Random Choice and Market Demand

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Abstract

I characterize mean (or market) demands derived from a general random choice model that does not require the use of preferences or maximizing behavior. I show that mean demands satisfy the compensated *Law of Demand* and all the properties of standard demand theory including symmetry and negative semi-definitiveness of the Slutsky matrix. The framework accommodates as special cases Gary Becker's (1962) model of irrational behavior and boundedly rational random choice models formulated in decision theory.

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1 Introduction

Gary Becker's classic and influential study of "irrational behavior" examined the market implications of a model in which individuals randomly choose among competing commodities subject to a linear budget constraint. A remarkable and surprising result is that, in response to a compensated own-price change, individuals on average find themselves satisfying the *Law of Demand*. Becker's (1962) example provided the first persuasive account of how traditional market demand predictions could be obtained under non-rational decision-making.

The framework in Becker (1962), however, is restrictive and not able to integrate realistic models of non-rational behavior. Existing results consider random choices drawn from a uniform distribution, for individuals who satisfy non-satiation, and for a two-commodity case. It is not at all obvious that random choice would yield well-behaved mean (or market) demand functions without these assumptions.[1] Moreover, the available framework does not easily fit into the mainstream probabilistic theories of choice behavior. This literature formulates descriptive random choice models to explain systematic violations of rational behavior; see, e.g., Gonzalez-Vallejo (2002), Luce (1959), Roe et al. (2001), and Tversky (1969, 1972).

This paper takes a fresh look at the market implications of random choice models using a new set of analytical tools based on the statistical analysis of truncations. I characterize mean demand functions without the distributional assumptions employed in the literature or the assumption of non-satiation, and for any general number of commodities. I show that the *Law of Demand* is more robust than previously considered in the literature. In particular, I show that random choice models satisfy a compensated *Law of Demand* for a general probability distribution function, for demands that lie in the interior of the budget

[1] It is well known that the use of two commodities yields symmetry and integrability results that do not generalize to more than two commodities. There are also examples of aggregate properties that follow from uniform distributions that are not obtained under alternative distributions. A famous example by Caplin and Spulber (1986) showed that individual price rigidity may be consistent with aggregate price flexibility. This result requires a uniform distribution of price-setting firms. Finally, non-satiation (e.g., the assumption that "more is preferred to less") is responsible for downward sloping indifference curves, for the direction of increasing utility in the indifference map, and for homogeneity and adding up conditions in utility maximizing models.

set, and for an arbitrary number of commodities.

I also show that mean demands are not only in agreement with the compensated *Law of Demand*, but also with all the propositions of standard demand theory. Surprisingly, random choice is enough to obtain the symmetry and negative semi-definiteness of the Slutsky matrix of substitution effects. A symmetric and negative semi-definite Slutsky matrix implies that mean demands under random choice are *observationally equivalent* to those based on the behavior of a "representative, rational agent." This equivalence shows that all the propositions of demand theory remain valid under fundamentally different assumptions about the individual decision-making process. It also suggests that alternative (and potentially more realistic) assumptions about individual decision-making do not necessarily yield a better understanding of aggregate behavior.[2]

Probabilistic choice models are of two general forms. "Random preference" models where preferences are selected randomly and choices are made to 'maximize' the randomly chosen preferences, and "random choice" models where choices itself are random variables; see, e.g., McFadden and Richter (1990, Section 3). The theorems presented here apply to mean demands from a general "random choice" model that does not require, but can easily accommodate the use of preferences or maximizing behavior. I do not attempt to stochastically rationalize the "observed" choice probabilities as rationality is not needed here. "Random choice" models are more general than "random preference" models because random choices are not necessarily stochastically rational; see, e.g., McFadden and Richter (1990, pp. 168-169).[3]

Random choices here rely on a general probability distribution function. However, I maintain the classical requirement of consumer sovereignty. I assume that the probability

[2] As stressed by Becker (1962), this result is a consequence of aggregation. This view of aggregation contrasts with the Sonnenschein-Mantel-Debreu Theorem. This well-known indeterminacy result shows that aggregation does not necessarily preserve individual rationality. The approach presented here provides examples under which aggregation rationalizes non-rational (e.g., irrational or boundedly rational) individual behavior.

