to predict how changes in the aggregate capital stock and in the distribution of workers influence future wages. The main advantage of assuming that \( X_t(\omega) \) is independently drawn each period is that the opportunity cost of capital will be independent of the aggregate state. Wages will also be independent of \( (K_t(\omega^{t-1}), \mu_t(\omega^{t-1})) \).

In this case, search decisions will only be based on location-specific variables. This implies that I can examine a stationary solution based on a “representative island.”

The independence assumption in aggregate demand and the independence in productivity shocks across locations also imply that there is no aggregate uncertainty. Therefore, in the solution of the dynamic problem analyzed later in this section, the sequence \( \{K_t, \mu_t\} \) is deterministic. (In the Appendix I show formally that \( \mu_{t+1} \) evolves deterministically.) In the stationary allocation, the aggregate capital stock will be a stationary point, \( K^* \), and that the distribution of workers will be a stationary distribution, \( \mu^* \).

**Worker search.** Workers are initially distributed according to \( \mu_0 \). Workers are allowed to move between locations after observing \( z_t(x) \), but before \( X_t(\omega) \) is known.\(^{15}\)

Movements are of two types: the variable \( s_t(x) \) represents workers who move out of \( x \), i.e., searchers. The variable \( a_t(x) \) represents workers who move in, i.e., worker arrivals. The purpose of worker search is to find the distribution of workers that maximizes (15), given \( \mu_0 \) and the search technology specified below. I assume that worker search and

\(^{14}\)If demand is persistent, the solution will involve a stationary distribution of aggregate capital and a distribution over the distribution of workers. Aggregate capital converges to a stationary point under i.i.d. shocks even under convex adjustment costs. Under adjustment costs, however, workers would need to use the aggregate capital stock in their decision-making process. The aggregate capital stock and the distribution of workers are not relevant in Lucas and Prescott [24] since they do not allow for capital mobility.

\(^{15}\)As in the basic model, if partial information about demand is available, all the results remain unchanged.
capital assignments are simultaneously decided.

Search is costly because searchers cannot work for one period, while in transit. Given the number of workers who stay in $x$, $l_t(x) - s_t(x)$, employment is

\[
e_t(x, \omega) = \begin{cases} 
  l_t(x) - s_t(x) & \text{if } x \leq X_t(\omega) \\
  0 & \text{otherwise}.
\end{cases} \tag{17}
\]

As in the basic model, if demand is positive in $x$, all available workers, $l_t(x) - s_t(x)$, will be employed. The number of workers in $x$ evolves as

\[
l_{t+1}(x) = l_t(x) + a_t(x) - s_t(x). \tag{18}
\]

Feasibility in worker search requires that the total number of workers searching be equal to the total number of arrivals so that $\mu_{t+1}([0, 1]) \equiv \int_0^1 l_{t+1}(x)dx = 1$. That is,

\[
\int_0^1 s_t(x)dx = \int_0^1 a_t(x)dx. \tag{19}
\]

**Solution.** The solution to the dynamic assignment problem consists of a bounded sequence of aggregate capital that maximizes (15). It also involves a bounded sequence of the distribution of workers consistent with the law of motion generated by worker search. For a given aggregate capital and a given distribution of workers, the capital assignments to each location $x$ take place as in the basic model. The only difference is that some workers may leave the location. The rest of this section examines the social planner solution of the dynamic assignment. The solution to the social planner’s dynamic problem will also be a competitive equilibrium.

Aggregate capital satisfies (16). Thus, aggregate capital will be such that $r_t = \rho + \delta$ for all $t \geq 1$.\(^{16}\) Capital assignments across locations are analogous to those described by (5), and hence they do not require any further comment. Search decisions involve two

\(^{16}\)I have ignored the case where $r_t > \rho + \delta$ even if all output is saved. If initial capital is too low, the representative household will choose to save all output until $r_t = \rho + \delta$, which will happen in finite time. For the results presented here, it is only necessary that this equality holds in the stationary case.
choices in each location: \( s_t(x) \) and \( a_t(x) \). Let \( s^*_t(x) \) and \( a^*_t(x) \) be the optimal number of workers who search and arrive at location \( x \), respectively. Let \( l^*_t(x) \) be the number of workers in location \( x \) once searchers arrive, just before the beginning of the next period. I consider a directed search assignment where workers direct their search to the most attractive locations. In a numerical analysis below, I examine random search based on the specification used by Alvarez and Veracierto [4]. Under random search, the arrival of workers will be independent of \( x \), e.g., \( a_t(x) = \bar{a}_t \) for all \( x \).

To save on notation, let \( w_t(x) \) be the expected marginal product of labor in \( x \),

\[
w_t(x) = q(x)z_t(x)f_t(k_t(x), l_t(x) - s_t(x)).
\] (20)

\( w_t(x) \) is also the equilibrium wage rate under uncertain and sequential trade, (14). The marginal product of labor depends on capital and it takes into account the fact that some workers may leave the location. Expected wages in sub-market \( x \) and period \( t + 1 \) are \( \mathbb{E}_t[w_{t+1}(x)] \), where the expectation is taken over the location-specific productivity shock, \( z_{t+1}(x) \), which is unknown at the time search decisions take place. (The expectation over demand conditions, e.g., \( q(x) \), is already included in \( w_{t+1}(x) \).)

Let \( \theta_t \) denote the Lagrange multiplier for the constraint (19), i.e., the opportunity cost of search. Since the aggregate state evolves deterministically, \( \theta_t \) is deterministic. Optimal search decisions involve two separate Euler equations. Worker search satisfies

\[
\left( \theta_t - w_t(x) - \frac{\mathbb{E}_t[w_{t+1}(x)]}{1 + \rho} \right) s^*_t(x) = 0,
\] (21)

where the term in parentheses equals zero if \( s^*_t(x) > 0 \). Worker arrival satisfies

\[
\left( \frac{\mathbb{E}_t[w_{t+1}(x)]}{1 + \rho} - \theta_t \right) a^*_t(x) = 0,
\] (22)

where the term in parentheses equals zero if \( a^*_t(x) > 0 \).

To understand the implications of (21) and (22), it is useful to consider the three standard cases that characterize worker search; see Lucas and Prescott [24] and Stokey,
Lucas, and Prescott ([35], section 13.8):

(i) Workers leave \( x \), i.e., \( s_t^*(x) > 0 \) and \( l_{t+1}^*(x) = l_t(x) - s_t^*(x) \). Thus, search takes place up until the point in which the workers who stayed behind are indifferent between staying and searching.

(ii) No additional worker arrives and no worker leaves \( x \), i.e., \( s_t^*(x) = 0 \) and \( l_{t+1}^*(x) = l_t(x) \). Workers in these locations are not willing to forego their current expected payoff in order to search. The distribution of workers in these locations remains unchanged.

(iii) Workers arrive in \( x \), i.e., \( a_t^*(x) > 0 \) and \( l_{t+1}^*(x) = l_t(x) + a_t^*(x) \). Since search is directed, these workers are assigned in such a way that the expected payoff from searching is equalized across locations (i.e., until the term in parentheses in (22) holds with equality).

The next proposition addresses the existence and uniqueness of a stationary solution to the dynamic assignment problem. A stationary allocation is a vector \((\mathbf{x}_0^*, \mu_0^*)\) such that \( \mathbf{x}_t = \mathbf{x}_0^* \), and \( \mu_t = \mu_0^* \). A stationary allocation also involves a series of functions describing the number of searchers and arrivals to each location, \( s_t^*(x) = s^*(x) \) and \( a_t^*(x) = a^*(x) \) respectively, and the capital assignment, \( k_t^*(x) = k^*(x) \). These functions depend on the opportunity cost of capital, \( \rho + \delta \), and on the state of each sub-market, e.g., the number of workers at the beginning of the period, \( l(x) \), and the augmented shock, \( q(x)z(x) \).

**Proposition 5** For a given initial capital stock and a distribution of workers \((K_0, \mu_0)\), there exists a unique stationary solution to the dynamic assignment problem.

**Proof (Idea).** The idea is that a stationary version of the dynamic assignment problem can be studied along the lines considered by Lucas and Prescott [24].

Let \( \varphi(x) \equiv q(x)z(x) \) denote an **augmented shock** in \( x \). Let

\[
g(\varphi(x), l(x)) \equiv \max_{k(x)} \{ \varphi(x)f(k(x), l(x)) - (\rho + \delta)k(x) \}.
\]

This indirect production function nets out the effect of capital mobility. Thus, search decisions can be made on the basis of the function \( g(\varphi(x), l(x)) \) alone, which only depends on the state of the local labor market, \( l(x) \) and \( \varphi(x) \). Since worker search and
capital assignments are simultaneously decided, capital will change optimally in response to workers leaving location $x$. The rest of the proof involves showing that the function $g(\varphi(x), l(x))$ satisfies the assumptions in Lucas and Prescott [24] and that the stationary allocation has their recursive representation. ■

**Discussion.** Equilibrium search models typically abstract from capital mobility and capital accumulation; see, e.g., Lucas and Prescott [24], Gouge and King [17], Alvarez and Shimer [5]. The use of the indirect production function $g(\varphi(x), l(x))$ provides a simple and tractable way to incorporate a firm’s choice of capital as well as factor substitution into models of equilibrium search.