[3] The relationship between "random preference" models and "random choice" models is the subject of *revealed stochastic preference theory*; see, e.g., Bandyopadhyay et al. (1999), Falmagne (1978), Gul and Pesendorfer (2006), McFadden and Richter (1990), and McFadden (2005). The goal of this literature is to identify the properties that "observed" choice probabilities must satisfy in order to guarantee that random choices are rationalized by some random preference and maximizing behavior. As in its deterministic counterpart, this approach makes direct assumptions on the choice behavior of individuals.

distribution function under which choices are generated is invariant with respect to the budget set. This assumption is the analog of assuming that tastes are invariant to the economic environment. Mean demands and "observed" choice probabilities at the market level, nonetheless, will necessarily vary with prices and income. With this framework, it is possible to analyze the market implications of many compelling choice theoretic behaviors, including certain aspects of *boundedly rational* behaviors. I list several examples based on prominent random choice models formulated by decision theorists.

Related literature. "Random choice" models originated in mathematical psychology. These models formulate axiomatic rules or choice heuristics based on reasonable restrictions on choice probabilities; see, e.g., Gonzalez-Vallejo (2002), Luce (1959), Roe et al. (2001), and Tversky (1969, 1972). These models, however, do not typically recognize the importance of budget constraints in determining choice behavior.

There is a large literature that studies "random preference" models in economics with a special emphasis on econometric applications; see, e.g., Anderson et al. (1992), Gul and Pesendorfer (2006), and McFadden (2005). These models typically rely on an *indirect utility function* that subsumes the direct maximization of a well-defined utility function. Indirect utilities vary with prices and income, and individuals are subject to optimization errors in the form of a random utility component. The approach in this paper is direct with respect to the budget constraint. Its main advantage is that there is no need to assume maximizing behavior at any stage of the decision process.

In the spirit of Becker (1962), Grandmont (1992) derived through aggregation mean demands with positive income effects, e.g., mean demands of the Cobb-Douglas type; see also Hildenbrand (1994, Chapter 2) and Kneip (1999). In Becker (1962), income effects are positive due to the use of a uniform distribution. This result does not generalize to all distribution functions. I do not examine the general equilibrium implications of random choice, but I provide the restrictions needed for positive income effects.[4]

Sanderson (1974, 1980) appear to be the only papers that relax some of the assump-

[4] Using a double-auction market and the uniform distribution, Gode and Sunder (1993, 1997) documented a high allocative efficiency of markets under random choice; see also Duffy (2006). Given the importance of positive income effects for the stability of equilibrium, it seems likely that their findings would not generalize to random choices consistent with negative income effects.

tions in Becker (1962).[5] He showed that an income increase leads to a first-order stochastic shift in the distribution of consumption opportunities. These studies, however, did not associate this shift with a statistical truncation or examined the implications of such shift. I show that the restrictions assumed by Sanderson (1974, 1980) are not needed *a priori* but arise due to the way feasible opportunities are truncated. I also provide integrability results not previously available and allow for non-satiation. The analysis of the Slutsky matrix is also novel.

The rest of the paper is organized as follows. Section 2 examines two-commodities and non-interior demands. Section 3 presents the general theorems. Section 4 presents several examples of random choice models. Section 5 concludes.

2 A simple example

It is useful to start with the case of two commodities. Individuals are "irrational" in the sense that there are no preferences or maximizing behavior. Later, I provide a general analysis for L commodities and discuss preferential choice models.

Let X_1 and X_2 denote the two commodities. There is a continuum of individuals who randomly choose their consumption. Individuals must divide their income w in a way such that their budget constraint is satisfied:

$$p_1 X_1 + p_2 X_2 \leq w, \tag{1}$$

subject to $X_1 \geq 0$, $X_2 \geq 0$, and with p_1 and p_2 as the given prices of X_1 and X_2. Income and prices are strictly positive and finite.

As in Becker (1962), only X_1 is randomly chosen over a range limited by the maximum amount that can be bought. Once X_1 is determined, the remaining income is spent in X_2. Throughout this section, X_1 is a random variable with a well-behaved distribution function $F(x_1)$ defined over $x_1 \in \mathbb{R}_+$. The density function is $f(x_1)$. First and second

[5] An application of random choice that originated in Bronars (1987) has been to use mean demands based on the uniform distribution in order to derive the *statistical power* for tests of revealed preferences. Beyond convenience, there is no reason for the use of uniform distributions. I discuss this point below.

moments of all random variables are assumed finite and analytical problems concerning results that hold on sets of measure zero are neglected throughout.

Let $\mathbf{X}_1^{\max} \equiv w/p_1$ represent the maximum quantity of X_1 that can be purchased given w and p_1. Choices of X_1 are subject to $0 \leq X_1 \leq \mathbf{X}_1^{\max}$. Let $\bar{x}_1(p,w)$ denote the mean demand for X_1 for a given value of prices $p = (p_1, p_2)$ and income w. The mean demand for X_1 is given by a *right-truncated mean* formula:

$$\bar{x}_1(p,w) \equiv \mathbb{E}\left[X_1 \middle| 0 \leq X_1 \leq w/p_1\right] = \frac{\displaystyle\int_0^{\mathbf{X}_1^{\max}} x_1 f(x_1) dx_1}{F(w/p_1)}. \tag{2}$$

This formula represents the conditional expectation of X_1 truncated by the fact that X_1 cannot take values above w/p_1, e.g., $F(w/p_1) \equiv \Pr[0 \leq X_1 \leq w/p_1]$.

As in utility maximizing models, the assumption of consumer sovereignty implies that $\bar{x}_1(p,w)$ depends on (p,w) due only to the limitations imposed by (1), e.g., $f(x_1)$ is independent of (p,w) as utility functions are typically assumed independent of (p,w). Changes in the function $f(x_1)$ can be introduced by assuming that these distributions are indexed by prices or income. Indexing $f(x_1)$ by (p,w) would be analogous to assuming that the direct utility function depends on prices or income.

Let $\bar{x}_2(p,w)$ denote the mean demand for X_2. X_2 satisfies $p_2 X_2 = w - p_1 X_1$ as long as $X_1 \leq w/p_1$. This yields:

$$\bar{x}_2(p,w) \equiv \frac{w - \bar{x}_1(p,w)p_1}{p_2}. \tag{3}$$

The Law of Demand. When p_1 changes, the standard Slutsky compensation implies that income varies by

$$\left.\frac{\partial w}{\partial p_1}\right|_{\bar{w}} = \bar{x}_1(p,w), \tag{4}$$

where $\bar{w}$ in (4) represents the income level associated with the compensated changes.

Using (4), I first show that

$$\left.\frac{\partial \bar{x}_1(p,w)}{\partial p_1}\right|_{\bar{w}} < 0. \tag{5}$$

The relevant derivative needed to evaluate (5) is:

$$\left.\frac{\partial \bar{x}_1(p,w)}{\partial p_1}\right|_{\bar{w}} = \frac{\partial \mathbb{E}\left[X_1 \middle| 0 \leq X_1 \leq \mathbf{X}_1^{\max}\right]}{\partial \mathbf{X}_1^{\max}} \left(\left.\frac{\partial \mathbf{X}_1^{\max}}{\partial p_1}\right|_{\bar{w}}\right). \tag{6}$$