Proposition 5 only relies on differentiability and concavity properties on $f(k(x), l(x))$. Consider, for example, the case of a Cobb-Douglas production function, $f(k(x), l(x)) = k(x)^\alpha l(x)^\beta$ with $\alpha + \beta < 1$. The indirect production function satisfies

$$g(\varphi(x), l(x)) = \varphi(x)^{1/(1-\alpha)}[(\rho + \delta)/\alpha]^{\alpha/(\alpha-1)}l(x)^{\beta/(1-\alpha)}.$$  \hspace{1cm} (23)

Proposition 5 also holds under a Leontief production function. Assume, for example, that $f(k(x), l(x)) = \{\min[k(x), l(x)]\}^\beta$ with $\beta \in (0, 1)$. The indirect production function will simply be $g(\varphi(x), l(x)) = \varphi(x)l(x)^\beta$. The stationary value of the capital stock, $K^*$, is such that the capital-labor ratio will equal one for all potentially employed workers, e.g., $K^* = \int_0^1 [l^*(x) - s^*(x)]dx$. In this case, the stationary allocation involves no excess supply of capital (or labor) such as that possible in the basic model under a Leontief production function. In the basic model, the aggregate capital stock was a given parameter, whereas here $K^*$ is optimally chosen.

The function $g(\varphi(x), l(x))$ can be used to highlight important differences with existing search models. In models without capital, a large number of workers lowers equilibrium wages and induces workers to search. When capital is movable, and labor and capital are complements, a large number of workers is an incentive for capital flows. Capital in-flows increase wages even if workers do not search. Thus, abstracting from capital mobility will likely overstate the importance of search unemployment. On the other hand, capital mobility amplifies the effect of shocks $\varphi(x)$. For example, the exponent on $\varphi(x)$ in (23),
$1/(1-\alpha)$, is larger than one. The reason is that a good location (e.g., a location with high values of $\varphi(x)$) attracts workers and capital directly, but also indirectly as both factors are complements. Thus, capital mobility will likely make wages more unequal increasing the benefit of search.

Search provides other insights as well. Worker search is decreasing in the augmented shock, $\varphi(x)$; see Lucas and Prescott ([24], Proposition 2) and Stokey, Lucas, and Prescott ([35], section 13.8). This implies that search is lower in more productive locations but also in locations with less uncertain demand, e.g., workers search to “insure” against demand uncertainty. Since the arrival at a new location does not guarantee employment, even is search is directed, this insurance is limited. Another consequence of this result is that workers will differ in the duration of their unemployment spells. (In Lucas and Prescott [24], all unemployment spells are of the same length.)

The theory also predicts that worker reallocation will take place slowly over time. While I do not examine the speed of adjustment to the “steady state,” transitions are likely to be slow.\footnote{Stokey, Lucas, and Prescott ([35], section 13.8b) best illustrate this point through the less-than-unitary slope of the policy function with respect to changes in the number of workers at the beginning of the period.} Notice also that sub-markets with a high fraction of searchers are sub-markets with fewer vacancies and job matches. This result follows from Proposition 2: $k^*(x)$ satisfies $\varphi(x)f_k(k^*(x), l^*(x) - s^*(x)) = \rho + \delta$. Since $f_{kl}(k,l) > 0$, as workers leave, the expected marginal product of capital in that sub-market declines. This implies that $k^*(x)$ is decreasing (parametrically) in $s^*(x)$, and (7) and (10) imply fewer vacancies and job matches.

The mean and variance of output, unemployment, vacancies, and job matches are characterized as in the basic model. For example, mean unemployment in $x$ is $u^*(x) = s^*(x) + (1 - q(x))[l^*(x) - s^*(x)]$. The first component is due to search frictions and the second to demand uncertainty, e.g., waiting unemployment. The variance of unemployment is $q(x) [1 - q(x)] \{l^*(x) - s^*(x)\}^2$, which, parametrically speaking, is a decreasing function of $s^*(x)$. In the general model, mean and variance are also positively related, but the relationship is weaker than in the model without search because search reduces
the dispersion of unemployment across sub-markets. Aggregate labor market outcomes can also be obtained as in the basic model.

Finally, notice that searchers are not attached to any particular sub-market. The search technology prevents workers from using the vacant jobs. Thus, searchers and vacant jobs are in different places, e.g., *mismatched*.

**Numerical results.** In this sub-section I obtain some numerical solutions for the above model. The main purpose of these exercises is to examine how the mean and cross sectional distribution of output, unemployment, and vacancies change in response to changes the technological and demand conditions of the economy.

The model can be parameterized easily. I assume an exponential distribution for \(X_t(\omega)\), i.e., \(q(x) = 1 - \exp\{-\chi x\}\). Productivity shocks are given by \(z_t(x) = z_{t-1}(x)^\theta \exp\{\xi_t\}\), with \(\sigma^2_\xi\) as the variance of \(\xi_t\). The variance of log-productivity is \(\sigma^2_\xi/(1 - \eta)\). Thus, an increase in \(\sigma^2_\xi\) and \(\eta\) make \(z_t(x)\) more volatile. I use the Cobb-Douglas production function consistent with (23). I consider each period to be a quarter. The time discount rate is \(\rho = 1.045^{1/4} - 1\), and the depreciation rate is \(\delta = 0.012\). I assume \(\beta = 0.60\) and \(\eta = 0.95\). These measures are conventional values in real business cycle models; see, e.g., Cooley ([13], pp. 22). For the volatility parameter, I use \(\sigma^2_\xi = 0.10\), which is roughly the value used by Alvarez and Veracierto ([4], Table 1). I assume that \(\alpha = 0.10\). Under a value of \(\chi = 0.05\), 2.9 percent of the capital stock remains vacant.\(^{18}\)

Table 1 reports the baseline results under directed search. Panel A reports the stationary mean values of labor market outcomes, capital, and output. In the baseline case, total vacancies (measured as a fraction of the capital stock) are 2.92 percent, whereas the total unemployment rate is 4.05 percent. Search unemployment is 1.20. Waiting unemployment is 2.85. The opportunity cost of search, \(\theta\), is about four times output per worker. The stationary value of the capital-output ratio is 5. Panel B uses the endogenous distribution of workers across locations to obtain aggregate measures of dispersion for output, employment, total unemployment, and vacancies. The specific numerical values in Panel B are not important. The most relevant aspect in Panel B is the direction of change.

\(^{18}\)The solution is based on a value function iteration with bounds for the employment grid that ensure that there is a unique invariant distribution, see Stokey, Lucas, and Prescott ([35], section 13.8).
across specifications.

<table>
<thead>
<tr>
<th>Table 1. Numerical simulations under directed search.</th>
<th>Alternative parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>Baseline</td>
<td>η = 0.9875</td>
</tr>
<tr>
<td>Total unemployment rate</td>
<td>4.05</td>
</tr>
<tr>
<td>Search unemployment</td>
<td>1.20</td>
</tr>
<tr>
<td>Waiting unemployment</td>
<td>2.85</td>
</tr>
<tr>
<td>Vacancy rate</td>
<td>2.92</td>
</tr>
<tr>
<td>Cost of search</td>
<td>0.72</td>
</tr>
<tr>
<td>Capital per worker</td>
<td>0.91</td>
</tr>
<tr>
<td>Output per worker</td>
<td>0.18</td>
</tr>
</tbody>
</table>

A. Stationary mean values

| B. Cross-sectional variance                          |                               |
|                                                     | Output                        | Employment                 | Unemployment             | Vacancies                |
|                                                     | 0.0021                        | 0.0072                     | 0.0079                   | 0.0028                   |
|                                                     | 0.0499                        | 0.2032                     | 0.1708                   | 0.0871                   |
|                                                     | 0.0084                        | 0.0335                     | 0.0439                   | 0.0545                   |
|                                                     | 0.0081                        | 0.0310                     | 0.0498                   | 0.0405                   |

Note.— The mean values and the cross-sectional variances are constructed using the stationary distribution of workers. The unemployment and vacancy rates are in percent.

Table 1 also studies the effects of changes in the structural characteristics of the economy. The first alternative specification, (1), increases the persistence of the location-specific productivity shocks from $\eta = 0.95$ to $\eta = 0.9875$, which is roughly the value used by Alvarez and Veracierto ([4], Table 1). The table shows that the two forms of unemployment increase. Higher persistence in the productivity shock also leads to higher capital accumulation, a higher vacancy rate, and higher output.

There are two reasons behind these changes. First, when persistence increases, a high realization of productivity in the present period increases the chances of a high realization of productivity in the future. As a consequence, expected wages increase and this makes worker search more attractive. To understand the reasons behind the increase in waiting unemployment, it is useful to first examine the changes in aggregate capital. Aggregate capital increases because an increase in persistence also raises the future expected marginal
product of capital. This change is an intertemporal incentive to accumulate more capital. To understand the changes in vacancies, notice that when the aggregate capital stock is low, capital should be concentrated in locations with low demand uncertainty, e.g., locations more likely to produce. As aggregate capital increases, more capital is assigned to locations with more uncertain demand. This explains why the vacancy rate increases. Finally, since capital and labor are complements, as more capital is assigned to locations with more uncertain demand, more workers will also locate in these locations (relative to the baseline case). These workers will be more likely to experience waiting unemployment. Notice that higher persistence is an incentive to locate in locations with more uncertain demand, but also an incentive for higher mobility.

There is a second reason behind the changes in (1). An increase in persistence of productivity makes \( z(\xi) \) more volatile. Higher volatility makes wages more unequal and this gives workers more incentives to search. The marginal product of capital will also be more unequal across locations. These changes also lead to capital and labor reallocations to locations with more uncertain demands.

The second specification, (2), increases the volatility of location-specific productivity shocks from \( \sigma_{\xi} = 0.10 \) to \( \sigma_{\xi} = 0.20 \). Higher dispersion in the productivity shocks also increases unemployment, and it leads to higher capital accumulation and a higher vacancy rate. Quantitatively, the effects on capital, vacancies, and output are similar to those of specification (1). An increase in the volatility of the shocks leads to higher cross-sectional dispersion in wages and the marginal product of capital. The main difference between (1) and (2) is that search unemployment increases more in response to changes in \( \sigma_{\xi} \) than to changes in \( \eta \).