The first term represents the effect of changes in $\mathbf{X}_1^{\max}$ on the mean demand and the second is the effect of a compensated change in p_1 on $\mathbf{X}_1^{\max}$. This first term satisfies:

$$\frac{\partial \mathbb{E}\left[X_1 \middle| 0 \leq X_1 \leq \mathbf{X}_1^{\max}\right]}{\partial \mathbf{X}_1^{\max}} = \frac{f(\mathbf{X}_1^{\max})}{F(\mathbf{X}_1^{\max})}\left(\mathbf{X}_1^{\max} - \bar{x}_1(p,w)\right) > 0, \tag{7}$$

which follows by a standard application of Leibnitz's rule for differentiation under the integral sign in (2). With $\Pr(X_1 < \mathbf{X}_1^{\max}) > 0$, it is clear that the previous expression is positive. From (1) and (4), the second term in (6) is:

$$\left.\frac{\partial \mathbf{X}_1^{\max}}{\partial p_1}\right|_{\bar{w}} = -\frac{\mathbf{X}_1^{\max} - \bar{x}_1(p,w)}{p_1} < 0. \tag{8}$$

(6), (7) and (8) yield the following proposition:

Proposition 1 *Suppose individuals randomly choose X_1 using a probability distribution function $F(x_1)$. Suppose further that X_2 is a residual. Then, the compensated Law of Demand holds for $\bar{x}_1(p,w)$ and $\bar{x}_2(p,w)$.*

The intuition behind Proposition 1 is really simple. The compensated *Law of Demand* holds due to an economic fact and a statistical fact. First, the maximum amount of X_1 that can be purchased declines as p_1 increases. Second, because the truncated mean $\bar{x}_1(p,w)$ is less than the truncation point $\mathbf{X}_1^{\max}$, the right-truncated mean increases as the point of truncation increases; see (7).

Proposition 1 is described graphically in Figure 1. The fact that some values of X_1 are not feasible truncates the distribution of X_1 at $\mathbf{X}_1^{\max}$. The budget constraint (1) implies that the dark-shaded area to the right of $\mathbf{X}_1^{\max}$ is not feasible at (p_1, p_2, w). The mean demand for X_1 in this case is $\bar{x}_1(p_1, p_2, w)$. Once prices change, the truncation point changes to $\mathbf{X}_1^{\max\prime} < \mathbf{X}_1^{\max}$ and to $\mathbf{X}_1^{\max\prime\prime} < \mathbf{X}_1^{\max}$ when income is compensated. In this case, the light-shaded area to the right of $\mathbf{X}_1^{\max\prime\prime}$ is no longer feasible and the mean demand