The previous exercises consider resource reallocations driven by technological factors, e.g., changes in the stochastic process that governs \( z(\xi) \). The last exercise, (3), considers a decline in aggregate demand or a rise in demand uncertainty, as in Proposition 4. I increase \( \chi \) from \( \chi = 0.05 \) to \( \chi = 0.10 \). Under higher demand uncertainty, unemployment and vacancies also increase. The main component behind the increase in unemployment is waiting unemployment, although search also becomes more attractive. The logic behind
these changes is that of the basic model. A decline in \( q(x) \) increases waiting unemployment and vacancies directly. Since workers search partly to “insure” against an uncertain demand, search unemployment also increases. In response to a shift in \( q(x) \), vacancies increase but the aggregate capital stock and total output decline. The decline in mean aggregate capital and output contrasts with the increase in these variables in specifications (1) and (2).

Panel B displays the microeconomic dispersion associated with the previous specifications. In all cases, the increase in mean unemployment and vacancies seen in Panel A are accompanied by an increase in the cross-sectional dispersion of these variables. These results are consistent with Proposition 4. Notice that the dispersion of output also increases in these specifications.

Overall, the labor market adjustments in specifications (1) to (3) are associated with “shifts” in the stationary values of unemployment and vacancies compared to the baseline case. There are obvious differences in terms of which type of unemployment contributes more to the increase in total unemployment, and differences in whether the shifts are pro- or counter-cyclical. However, a process of resource reallocation increases unemployment and vacancies. Since the number of workers searching and the number of vacant jobs increase, and since searchers and vacancies are in different locations, mismatch increases in this dimension. These reallocations are also associated with higher overall labor market dispersion, which is another dimension of mismatch.

Table 1 uses a directed search specification. Table 2 reproduces Table 1 under random search, as considered in Alvarez and Veracierto [4]. I maintain all parameter values as in the numerical simulations of Table 1. Under random search, the Euler equation (21) also holds but the arrival of workers in (22) is not directed. Instead, \( a(x) = \bar{a} \) for all \( x \in [0,1] \). The law of motion (18) and the feasibility condition (19) change accordingly.

There are two noticeable aspects in Table 2. First, search unemployment is higher under random search than under directed search. Notice, however, that there are no significant differences in the opportunity cost of search. The reason is that under directed or random search, the decision to leave a location is based on the same Euler equation,
The second noticeable feature is that random search causes capital and output to be considerably lower in Table 2 relative to Table 1. The reason is that random search induce more worker and capital mobility than directed search.

Table 2. Numerical simulations under random search.

<table>
<thead>
<tr>
<th>Alternative parameter values</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total unemployment rate</td>
<td>5.90</td>
<td>12.29</td>
<td>14.03</td>
</tr>
<tr>
<td>Search unemployment</td>
<td>3.21</td>
<td>8.99</td>
<td>11.06</td>
</tr>
<tr>
<td>Waiting unemployment</td>
<td>2.69</td>
<td>3.30</td>
<td>2.97</td>
</tr>
<tr>
<td>Vacancy rate</td>
<td>2.49</td>
<td>2.78</td>
<td>3.21</td>
</tr>
<tr>
<td>Cost of search</td>
<td>0.75</td>
<td>1.18</td>
<td>1.09</td>
</tr>
<tr>
<td>Capital per worker</td>
<td>0.39</td>
<td>0.56</td>
<td>0.59</td>
</tr>
<tr>
<td>Output per worker</td>
<td>0.08</td>
<td>0.11</td>
<td>0.11</td>
</tr>
<tr>
<td>A. Stationary mean values</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B. Cross-sectional variance</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note.— The mean values and the cross-sectional variances are constructed using the stationary distribution of workers. Random search is based on the specification in Alvarez and Veracierto [4]. The unemployment and vacancy rates are in percent.

Despite the previous differences, the labor market adjustments share the features seen in Table 1. The process of resource reallocation in specifications (1) to (3) increases unemployment and vacancies (e.g., the Beveridge curve shifts). Labor market dispersion also increases during these reallocations. Thus, the conclusions obtained under directed search appear to be robust to the alternative random search protocol.

4 Some implications of the theory

This section examines the implications of the theory for the medium-run behavior of aggregate unemployment and vacancies, and for the dispersion of labor market outcomes.
The Beveridge curve. I first examine the “medium-run Beveridge curve.” I use seasonally adjusted unemployment rates from the Bureau of Labor Statistics (BLS). To measure vacancies, I use the Conference Board help-wanted advertising index, the best available U.S. vacancy rate proxy prior to 2000. After December 2000, I use the job openings rate from the Job Openings and Labor Turnover Survey (JOLTS). The vacancy index is re-scaled using periods of common overlap.19

Table 3 estimates the trend components of unemployment and vacancies for alternative sample periods. Panel A starts in the first quarter of 1951 and ends in the last quarter of 1997. The help-wanted index has declined since the mid-2000s due to online job posting and the use of other recruitment methods. The sample in Panel A is not influenced by these changes in advertisement or by potential problems due to the merging procedure with JOLTS. Panel B ends in the last quarter of 2007 (before the 2007-2009 recession). Because filtering for medium-run trends is sensitive to end points, this sample period provides results that are not affected by the sharp rise in unemployment during the 2007-2009 recession. Panel C includes data up until the last quarter of 2010.

Table 3 decomposes unemployment and vacancies using the Hodrick-Prescott (HP) filter, the Baxter-King (BK) filter, and a third-degree polynomial in time. With a smoothing parameter of $\lambda = 0$, the HP filter returns the raw data. I focus on Panel B. These results are virtually identical to those in Panel C. The results in Panel A provide stronger support for the points discussed here.

Table 3 suggests two basic points: (i) The trend components of unemployment and vacancies are as volatile as the cyclical components. Under the HP filter with $\lambda = 10,000$, the standard deviation of the trend components is about 75 percent of the standard deviation of the raw series. These values are similar to those in the BK filter. Under $\lambda = 100,000$, the standard deviation of the trend component of unemployment is of the same order of magnitude as the standard deviation of the cyclical component. (A third-order polynomial in time yields similar results.) Under $\lambda = 150,000$, the standard deviations are also of similar magnitudes.

19 Unemployment rates are available in www.bls.gov. I would like to thank Ken Goldstein for making help-wanted advertising index available. JOLTS is also available from the BLS.
Table 3. The medium-run [short-run] Beveridge curve.

<table>
<thead>
<tr>
<th></th>
<th>Hodrick-Prescott filter with smoothing parameter $\lambda$ equal to</th>
<th>Baxter-King band-pass filter</th>
<th>Third-order polynomial in time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0$</td>
<td>$10,000$</td>
<td>$100,000$</td>
</tr>
<tr>
<td>Corr.($\ln \bar{U}_t, \ln \bar{V}_t$)</td>
<td>0.166</td>
<td>0.688</td>
<td>0.920</td>
</tr>
<tr>
<td>[−0.918]</td>
<td>[−0.900]</td>
<td>[−0.891]</td>
<td>[−0.944]</td>
</tr>
<tr>
<td>Std.Dev.($\ln \bar{U}_t$)</td>
<td>0.286</td>
<td>0.219</td>
<td>0.187</td>
</tr>
<tr>
<td>[0.153]</td>
<td>[0.190]</td>
<td>[0.197]</td>
<td>[0.119]</td>
</tr>
<tr>
<td>Std.Dev.($\ln \bar{V}_t$)</td>
<td>0.375</td>
<td>0.324</td>
<td>0.313</td>
</tr>
<tr>
<td>[0.169]</td>
<td>[0.190]</td>
<td>[0.194]</td>
<td>[0.137]</td>
</tr>
</tbody>
</table>

| B. Sample: 1951.I-2007.IV |
| Corr.($\ln \bar{U}_t, \ln \bar{V}_t$) | 0.091 | 0.625 | 0.851 | 0.873 | 0.436 | 0.896 |
| [−0.916] | [−0.900] | [−0.894] | [−0.942] | [−0.804] |
| Std.Dev.($\ln \bar{U}_t$) | 0.271 | 0.203 | 0.173 | 0.169 | 0.233 | 0.165 |
| [0.151] | [0.184] | [0.190] | [0.113] | [0.214] |
| Std.Dev.($\ln \bar{V}_t$) | 0.351 | 0.299 | 0.289 | 0.287 | 0.305 | 0.289 |
| [0.164] | [0.186] | [0.188] | [0.128] | [0.199] |

| C. Sample: 1951.I-2010.IV |
| Corr.($\ln \bar{U}_t, \ln \bar{V}_t$) | 0.016 | 0.511 | 0.790 | 0.837 | 0.309 | 0.812 |
| [−0.909] | [−0.878] | [−0.869] | [−0.941] | [−0.717] |
| Std.Dev.($\ln \bar{U}_t$) | 0.282 | 0.210 | 0.168 | 0.161 | 0.247 | 0.150 |
| [0.157] | [0.195] | [0.202] | [0.115] | [0.239] |
| Std.Dev.($\ln \bar{V}_t$) | 0.349 | 0.297 | 0.286 | 0.284 | 0.305 | 0.288 |
| [0.165] | [0.185] | [0.188] | [0.128] | [0.196] |

Note.— $\ln \bar{U}_t$ and $\ln \bar{V}_t$ denote the log of the trend component in quarterly unemployment and vacancies, respectively. In square brackets are the corresponding values for the cycle components, i.e., $\ln U_t - \ln \bar{U}_t$. A smoothing parameter of zero in the HP filter returns the unfiltered data. The BK filter passes through the time series components between 6 and 32 quarters with a window of 12 quarters.