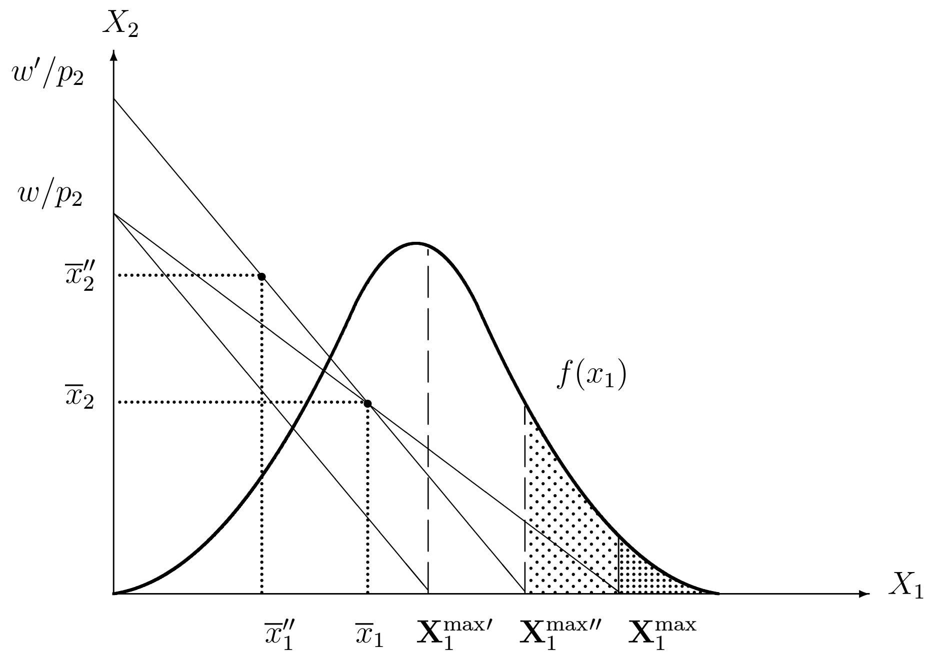

Figure 1: The mean demand for X_1, $\bar{x}_1(p_1, p_2, w)$, is drawn from $f(x_1)$. Under (p_1, p_2, w), the truncation point is $\mathbf{X}_1^{\max} = w/p_1$. For uncompensated changes, the truncation point is $\mathbf{X}_1^{\max\prime}$. For compensated changes, the truncation point is $\mathbf{X}_1^{\max\prime\prime}$.

is reduced to $\bar{x}_1''$, which represents $\bar{x}_1(p_1', p_2, w')$. For compensated and uncompensated changes, Figure 1 shows that the maximum quantity of X_1 that can be purchased declines. This decline in $\mathbf{X}_1^{\max}$ lowers the mean of X_1.[6]

Discussion. Several points should be noted.[7] (i) Proposition 1 does not depend on particular assumptions about $F(x_1)$. However, if $F(x_1)$ is uniform, as in Becker (1962), (2) and (3) yield

$$\bar{x}_1(p, w) = \frac{1}{(w/p_1)} \int_0^{w/p_1} x_1 dx_1 = \frac{1}{2} \frac{w}{p_1}, \tag{9}$$

and $\bar{x}_2(p, w) = (1/2)(w/p_2)$. These mean demands are identical to those of a "representative agent" who maximizes a Cobb-Douglas utility function.

[6] Sanderson (1974, 1980) noted that an increase in w expands the choice set in a first order stochastic sense. He did not, however, examine the implications of this change or noticed that this result follows due to the truncation of $F(x_1)$. Sanderson (1974, 1980) assumed changes in the distribution function underlying the random choices. Since the relevant distribution is $F(x_1)/F(\mathbf{X}_1^{\max})$, the stochastic dominance orderings assumed in Sanderson (1974, 1980) are always obtained here. For example, $F(x_1)/F(\mathbf{X}_1^{\max\prime})$ stochastically dominates $F(x_1)/F(\mathbf{X}_1^{\max})$ in the first degree sense in $[0, \mathbf{X}_1^{\max\prime}]$.

[7] It is perhaps obvious, but neither a Hicksian compensation nor standard duality results are possible here because demands do not rely on utility functions.

(ii) Because the Slutsky equation can be defined for any demand function, such decomposition can be established here:

$$\frac{\partial \bar{x}_1(p,w)}{\partial p_1} = \left.\frac{\partial \bar{x}_1(p,w)}{\partial p_1}\right|_{\bar{w}} - \bar{x}_1(p,w)\left(\frac{\partial \bar{x}_1(p,w)}{\partial w}\right). \tag{10}$$

As evident in Figure 1, X_1 *is on average a normal good.* Income effects satisfy

$$\frac{\partial \bar{x}_1(p,w)}{\partial w} = \frac{\partial \mathbb{E}\left[X_1 | \, 0 \leq X_1 \leq \mathbf{X}_1^{\max}\right]}{\partial \mathbf{X}_1^{\max}} \frac{1}{p_1} > 0. \tag{11}$$

Income effects are not necessarily positive for $\bar{x}_2(p,w)$ or in the $L-$commodity case.