(ii) Unemployment and vacancies are strongly negatively associated over the business cycle. In the medium-run, however, unemployment and vacancies are strongly positively associated. The correlation between the trend components with the HP filter and $\lambda = 10,000$ is 0.625. The correlation between trends depicted in Figure 1 (which uses $\lambda = 100,000$) is 0.851. This correlation mirrors the correlation between cyclical components, $-0.900$. The mirroring of the correlations is clearer with $\lambda = 150,000$; 0.873 versus
Panel A, provides an even sharper contrast of the correlations of medium-run trends and the correlation of the cyclical components.

The main implication of (i) is that a significant fraction of the volatility of unemployment and vacancies is due to fluctuations in their medium-run trends rather than to fluctuations in their business cycle components. Consistent with (i), the focus of the theory has been on stationary allocations and on how resource reallocations affect the stationary values of unemployment and vacancies.

In terms of (ii), notice that during a business cycle, unemployment and vacancies move along the Beveridge curve; see, e.g., Shimer [34]. However, the strong positive correlation between medium-run trends shows that the Beveridge curve itself tends to shift. Although (ii) is sensitive to the smoothing procedure, its main implication is that the medium-run Beveridge curve is positively sloped. This finding is important because a negatively sloped Beveridge curve has been used as evidence against the importance of labor market imbalances or mismatch; see, e.g., Abraham and Katz ([1], pp. 513-515), Schioppa ([33], pp. 17-19), and Petrongolo and Pissarides ([30], pp. 400).

The focus of the existing literature on unemployment and vacancies has been almost exclusively on business cycle frequencies. Empirically, (ii) suggests that resource reallocations have the greatest impact on labor markets at time scales longer than those of a business cycle. To explain a positively sloped medium-run Beveridge curve, the theory essentially posits the need to reallocate both capital and labor from declining firms. As Proposition 4 suggests for the basic model, and as illustrated in Tables 1 and 2, the complementarity between capital and labor implies that the stationary values of unemployment and vacancies increase due to reallocations driven by changes in productivity or by changes in demand uncertainty.

There are several difficulties with the measurement of unemployment and vacancies (e.g., the help-wanted index). Table 3 shows that removing recent observations, which are subject to changes in advertisement technologies, yields stronger results. Medoff [25]

---

20Blanchard and Diamond [7] is an important exception. They argue that half of the shifts in the Beveridge curve are due to reallocation shocks, and the other half are “due to an unexplained deterministic trend,” Blanchard and Diamond ([7], pp. 4-5). During the 1990s and until 2007, medium-run trends for unemployment and vacancies declined jointly. This decline is not consistent with a deterministic trend.
has shown that the shifts in the U.S. Beveridge curve in the 1970s are supported by cross-sectional evidence and by evidence based on discharges and quits. See also Abraham [2], Blanchard and Diamond [7], Bleakley and Fuhrer [9], and Daly et al. [14] for additional evidence on the shifts in the Beveridge curve. Further, the majority of Western European countries have experienced multiple shifts in the Beveridge curve during the post-war period. With the exception of Norway and Sweden, the Beveridge curve shifted to the right by a sizable amount from the 1960s to the early 1980s; see, e.g., Nickell et al. ([28], Figure 1). This shift is similar to that evident in Figure 1.

**Labor market dispersion.** The model implies that resource reallocations will not only increase unemployment and vacancies, but also lead to higher labor market dispersion. In this sub-section, I examine empirically the association between means and dispersion for unemployment and vacancies.

Figure 2 displays the unfiltered monthly unemployment rate and the standard devi-
The strong positive association between means and dispersion in Figure 2 is not limited to the dispersion across industries. Table 4 displays the correlations between the unemployment rate and several measures of labor market dispersion. Given the reduced availability of dispersion measures for vacancies, my main focus is on unemployment rates. Table 4 uses monthly data until December 2010. The values in square brackets are based on the sample that ends in December, 2007.

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21 The underlying data used for measures of dispersion are the monthly files of the Current Population Survey (CPS), available since January, 1976. Dispersion measures are calculated based on workers characteristics prior to the current unemployment spell. I would like to thank Rob Valletta and Katherine Kuang from the Federal Reserve Bank of San Francisco for making these data available; see Daly et al. [14] for additional information about these data.
The most salient feature of Table 4 is the strong positive correlation between aggregate unemployment and its cross-sectional dispersion across industries, occupations, and states. Table 4 also shows a positive correlation between all measures of unemployment dispersion. For example, Lilien’s [23] index of employment growth dispersion is positively correlated with the aggregate unemployment rate. (Lilien’s [23], however, index is negatively correlated with the unfiltered aggregate vacancy rate, as first documented by Abraham and Katz [1].) The fact that multiple measures of unemployment dispersion are positively associated suggests that labor market dispersion typically increases simultaneously along multiple dimensions.

Table 5. Correlation between mean vacancies and its dispersion.

<table>
<thead>
<tr>
<th></th>
<th>U.S. data</th>
<th>U.K. data</th>
</tr>
</thead>
<tbody>
<tr>
<td>ln $V_t$</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>ln $\sigma_V^I$</td>
<td>0.635</td>
<td>0.870</td>
</tr>
</tbody>
</table>

Groups 8 12
Obs. 120 255

Note.— (I) denotes Industries and (S) Regions. The dispersion of vacancies measures the standard deviation of the job openings rate in JOLTS. For U.K. data, the correlation is between the log-average number of unfilled vacancies across 12 regions (seasonally adjusted) and its standard deviation. This data begins in January 1980 and ends in April 2001 because of computer problems at the time. Data available from www.nomisweb.co.uk.

To examine the association between the mean and the dispersion of vacancies, Table 5 relies on JOLTS for the post December 2000 period. I examine the correlation of the monthly aggregate job vacancy rate and its dispersion across 8 industries. Table 5 also presents the correlation between the monthly aggregate and the regional dispersion of unfilled vacancies across 12 regions in the U.K., from January 1980 to April 2001. As in Table 4, Table 5 shows that cross-sectional measures of dispersion are strongly positively

22The correlation between the job-hiring rate in JOLTS and its cross-sectional dispersion across eight industries is 0.46. This is consistent with the predictions of the model, although I focused on the total number of job matches and not on new matches alone.
correlated with aggregate vacancies. Moreover, in the U.S., the trend component of 
vacancies, $\ln \bar{V}_t$, is positively correlated with the trend component of $\ln \sigma^p_U$, 
$\ln \sigma^p_{U^G}$, and $\ln \sigma^S_{U^G}$. (These estimates, available upon request, are not surprising 
given Tables 3 and 4.)

Overall, Tables 4 and 5 verify an important prediction of the model: both unemploy-
ment and vacancies covary positively with their cross-sectional dispersion. In the model, 
changes in the structure of the economy are sectorally unbalanced (e.g., Proposition 4 and 
Tables 1 and 2). In this fashion, the model rationalizes the positive correlation between 
mean unemployment and vacancies, and their cross-sectional dispersion.23

5 Some elaborations of the model

Congestion. It is possible to incorporate congestion in the market assignment. As in 
Lagos [21], assume that the technology for the basic model is Leontief, i.e., $f(k, l) = 
\min[k, l]$. A motivation behind it is that the short-side of the market is served: if there 
are $k(x) > l(x)$ jobs in $x$, $k(x) - l(x)$ jobs will not “meet” workers; these jobs will remain 
unused. The basic capital assignment problem is:

$$
F(K, \mu) \equiv \max_{k(x)} \left\{ \int_0^1 \varphi(x) \min[k(x), l(x)] dx : \int_0^1 k(x) dx \leq K \right\}.
$$

(24)

As in Lagos [21], assume that the market (not a social Planner) equalizes average 
productivities:

$$
(\varphi(x) \min \left[ 1, \frac{l(x)}{k^*(x)} \right] - r) k^*(x) = 0.
$$

(25)

If $\varphi(x) < r$, then $k^*(x) = 0$, because these locations do not cover the cost of capital. 
If $\varphi(x) > r$, then $\varphi(x)l(x)/k^*(x) = r$. These assignments are directed toward markets 
with higher values of $\varphi(x)l(x)$. Since $\varphi(x)/r > 1$, these sub-markets will have an excess 
supply of capital, i.e., $k^*(x) > l(x)$. If aggregate capital $K$ is small (in a sense made

23Changes in labor market dispersion are informative about changes in aggregate labor markets. In 
an Appendix not for publication, I measure empirically the contribution of labor market dispersion to 
movements along the short-run Beveridge curve, and to shifts in this curve. The analysis, based on 
partial correlations, finds that the main contribution of dispersion measures is in explaining changes in 
medium-run trends. Lilien’s [23] index, does not help explain the shifts in the aggregate Beveridge curve.
Proposition 6 The market assignment (25) exhibits an excess supply of capital in sub-markets with \( \varphi(x) > r \), and an excess supply of workers in markets with \( \varphi(x) < r \). If \( \min_{x \in [0,1]} \varphi(x) \geq r \), workers are in excess supply if \( rK < \int_0^1 \varphi(x)l(x)dx \).

In addition to demand uncertainty, captured by \( \varphi(x) \), unemployment and vacancies in this proposition are due to local shortages of capital and labor. In the Leontief case both \( r \) and \( K \) are free parameters. Thus, an excess supply of workers has two possible causes: either \( r \) is too high or \( K \) is too low, which is the only case Lagos ([21], Proposition 2) discussed.

Unemployment in the previous market assignment is inefficient due to a coordination problem. The optimal allocation is such that \( k^*(x) = l(x) \) in \( \varphi(x) \geq r \), given capital availability; see, e.g., Akerlof [3]. In the market assignment (25), however, \( k^*(x) > l(x) \) in \( \varphi(x) \geq r \). Agents do not take into account the marginal reduction in worker-job meeting rates imposed as more capital is assigned to productive sub-markets. Thus, too much capital is directed toward the most productive locations leaving workers in less productive locations unemployed. In this case, redirecting capital to the least productive locations will improve labor market conditions. Notice, however, that the excess supply of capital and labor in Proposition 6 requires a Leontief technology. In a CES function, \( f(k, l) = [\alpha k^a + \beta l^b]^{1/e} \), Proposition 6 applies if \( a \to -\infty \). If \( a > -\infty \), there will be no excesses of workers and jobs.\(^{24}\)

Rest unemployment. Other types of unemployment could be studied. Jovanovic [20], Gouge and King [17], and Alvarez and Shimer [5] have examined rest unemployment, in which workers have an “outside option” with a payoff \( b > 0 \). These authors consider this modification because in its absence, search unemployment would vary positively with improvements in productivity, as in specifications (1) and (2) in Tables 1 and 2.

\(^{24}\)When \( a > -\infty \), there will be no excess supply of capital. If \( -\infty < a < 0 \), \( f_k(0, l) = \alpha^{1/e} \). In this case, the first order condition is \( [\varphi(x)f_k(k^*(x),l(x)) - r]k^*(x) = 0 \). Thus, \( k^*(x) = 0 \) if \( \varphi(x)\alpha^{1/e} < r \). Workers in these locations may be considered unemployed. This type of unemployment was studied by Akerlof [3].
If workers value leisure or non-market activities such as rest, workers with potential wages below $b$ will rest. Workers with potential wages above $b$ may still become unemployed because resting and waiting are conceptually very different. The central difference between search, rest, and waiting is the degree of separation from the labor market. Searchers are in transition between labor markets. Individuals who rest are not in the labor market (or marginally attached to it), so they will not work in the current period. Workers who are waiting are in the labor market, and hence they are more likely to find employment than the other two types of workers. The data seems to support this prediction; see, e.g., Jones and Riddell [19].

**Aggregate uncertainty.** I focused on a stationary allocation without aggregate uncertainty. In an Appendix not for publication I examine a general case that allows the aggregate state to be uncertain. Following Miao [26], I establish the existence of a solution in a much more general case. Uniqueness, however, cannot be established in general. Numerical solutions, such as those described in Cooley ([13], chapter 4) for models of heterogeneous agents, could be considered to incorporate aggregate shocks to the quantitative analyses of the present model.

### 6 Conclusions

This paper constructed a general equilibrium model of mismatch. In the model, aggregate demand is uncertain. Unemployment and vacancies arise due to the inability of market participants to fully anticipate and respond to changes in demand. Workers with certain qualifications or in specific locations are unemployed because there is no demand for the commodities they produce. Workers cannot relocate or learn new skills instantaneously. Movements to a different sub-market are possible but, as in Lucas and Prescott [24], these movements take time and involve a period of (search) unemployment.

The model focused on the medium-run behavior of the labor market. In U.S. data, the volatility of the medium-run components of unemployment and vacancies is as important as the volatility of business cycle components, and unemployment and vacancies
tend to covary positively in the medium run. These features of the data suggest changes in the structure of the labor market that appear to shift the Beveridge curve. The theory accounts for these shifts as a consequence of capital and labor reallocations. The reallocation of capital and labor leads to a simultaneous increase in unemployment and vacancies due to their complementarity in production. These changes are sectorally unbalanced because demand uncertainty and productivity differ across sub-markets. Thus, episodes of high unemployment or high vacancies are accompanied by high cross-sectional dispersion. Mismatch increases in the sense of higher imbalances, e.g., higher dispersion in unemployment and vacancies across labor markets, but also in the sense that the number of unemployed workers and vacant jobs in different locations increases.

Although the model has many interesting theoretical and empirical implications, several simplifying assumptions were made. Cyclical fluctuations in unemployment and vacancies appear to be as important as secular changes. The paper, however, focused entirely on “steady state” comparisons. A natural next step would be to seek an integrated view of the labor market. I also abstracted from risk considerations. The all-or-nothing structure of production is a tractable assumption that yields simple predictions about higher moments of labor market outcomes not possible under more general assumptions. The model also focused on perfectly competitive sub-markets and assumed away matching frictions within each sub-market. The present framework, however, could be used for thinking about labor market policies that interfere with the creation of jobs and the reallocation of capital and labor.

References


Appendix A

Aggregate variances. Expression (12) ignores covariance terms in the variance of vacancies. In general,

\[ \mathbb{E}_{\omega} \left[ \left( \int_{0}^{1} [v_t(x, \omega) - v_t(x)] \, dx \right)^2 \right]. \quad (A1) \]

Upon expansion, (A1) yields

\[ \begin{align*}
Var_{\omega}[\tilde{V}_t(\omega)] &= \mathbb{E}_{\omega} \left[ \int_{0}^{1} [v_t(x, \omega) - v_t(x)]^2 \, dx \\
&\quad + 2 \int_{0}^{1} \int_{0}^{1} [v_t(x, \omega) - v_t(x)][v_t(x', \omega) - v_t(x') \, dx \, dx'] \right]. \quad (A2)
\end{align*} \]

The first term in (A2) is the aggregate over individual variances in (12). The second term depends on the covariance structure. To solve for this term, notice that under Bernoulli random variables this covariance becomes

\[ \Pr(\{\omega : v_t(x', \omega) - v_t(x')\}) = 1]Pr(\{\omega : v_t(x', \omega) - v_t(x')\}) = 1]v_t(x)v_t(x'). \]

Thus,

\[ \begin{align*}
Var_{\omega}[\tilde{V}_t(\omega)] &= \int_{0}^{1} Var_{\omega}[v_t(x, \omega)] + 2 \int_{0}^{1} \int_{0}^{1} \{(1 - q_t(x))k_t^*(x)\}^2 \{(1 - q_t(x'))k_t^*(x')\}^2 \, dx \, dx'.
\end{align*} \]

Since the covariance depends on quadratic terms for \((1 - q_t(x))\), which are likely to be small, it is reasonable to use the approximation (12) in the text. Notice that the approximation will be exact if the changes in demand across locations were independent.

The absence of aggregate uncertainty. In the dynamic model, I treat the aggregate state \((K_t, \mu_t)\) as a deterministic object despite the presence of location-specific productivity shocks. In here I show that the relevant state variables evolve deterministically.

Let the transition function for the shock \(z_t(x)\) be \(\Phi([a, b], z_t(x)) \equiv \Pr(\{\omega' : z_{t+1}(x, \omega') \in [a, b]|z_t(x)\})\), i.e., the probability of observing future productivity of location \(x\) in the interval \([a, b]\) given \(z_t(x)\). To be explicit about the role of idiosyncratic shocks, let \(l_t(z_t(x))\)
be the labor force in location $x$ at the beginning of period $t$. Let $\Omega$ represent the possible states of nature. At the end of period $t$, when searchers arrive, the mass of workers in $[a, b]$ is $\int_a^b l_{t+1}(z_t(x))dx$. The distribution of the labor force at the beginning of $t+1$, after shocks are realized, is

$$\tilde{\mu}_{t+1}([a, b], \omega') = \int_{\Omega} \int_a^b l_{t+1}(z_{t+1}(x, \omega'))\Phi(d\omega', z_t(x))dx.$$  \hspace{1cm} (A3)

This distribution is a random measure as it depends on the location-specific shocks. By assuming that individual shocks do not lead to aggregate uncertainty, in the sense considered by Bergin and Bernhardt [6], I can treat the distribution of the labor force as a deterministic state variable. For this assumption to hold, the distribution of individual shocks across locations, for a given $\omega'$, should be the same as the distribution over states $\Omega$ for any given location. This interpretation is in the spirit of the Law of Large Numbers for the continuum; see, e.g., Bergin and Bernhardt [6]. In this case, $\tilde{\mu}_{t+1}([a, b], \omega')$ in (A3) can be written as:

$$\mu_{t+1}([a, b]) = \int_0^1 l_{t+1}(z_t(x))\Phi([a, b], z_t(x))dx,$$  \hspace{1cm} (A4)

which is deterministic.

Although $\mu_t$ is deterministic, it has no “nice” properties. For example, more than one future distribution $\mu_{t+1}$ may be consistent with the current distribution $\mu_t$ (i.e., $\mu_{t+1}$ is not a contraction mapping). This problem is typical of dynamic economies with frictions and heterogeneous agents.

**Proof of Proposition 1 (Uniqueness).** The social planner solves

$$F_t(K_t, \mu_t) \equiv \max_{k_t(x), l_t(x)} \int_0^1 q_t(x)z_t(x)f(k_t(x), l_t(x))dx + r_t \left[ K_t - \int_0^1 k_t(x)dx \right].$$  \hspace{1cm} (A5)

To see that there is no other feasible assignment of capital that brings higher value note that concavity implies $f(\hat{k}_t(x), l_t(x)) - f(k^*_t(x), l_t(x)) \leq f_k(k^*_t(x), l_t(x))[\hat{k}_t(x) - k^*_t(x)]$. Assume that $\hat{k}_t(x)$ is another feasible assignment of capital. From the first order condition
(5) and the previous expression, it is possible to see that
\[ \int_0^1 q_t(x)z_t(x)[f(\hat{k}_t(x), l_t(x)) - f(k^*_t(x), l_t(x))]dx \leq r_t \int_0^1 [\hat{k}_t(x) - k^*_t(x)]dx = 0. \]

Thus, \( k^*_t(x) \) dominates any other feasible assignment. Notice from the Lagrangean (A5) that \( \partial F_t(K_t, \mu_t) / \partial K_t = r_t \), which follows due to the envelope theorem. ■