(iii) Proposition 1 examined mean demands. Similar results hold for other measures of central tendency. Let $\bar{x}_1^{\mu}(p,w)$ denote the *median demand* for X_1, defined by $F(\bar{x}_1^{\mu}(p,w)) = F(\mathbf{X}_1^{\max})/2$. Thus,

$$\frac{\partial \bar{x}_1^{\mu}(p,w)}{\partial \mathbf{X}_1^{\max}} = \frac{1}{2}\frac{f(\mathbf{X}_1^{\max})}{f(\bar{x}_1^{\mu}(p,w))} > 0. \tag{12}$$

As in (6), (12) is enough to verify the compensated *Law of Demand.* The same result holds for the mid-range since this statistic treats $F(x_1)$ as a uniform distribution. Similar results cannot be established for the mode, but the mode is not meaningful here.[8]

(iv) The framework is a random choice model; see, e.g., McFadden and Richter (1990, pp. 165-166). At the market level, the "observed" probability that X_1 is at least x_1, when individuals are restricted by a budget set (p,w), is $\pi_{p,w}(x_1) \equiv F(x_1)/F(w/p_1)$. Random choices are *not* generated by a random preference model, and the purpose of the analysis is *not* to stochastically rationalize the choice probabilities $\pi_{p,w}(x_1)$. Proposition 1 simply presents comparative statics for $\bar{x}_1(p,w)$.

(v) Following Bronars (1987), many empirical studies have used random choices to test the validity of preference maximization. Figure 1 illustrates this procedure. When p_1 increases, there is a fraction of individual violations of the compensated *Law of Demand.*

[8] Let $\bar{x}_1^{\mathrm{m}}(p,w)$ denote the *mode* of X_1. Under standard conditions, the mode is defined by $\partial f(\bar{x}_1^{\mathrm{m}}(p,w))/\partial x_1 = 0$. The mode does not depend on $\mathbf{X}_1^{\max}$ unless it is non-interior (i.e., if $\bar{x}_1^{\mathrm{m}}(p,w) = \mathbf{X}_1^{\max}$). In such a case, the analysis presented here holds but the results are trivial.

These violations are represented by the probability of observing an increase (rather than a reduction) in demands: $\Pr(\bar{x}_1(p_1, p_2, w) \leq X_1 \leq \mathbf{X}_1^{\max\prime\prime}) = F(\mathbf{X}_1^{\max\prime\prime}) - F(\bar{x}_1(p_1, p_2, w))$. This probability can be used to construct the statistical power for tests of "rational behavior." The uniform distribution used by Bronars (1987), however, assumes that individual violations of the *Law of Demand* are not systematic. I will later on list some examples that account for systematic violations of individual rationality.

(vi) I have focused on X_1 but the *Law of Demand* also holds for X_2. In Appendix A I show the negativity of the compensated own price effects. I also show that one needs additional assumptions on $F(x_1)$ to establish positive income effects for $\bar{x}_2(p, w)$.

Finally, (vii) mean demands (2) and (3) satisfy all of the properties of standard demand theory. Homogeneity follows by construction as only $\mathbf{X}_1^{\max}$ determines $\bar{x}_1(p, w)$; see (2). $\bar{x}_2(p, w)$ is also homogeneous; see (3). Adding up conditions are satisfied by construction. Mean demands are also symmetric with respect to cross-price changes, which implies that mean demands can be *rationalized*, e.g., integrated. This last result, however, is to be expected because I examined two commodities and non-interior demands. Symmetry here follows due to homogeneity and adding up, and hence, it has no independent value.[9]

3 The general theorems

In this section, individuals randomly choose their consumption of $L > 1$ commodities using a general probability distribution function. Section 4 presents particular distribution functions consistent with non-rational random choice models.

It is useful to contrast the method of analysis considered here with the traditional Lagrange approach. Lagrange multipliers provide a strategy that ensures choices always lie in the feasible budget set. Individual choices here *cannot* be restricted through Lagrange multipliers since there is no optimizing behavior. To ensure feasibility, I will instead rely on a statistical truncation that generalizes the example above.

Let $X \equiv (X_1, ..., X_L)$ represent a commodity vector and let $p \equiv (p_1, ..., p_L)$ denote the

[9] This fact is well known; see, e.g., Mas-Colell et al. (1995, p. 36 and Exercise 2.F.15) and Katzner (1970, Theorem 4.1-2). Later sections establish symmetry and integrability in general cases.

corresponding price vector. The budget constraint generalizes (1):

$$p \cdot X = \sum_{\ell=1}^{L} p_\ell X_\ell \leq w. \tag{13}$$

Individuals randomly choose their consumption using a well-behaved probability distribution function $F(x_1, ..., x_L) \equiv \Pr(X_1 \leq x_1, ..., X_L \leq x_L)$ with a density $f(x_1, ..., x_L)$. Let $x \equiv (x_1, ..., x_L)$. I assume that $x \in \Omega \subseteq \mathbb{R}_+^L$, which is a standard *consumption set*; see Mas-Colell et al. (1995, p. 18). With L commodities, the budget constraint (13) may hold as an inequality depending on whether X_L is chosen randomly or determined as a residual. I will treat both cases separately.

The case of $L = 2$. Consider first the case of $L = 2$ and interior demands. Choices of X_1 are feasible if $X_1 \leq \mathbf{X}_1^{\max}$ with $\mathbf{X}_1^{\max} \equiv w/p_1$. The choice of X_2 takes into account that some income may be devoted to X_1. In particular, X_2 is feasible if $p_2 X_2 \leq w - p_1 X_1$ for feasible values of X_1. Thus, for each possible value $x_1 \leq \mathbf{X}_1^{\max}$, the *maximum feasible consumption* of X_2 is $\mathbf{X}_2^{\max}(x_1) \equiv (w - p_1 x_1)/p_2$.

The fact that some values of X_1 and X_2 are not feasible truncates the distribution $F(x_1, x_2)$. In particular, let $F(p, w) \equiv F(\mathbf{X}_1^{\max}, \mathbf{X}_2^{\max}(x_1))$ denote the probability that X_1 and X_2 are feasible given (p, w). Then,

$$F(p, w) \equiv \Pr\left[0 \leq p_1 X_1 + p_2 X_2 \leq w\right] = \int_0^{\mathbf{X}_1^{\max}} \left[\int_0^{\mathbf{X}_2^{\max}(x_1)} f(x_1, x_2) dx_2\right] dx_1. \tag{14}$$

The mean demand for X_1 is $\bar{x}_1(p, w) \equiv \mathbb{E}\left[X_1 | \, 0 \leq p_1 X_1 + p_2 X_2 \leq w\right]$, or

$$\bar{x}_1(p, w) = \frac{\displaystyle\int_0^{\mathbf{X}_1^{\max}} \left[\int_0^{\mathbf{X}_2^{\max}(x_1)} x_1 f(x_1, x_2) dx_2\right] dx_1}{F(p, w)}, \tag{15}$$

which, as in (2), is the truncated mean of X_1.

One way to interpret the double integrals in (14) and (15) is the following. As before, integration for x_1 is given by the exterior integral over $[0, \mathbf{X}_1^{\max}]$, with $\mathbf{X}_1^{\max}$ as the maximum feasible consumption of X_1. Integration for x_2 is given by the interior integral and it is limited by $\mathbf{X}_2^{\max}(x_1)$, which has to be integrated upon with respect to the possible values of x_1. Simply put, $F(p, w)$ integrates over the triangular area associated with the budget set. The fact that $(X_1, X_2) \leq (\mathbf{X}_1^{\max}, \mathbf{X}_2^{\max}(x_1))$ ensures that *every realization of X_1 and X_2 satisfy the budget constraint* (13).