**Proof of Proposition 2.** For vacancies, algebra shows that
\[
\frac{\partial v_t(x)}{\partial q_t(x)} = \left[ 1 - q_t(x) \right] k^*_t(x) \left\{ \frac{f_k(k^*_t(x), l_t(x))}{-f_{kk}(k^*_t(x), l_t(x))k^*_t(x)} - \frac{q_t(x)}{1 - q_t(x)} \right\}, \text{ and}
\]
\[
\frac{\partial \text{Var}_w[v_t(x, \omega)]}{\partial q_t(x)} = \left[ 1 - q_t(x) \right] \left\{ k^*_t(x) \right\}^2 \left\{ \frac{f_k(k^*_t(x), l_t(x))}{-f_{kk}(k^*_t(x), l_t(x))k^*_t(x)} - \frac{2q_t(x) - 1}{2(1 - q_t(x))} \right\}. \]

If the elasticity of the marginal product of capital is bounded, the previous expressions will be negative, i.e., \(-f_{kk}(k, l)k/f_k(k, l) > (1 - q_t(x))/(q_t(x) - 1/2)\). Notice, for example, that if \( f(k, l) = k^{\alpha}l^\beta \), \(-f_{kk}(k, l)k/f_k(k, l) = 1 - \alpha \). A decline in \( q_t(x) \) will unambiguously increase mean vacancies if \( 1 - \alpha \geq (1 - q_t(x))/q_t(x) \). Dispersion will also increase if \( 1 - \alpha \geq (1 - q_t(x))/q_t(x) - 1/2 \). Since \( q_t(x) \) is likely to be close to one, these bounds are not tight. If \( f(k, l) = \min[k, l] \), mean vacancies and their dispersion will also increase unambiguously if \( q_t(x) \) declines. ■

**Proof of Proposition 3.** Take \( \lambda K_t \) and \( \lambda l_t(x) \) for all \( x \) and \( \lambda > 0 \). Expression (9) yields \( U_t(\lambda K_t, \lambda \mu_t) = \lambda U_t(K_t, \mu_t) \). Constant returns in \( f(k, l) \), (3), and (5) imply that \( r_t \) is unchanged and so (7), and (13) yield \( V_t(\lambda K_t, \lambda \mu_t) = \lambda V_t(K_t, \mu_t) \), and \( M_t(\lambda K_t, \lambda \mu_t) = \lambda M_t(K_t, \mu_t) \). ■

**Proof of Proposition 4.** For unemployment, \( \partial u_t(x)/\partial q_t(x) = -l_t(x) < 0 \), and \( \partial \text{Var}_w[u_t(x, \omega)]/\partial q_t(x) = (1 - 2q_t(x))(l_t(x))^2 \). Thus, \( U_t(K_t, \mu_t) \) and \( \int_0^1 Var_w[u_t(x, \omega)]dx \) change in the same direction for \( q_t(x) \geq 1/2 \), as assumed here. Aggregation over the previous effects yields a similar result for \( V_t(K_t, \mu_t) \) and \( \int_0^1 Var_w[v_t(x, \omega)]dx \). ■

**Proof of Proposition 5.** The proof is in three steps: (i) I show that the indirect production function in the dynamic assignment satisfies the assumptions in Lucas and Prescott [24]. (ii) I show that the stationary case corresponds to the case studied by these
authors. (iii) I show that the distribution of capital can be determined separately once
the stationary joint distribution of labor and productivity shocks is found.

(i) I first show that the production function satisfies assumptions (1) and (2) in Lucas
and Prescott [24]. Recall that \( g(\phi(x), l(x)) \equiv \max_{k(x)} \{ \phi(x)f(k(x), l(x)) - (\rho + \delta)k(x) \} \)
with \( \phi(x) \equiv q(x)\phi(x) \). The augmented shock \( \phi(x) \) inherits the properties of \( \phi(x) \) since
\( X_t(\omega) \) is independently drawn. This implies that \( \phi(x) \) has a unique stationary distribution.
The first order condition for \( f(\phi(x), l(x), \rho + \delta) \) be the function associated with the
optimal assignment of capital. Thus, \( \phi(x), l(x)) \equiv \phi(x)f(G(\phi(x), l(x), \rho + \delta), l(x)), \)
which only depends on \( \phi(x) \) and \( l(x) \).

For convenience suppress the index \( x \). First, the envelope theorem yields \( g_t(\phi, l) =
\phi f_t(G, l) > 0 \). Thus, \( g_{tt}(\phi, l) = \phi f_{tt}(G, l) + \phi f_{kl}(G, l)G_t, \) and \( g_{t\phi}(\phi, l) = f_t(G, l) +
\phi f_{kl}(G, l)G_\phi \). Further, \( G_t = -f_{kl}(G, l)/f_{kk}(G, l) \) and \( g_{tt}(\phi, l) = \phi [f_{tt}(G, l)f_{kl}(G, l) -\)
f_{kl}(G, l)\phi(f_{kl}(G, l)]/f_{kk}(G, l) \). The term in square brackets is positive (due to strict concavity in
\( f \)). Further, \( G_\phi > 0 \). Hence,

\[ g_t(\phi, l) < 0 \text{ and } g_{t\phi}(\phi, l) > 0. \] (A6)

Expression (A6) corresponds to assumption (1) in Lucas and Prescott [24]. Further, notice that \( \lim_{\phi \to 0} g_t(\phi, l) = 0 \). This corresponds to their assumption (2). Thus, the
indirect marginal product \( g_t(\phi, l) \) satisfies the assumptions of the marginal product in
Lucas and Prescott [24].

(ii) Next I show that in the stationary case, the worker assignment problem can be
represented as in Lucas and Prescott [24]. Let \( K^* \) be the stationary value of capital in
the dynamic problem. This value has to be determined as part of the solution (see step
(iii)). Consider the sequential worker assignment problem for periods \( \tau \geq T \) such that
\( r_\tau \approx (\rho + \delta) \) and \( K_\tau \approx K^* \) for all \( \tau \geq T \). This problem solves

\[ \max_{\{s_\tau(x), \omega_\tau(x)\}_{x, \tau \geq T}} \mathbb{E}_{\tau, \omega} \left\{ \sum_{\tau = T}^{\infty} (1 + \rho)^{-\tau} \left[ \int_{0}^{1} g(\phi_\tau(x), l_\tau(x) - s_\tau(x))dx - \delta K^* \right] \right\}, \]

A4
subject to (18) and (19). This problem is the analog of (15).

The previous problem can be studied using a “representative island” because the Lagrange multiplier associated with the feasibility constraint (19) is independent of $x$ and because the influence of the distribution of the labor force on $r_x$ has been suppressed. The analysis of a representative island yields the same results as those in Lucas and Prescott [24]. Their existence results and characterizations apply here.

(iii) The only aspect to be determined is the distribution of the capital assignments. Suppose that the shocks and the labor force are distributed across locations according to $\Psi(\varphi(x), l(x))$. Except for the case when the stationary allocation involves no worker search, the process $(\varphi(x), l(x))$ has a unique stationary distribution $\Psi^*$; see Lucas and Prescott ([24], Proposition 3) and Stokey, Lucas, and Prescott ([35], section 13.8).

Let $s^*(\varphi(x), l(x))$ be the optimal search decision given a state $(\varphi(x), l(x))$. Recall that $G(\varphi(x), l(x), \rho + \delta)$ is the optimal capital assignment given $\varphi(x)$ and $l(x)$, and an opportunity cost $\rho + \delta$. Then,

$$K^* = \int_0^1 G(\varphi(x), l(x) - s^*(\varphi(x), l(x)), \rho + \delta)\Psi^*(x)dx,$$

which uniquely determines the aggregate capital stock, $K^*$. ■

Proof of Proposition 6. From the market assignment rule (25) it follows that workers in locations $\varphi(x) < r$ will be in excess supply. These workers do not cover the opportunity cost of capital. In all other locations, $\varphi(x) \geq r$, (25) implies that $k^*(x) > l(x)$, and so capital will be in excess supply. The total capital assigned is

$$\int_{\varphi^{-1}(r)}^1 k^*(x)dx = \frac{1}{r} \int_{\varphi^{-1}(r)}^1 \varphi(x)l(x)dx. \quad (A7)$$

If $\min_{x \in [0,1]} \varphi(x) \geq r$, then $\varphi^{-1}(r) \to 0$. If the right-hand-side of (A7) is larger than $K$, capital will be rationed. ■
Measuring the importance of reallocational shocks. It is not possible to identify the causes of changes in labor market dispersion as being a rise in demand uncertainty or other (reallocational) shocks to the supply-side of the economy. Thus, the main objective of the following exercise is not to causally explain unemployment movements, but rather to see if the factors that influence labor market dispersion are quantitatively important for understanding aggregate unemployment and vacancies.

For that purpose, consider a reduced-form Beveridge curve:

$$\ln U_t = \psi \ln V_t + \ln \bar{U}_t + \varepsilon_t,$$

where $\psi$ is the slope of the short-run Beveridge curve and $\ln \bar{U}_t$ is the medium-run component, which must be estimated jointly with $\psi$. I consider a third-order polynomial in time represented by $\ln \bar{U}_t = \tau_1 t + \tau_2 t^2 + \tau_3 t^3$. This specification provides a simple but flexible method to estimate medium-run trends.\(^{25}\)

Consider next an alternative reduced-form Beveridge curve:

$$\ln U_t = \psi' \ln V_t + \ln \bar{U}'_t + \beta_\sigma \ln \sigma_U + \varepsilon'_t,$$

where the dispersion of unemployment, $\ln \sigma_U$, has been controlled for. Thus, (B2) estimates the short-run correlation between unemployment and vacancies as well as the medium-run trend of unemployment after removing the influence of variables that affect microeconomic dispersion. For example, if the variables that affect microeconomic dispersion bear no relationship with the variables that influence aggregate labor markets, then $\psi'$ should be equal to $\psi$. However, if aggregate unemployment, vacancies, and mi-

---

\(^{25}\)Alternative specifications such as a linear trend are less flexible, but available upon request. (The Hodrick-Prescott trend cannot be estimated jointly with $\psi$.) Notice that (B1) examines the Beveridge curve from the unemployment side. I examined similar estimates using vacancies as the dependent variable and controlling for the dispersion of vacancies, $\ln \sigma_V$. The sample in these estimates is limited, as JOLTS began in December, 2000. During that period, there were no major recessions.
croeconomic labor market dispersion are jointly determined, then $|\psi'| < |\psi|$ because part of the variation in vacancies and the medium-run trend will be captured by $\beta_\alpha$.

Table B1. Reduced-form Beveridge curve and the contribution of labor market dispersion.

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</thead>
<tbody>
<tr>
<td>$\ln V_t$</td>
<td>-0.37</td>
<td>-0.76</td>
<td>0.10</td>
<td>-0.28</td>
<td>-0.16</td>
<td>-0.47</td>
<td>-0.40</td>
<td>-0.62</td>
<td>-0.16</td>
<td>-0.76</td>
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<td>(0.04)</td>
<td>(0.01)</td>
<td>(0.01)</td>
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<td>(0.02)</td>
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<td>(0.01)</td>
<td>(0.04)</td>
<td>(0.01)</td>
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</tr>
<tr>
<td>$\ln \bar{U}_t</td>
<td>_{t=1}$</td>
<td>0.86</td>
<td>0.24</td>
<td>0.31</td>
<td>0.46</td>
<td>0.86</td>
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<tr>
<td>$R^2$</td>
<td>0.12</td>
<td>0.92</td>
<td>0.86</td>
<td>0.95</td>
<td>0.90</td>
<td>0.94</td>
<td>0.89</td>
<td>0.94</td>
<td>0.31</td>
<td>0.92</td>
</tr>
</tbody>
</table>


| $\ln V_t$                      | -0.46 | -0.79 | 0.14 | -0.24 | -0.15 | -0.40 | -0.36 | -0.59 | -0.24 | -0.78 |
| (0.03)                          | (0.01) | (0.01) | (0.03) | (0.01) | (0.02) | (0.01) | (0.01) | (0.04) | (0.01) |
| $\ln \bar{U}_t|_{t=1}$          | 1.02 | 0.22 | 0.30 | 0.48 | 1.01 |
| {0.00}                          | {0.00} | {0.00} | {0.00} | {0.00} | {0.00} |
| $R^2$                           | 0.21 | 0.89 | 0.87 | 0.95 | 0.91 | 0.94 | 0.90 | 0.94 | 0.36 | 0.89 |

Note.— Robust standard errors in parentheses. $\ln \bar{U}_t|_{t=1} = \tau_1 + \tau_2 + \tau_3$. In curly brackets is the $p$-value of an $F$-test on the significance of $\tau_{1,2,3}$. All samples begin in January, 1976.

The logic of partial correlations and common dependence implies that $(\psi - \psi')/\psi$ estimates the contribution of variables that influence microeconomic dispersion to the movements along the short-run Beveridge curve, whereas the ratio $(\ln \bar{U}_t|_{t=1} - \ln \bar{U}'_t|_{t=1})/\ln \bar{U}_t|_{t=1}$ estimates such contribution to the medium-run component. Table B1 presents the estimates of (B1) and (B2). I report the value of the point estimates for the medium-run trend at $t = 1$, e.g., $\ln \bar{U}_t|_{t=1}$. Their significance is assessed with a joint $F$-test whose $p$-values are reported in curly brackets. Despite slight differences in the estimates, the conclusions that arise prior to the 2007-2009 recession are robust.
Specification (1) yields a point estimate for $\psi$ of $-0.37$ and an $R^2$ of $0.12$. Specification (2) yields $\psi = -0.76$ and the $R^2$ is $0.92$. These estimates show that vacancies account only for a small proportion of the variability of unemployment (in the $R^2$ sense), and that the correlation between unemployment and vacancies is stronger in the short run than in the medium run. The most useful contrast in Table A1 is between (2) and specifications (4), (6), and (8). Comparing (2) and (4) shows that factors affecting unemployment dispersion across industries contribute 63 percent ($= 1 - 0.28/0.76$) to the movements along the Beveridge curve and 72 percent ($= 1 - 0.24/0.86$) to the medium-run trends. The point estimates in (6) yield lower contributions to movements along the curve, 47 percent, and to changes in the trend, 63 percent. The same contributions in (8) are 20 and 40 percent. The contribution of Lillien’s [23] index of dispersion in (10) is zero.

Overall, the cross-sectional dispersion of unemployment helps explain about half of the movements along the short-run Beveridge curve and about 60 percent of the medium-run changes in aggregate unemployment. Thus, Table B1 suggests that the factors that influence microeconomic dispersion are likely to play a central role in the aggregate labor market.

A competitive market representation of the dynamic assignment. Here I sketch the equivalence between a competitive market allocation and the social Planner allocation studied in the text. Prescott and Rios-Rull [3] have extended Arrow-Debreu classical competitive analysis to economies with “islands” such as the one studied here.

The equivalence just mentioned is perhaps not surprising because worker search is appropriately priced by $\theta_t$. In a competitive market, which can be defined along the same lines of the basic model, the representative household takes prices $\{w_t(x), r_t, \theta_t\}_{x,t}$ as given and maximizes the present discounted value of expected aggregate consumption with respect to $\{s_t(x), a_t(x), K_{t+1}(\omega^t)\}_{x,t}$, subject to:

$$C_t(\omega^t) + K_{t+1}(\omega^t) \leq (1 + r_t(\omega^{t-1}) - \delta)K_t(\omega^{t-1}) + \int_0^1 \{w_t(x) [l_t(x) - s_t(x)]\} dx + \Pi_t(\omega^t),$$

(17) and (18) for all $t \geq 0$. Firms also take prices as given. Profits are a residual and
given by $\Pi_t(\omega^t)$. Static capital assignments are analogous to the ones studied in the basic model. Finally, market clearing in each period takes place as before. In addition, the total number of searchers must equal the total number of arrivals, i.e., (19). The equivalence between a competitive market and the (constrained) social Planner assignment can be established along familiar lines: the Euler equations, static optimality conditions, and feasibility conditions are the same in both approaches.

**Details on the equivalence to Lucas-Prescott.** Consider the dynamic assignment problem for periods $\tau \geq T$ such that $r_\tau \simeq (\rho + \delta)$ and $K_\tau \simeq K^*$ for all $\tau \geq T$. This problem solves

$$
\max_{\{s_\tau(x), a_\tau(x)\}_{x, \tau \geq T}} \mathbb{E}_{t, \omega} \left\{ \sum_{\tau = T}^{\infty} (1 + \rho)^{-\tau} \left[ \int_0^1 g(\varphi_\tau(x), l_\tau(x) - s_\tau(x))dx - \delta K^* \right] \right\},
$$

subject to (18) and (19).

To see the equivalence suppress the index $x$. The functional equation for this problem is:

$$
W_\tau(\varphi, l) = \max_{n, \omega} \{g(\varphi, l - s) + (1 + \rho)^{-1}\mathbb{E}_\omega[W(\varphi', l')]\}, \text{ subject to (18) and (19).}
$$

The first order conditions are:

$$
\{\theta - g_l(\varphi, l - s^*) - (1 + \rho)^{-1}\mathbb{E}_\omega[W_l(\varphi', l')]s^* = 0, \{(1 + \rho)^{-1}\mathbb{E}_\omega[W_l(\varphi', l')] - \theta\}a^* = 0. \text{ They are analogous to cases (i)-(iii) studied in the text.}
$$

The envelope condition is $W_l(\varphi, l) = g_l(\varphi, l - s^*) + (1 + \rho)^{-1}\mathbb{E}_\omega[W_l(\varphi', l')]$. Associated with the previous first order and envelope conditions there is a functional equation for $W_l(\varphi, l)$ which is

$$
W_l(\varphi, l) = \max\{\theta, g_l(\varphi, l) + (1 + \rho)^{-1}\mathbb{E}_\omega[W_l(\varphi', l')]\}.
$$

For instance, in case (i), $W_l(\varphi, l) = \theta$. In case (ii), $W_l(\varphi, l) = g_l(\varphi, l) + (1 + \rho)^{-1}\mathbb{E}_\omega[W_l(\varphi', l')]$. In case (iii), $W_l(\varphi, l) = g_l(\varphi, l) + \theta$. The recursive representation above is that of Lucas and Prescott [24]. This functional equation has a unique solution which is nondecreasing in $\varphi$ and nonincreasing in $l$, see, e.g., Lucas and Prescott ([24], Proposition 1).

To obtain the stationary distribution of the labor force, let $l_{\tau+1}^*(\varphi_\tau(x), l_\tau(x))$ be the
labor force at the end of the period when searchers arrive. This variable depends on the augmented productivity shock and the initial population. Let \( \mathbb{I}\{\cdot\} \) represent the indicator function. Let \( \Gamma(\cdot) \) be the transition function that describes the law of motion of the state.\(^{26}\) Transitions from \( (\varphi(x), l(x)) \) are described by \( \Gamma(\varphi'(x), l'(x)|\varphi(x), l(x)) \equiv \text{Pr}(\{\omega : \varphi_{\tau+1}(x, \omega) \leq \varphi'(x), l_{\tau+1}(\varphi, x, l_{\tau}) \leq l'(x)|\varphi_{\tau}(x) = \varphi(x), l_{\tau}(x) = l(x)\}) \), which corresponds to \( \Gamma(\varphi'(x), l'(x)|\varphi(x), l(x)) = \Phi([0, \varphi'(x)], \varphi(x))\mathbb{I}\{l_{\tau+1}(\varphi(x), l(x)) \in [0, l'(x)]\}\).

The transition function defines an operator \( P \) on distribution functions: suppose that the productivity shocks and the labor force are distributed across locations according to \( \Psi(\varphi(x), l(x)) \). Then, the joint distribution of these variables in the next period is \( P(\Psi(\varphi'(x), l'(x)) = \int_{0}^{1} \Gamma(\varphi'(x), l'(x)|\varphi(x), l(x))\Psi(\varphi(x), l(x))dx \). Except for the case when the stationary allocation involves no worker search, the process \((\varphi(x), l(x))\) has a unique stationary distribution \( \Psi^* \) which is the solution to \( P(\Psi^*) = \Psi^* \) see, e.g., Lucas and Prescott ([24], Proposition 3) and Stokey, Lucas, and Prescott ([35], section 13.8). This shows that there is a unique stationary distribution of the labor force derived from \( \Psi^*(\varphi(x), l(x)) \). That is, \( \mu^*([a, b]) = \int_{a}^{b} l(x)\Psi^*(x)dx \).

**An economy with aggregate productivity shocks.** Here I examine the dynamic assignment problem under aggregate productivity shocks. It is necessary to treat the distribution of jobs as a state variable, see, e.g., Kubler and Schmedders [2] and Miao [26]. The analysis differs from the deterministic case because uniqueness of stationary solutions cannot be established. Thus, I study non-stationary solutions. The notation here follows closely the main text.

Aggregate productivity shocks \( Z_t \) are defined on a countable space \((\mathbb{Z}, \mathcal{Z})\). Let \( Z^t = (Z_t, Z^{t-1}) \) be a history in \( \mathbb{Z}^t = \mathbb{Z} \times \mathbb{Z}^{t-1} \). The value of \( Z_0 \) is given. Aggregate shocks are a Markov process with transition function, \( Q : \mathbb{Z} \times \mathbb{Z} \rightarrow [0, 1] \). The transition function has the Feller property as defined by Stokey, Lucas, and Prescott ([35], chap. 12). Let \( \mathbb{P}([0, 1], \mathcal{B}) \) be the set of probability measures on \([0, 1]\) and define \( \mathbb{S} \equiv \mathbb{K} \times \mathbb{Z} \).

\(^{26}\)The transition \( \Gamma \) does not depend on the aggregate capital stock \( K_{\tau} \) and the distribution of the labor force \( \mu_{\tau} \). Otherwise, the transition of the state would have to incorporate their law of motion. Once these (deterministic) variables are included, there is no general uniqueness result such as the one discussed here. Existence can be established in more general cases such as in the case of aggregate productivity shocks. This case is discussed immediately below in this Appendix.
The state in this economy is the aggregate capital stock, the aggregate productivity shock, and the distribution of the labor force across locations. The state is a pair \((S_t, \mu_t)\) that lies in \(S^t \times \mathbb{P}([0,1])\). The capital assignment to \(x\) is a function \(k_t(x, S^t, \mu_t)\) with \(k_t : [0,1] \times S^t \times \mathbb{P}([0,1]) \to \mathbb{R}_+\). Similar notation defines the assignment of workers.

Given \(S^t\) and \(\mu_t(S^{t-1}, \mu_{t-1})\), capital, consumption and aggregate production are feasible if

\[
\mathcal{E}_t(S^t, \mu_t) + \mathcal{L}_{t+1}(Z^{t+1}, \mu_{t+1}) - (1-\delta)K_t(Z^{t-1}, \mu_{t-1}) \leq Z_t F_t(S^t, \mu_t).
\]

The objective of the problem is to maximize the present expected value of aggregate consumption subject to the feasibility conditions in the text: (3), (17), (18), and (19).

**Proposition 7** There exists a solution to the dynamic assignment problem with aggregate productivity shocks.

**Proof.** The proof of existence is constructive and it describes the solution and its properties in two steps. The first step conditions on a sequence of aggregate distributions \(\{\mu\}\) because, unlike Lucas and Prescott [24], decisions in period \(t\) depend on the expectation of the distribution of the labor force in \(t + 1\). In the second step, I study the consistency requirements of the assignment that make this conditioning valid. This approach follows Bergin and Bernhardt [6] and Miao [26].

The first step has three parts: (i-a) I examine capital accumulation decisions, (i-b) I examine search decisions, and (i-c) I derive the law of motion for the distribution of the labor force.

(i-a) For a given \(\{\mu\}\) and a history \(S^t \in S^t\), capital accumulation decisions satisfy the following Euler equation:

\[
\rho + \delta = \int_{S^{t+1}} r_t(S^{t+1}; \mu_{t+1}) P_{\mu}(S^t, dS_t + 1),
\]

with \(P_{\mu} : (\mathbb{S}, \mathbb{S}) \to [0,1]\) as the transition function for \(S_t = (K_t, Z_t)\). Capital accumulation decisions are continuous and the transition function \(P_{\mu}(S^t, dS_{t+1})\) has the Feller property. Since I have conditioned the decisions on \(\{\mu\}\), the construction of the transition function for \(S_t\) is standard, see Stokey, Lucas, and Prescott ([35], Theorem 9.13). The Feller property follows from Stokey, Lucas, and Prescott ([35], Theorem 9.14).
(i-b) Search decisions also satisfy a series of Euler equations as those in the text. For a given \{\mu\} and a history \(S^t \in S^t\), there exists a continuous worker assignment \(\{s_t(x, S^t, \mu_t, a_t(x, S^t, \mu_t)\}_{x \in [0,1]}\) that satisfies the following Euler equations:

\[
\begin{align*}
\left\{ w_t(z_t(x), S^t, \mu_t) + \mathbb{E}_{t, \omega'} \left[ \frac{w_{t+1}(z_{t+1}(x, \omega'), S^{t+1}, \mu_{t+1})}{1 + \rho} \right] - \theta_t(S^t, \mu_t) \right\} s_t(x, S^t, \mu_t) &= 0, \\
\left\{ \theta_t(S^t, \mu_t) - \mathbb{E}_{t, \omega'} \left[ \frac{w_{t+1}(z_{t+1}(x, \omega'), S^{t+1}, \mu_{t+1})}{1 + \rho} \right] \right\} a_t(x, S^t, \mu_t) &= 0.
\end{align*}
\]

Expectations are given by:

\[
\mathbb{E}_{t, \omega'}[H(z_{t+1}(x, \omega'), S^{t+1}, \mu_{t+1})] \equiv \int_{\Omega} \int_{S^{t+1}} H(z_{t+1}(x, \omega'), S^{t+1}, \mu_{t+1}) P_\mu(S^t, dS_{t+1}) \Phi(x, d\omega'),
\]

for an arbitrary function \(H(z_{t+1}(x, \omega'), S^{t+1}, \mu_{t+1})\). These expectations are defined over location-specific and aggregate shocks because the distribution of the labor force has been conditioned upon. The interpretation of the previous Euler equations is similar to that in the text.

(i-c) Let \{\mu\} represent the sequence \{\mu_t(S^t, \mu_{t-1}; S_0, \mu_0)\}. Recall that \(\mu_t\) is a function from \(S^t \times \mathbb{P}([0,1])\) to \(\mathbb{P}([0,1])\) and let \(\mathbb{P}([0,1])^{S^t}\) denote the set of such functions. Let \(\mathbb{P}^\infty([0,1]) \equiv \times_{t=1}^\infty \mathbb{P}([0,1])^{S^t}\). Notice that \{\mu\} lies in \(\mathbb{P}^\infty([0,1])\). At the beginning of period \(t + 1\), distribution of the labor force is:

\[
\mu_{t+1}([a, b], S^t, \mu_t) = \int_{\Omega} \int_{a}^{b} l_{t+1}(z_{t+1}(x, \omega'), S^t, \mu_t) \Phi(x, d\omega') dx,
\]

where the second line follows by the no aggregate uncertainty condition described in Bergin and Bernhardt [6]. Expression (B3) is deterministic conditional on a history \(S^t\). This completes the derivation of the evolution of the state of the economy.

(ii) A solution of the dynamic assignment problem requires consistency between decisions that conditioned on \{\mu\} and the sequence of distributions \{\mu\} from (B3). Since
equation (B3) defines a mapping $T : \mathbb{P}^\infty([0, 1]) \to \mathbb{P}^\infty([0, 1])$, a topological fixed point $\mu^* = T(\mu^*)$ is a solution to the assignment problem.

The rest of the proof follows the arguments in the proof of Miao ([26], Theorem 1) and applies Brouwer-Schauder-Tychonoff Fixed-Point Theorem to $T$. One needs to show that the domain of $T$ is compact, that $T$ maps $\mathbb{P}^\infty([0, 1])$ into itself, and that $T$ is continuous. Because $Z_t F_t(S^t, \mu_t)$ acts as a typical aggregate production function, the aggregate capital is uniformly bounded or there is a maximum sustainable stock of capital. To see this, consider a policy in which consumptions are always zero. The condition that $\lim_{k \to \infty} f_k(k, l) + 1 - \delta < 1$ implies that there is a $K < +\infty$ that uniformly bounds $K_t$.

The space of aggregate shocks $Z$ is also bounded and compact by assumption. This makes $S$ bounded and compact. Because the space $[0, 1]$ is compact, $\mathbb{P}([0, 1])$ is also compact, see, e.g., Aliprantis and Border ([1], Theorem 15.11). Finally, recall that $\{\mu\}$ is a function from $\mathbb{P}^\infty([0, 1])$. Endowed with the product (or pointwise convergence) topology, $\mathbb{P}^\infty([0, 1])$ and the set $S^\infty \times \mathbb{P}^\infty([0, 1])$ are compact, see, e.g., Aliprantis and Border ([1], Theorem 2.69).

Because feasibility in worker search is such that searchers arrive before the end of the period, $T(\mu)$ lies in $\mathbb{P}^\infty([0, 1])$. That is, the mapping $T$ returns a distribution function in $\mathbb{P}^\infty([0, 1])$. Finally, $T$ is continuous because the search decisions and job assignments are continuous. This establishes the existence of a solution to this general problem. ■

References