The integrals (14) and (15) are not taken with respect to rectangular regions. One can change the order of integration by changing the limits of integration. Economically, changing these limits amounts to a re-labeling of the different commodities because there is no order in consumption, e.g., X_1 and X_2 are chosen simultaneously.[10]

L **commodities.** Let $\mathbf{X}^{\max} \equiv (\mathbf{X}_1^{\max}, ..., \mathbf{X}_L^{\max}(x_1, ..., x_{L-1}))$ denote the vector of *maximum feasible consumptions*. This vector satisfies: $p_1\mathbf{X}_1^{\max} = w$, and

$$p_\ell \mathbf{X}_\ell^{\max}(x_1, ..., x_{\ell-1}) = p_{\ell-1}[\mathbf{X}_{\ell-1}^{\max}(x_1, .., x_{\ell-2}) - x_{\ell-1}], \tag{16}$$

for $\ell > 1$. For example, the maximum feasible consumption of X_2 is $p_2\mathbf{X}_2^{\max}(x_1) = p_1(\mathbf{X}_1^{\max} - x_1)$. In turn, X_3 is limited by: $p_3\mathbf{X}_3^{\max}(x_1, x_2) = p_2(\mathbf{X}_2^{\max}(x_1) - x_2)$, which implies that $p_3\mathbf{X}_3^{\max}(x_1, x_2) = (w - p_1x_1 - p_2x_2)$.

The maximum consumption of X_L satisfies $p_L\mathbf{X}_L^{\max}(x_1, ..., x_{L-1}) = w - \sum_{\ell=1}^{L-1} p_\ell x_\ell$, which ensures that (13) holds, e.g., if $X_L = \mathbf{X}_L^{\max}(x_1, ..., x_{L-1})$, then (13) will hold with equality. Further, $\mathbf{X}_k^{\max}(x_{1,}..., x_{k-1})$ depends on the potential values of additional commodities because a positive consumption for commodities $\ell < k$ limits the maximum amount that can be consumed of commodity k. Specifically, $\mathbf{X}^{\max}$ is ordered in the sense that $p_\ell\mathbf{X}_\ell^{\max}(x_1, ..., x_{\ell-1})$ is a subset of $p_{\ell-1}\mathbf{X}_{\ell-1}^{\max}(x_1, .., x_{\ell-2})$. A key implication of this ordering is the following. Suppose that the realization of X_ℓ equals its maximum feasible value, e.g., $X_\ell = \mathbf{X}_\ell^{\max}(x_1, ..., x_{\ell-1})$. Since negative consumptions are ruled out, (16) implies that $\mathbf{X}_k^{\max}(x_1, ..., x_{k-1}) = 0$ for all $k > \ell$. The logic is simply that there is no income left for commodities $k > \ell$.

[10] In utility maximization models, the order in which first order conditions is obtained is irrelevant. Here, one can assume that X_2 is chosen first subject to $p_2X_2 \leq w$ and then X_1 is chosen among $(w - p_2X_2)/p_1$ for feasible values of X_2. The equivalence between both procedures follows due to a strong form of Fubini's Theorem; see, e.g., Fikhtengol'ts (1965, Vol. II, Section 344). Mean demands coincide with the physical notion of *barycenter* or center of gravity. Through this analogy, one can also verify that the order of integration under interior demands can be changed without consequence.

Mean demands. Let $F(p,w) \equiv F(\mathbf{X}_1^{\max},..,\mathbf{X}_L^{\max}(x_1,...,x_{L-1}))$. The truncation that ensures that (13) is satisfied is given by

$$F(p,w) = \int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_L^{\max}(x_1,...,x_{L-1})} f\left(x_1,...,x_L\right) dx_1...dx_L, \tag{17}$$

where the integration is over the $L-$dimensional simplex associated with (13). (17) is analogous to integrating over the line segment $[0, w/p_1]$ in (2) and over the triangular area of the budget set in (14).

The mean demand for commodity ℓ is $\bar{x}_\ell(p,w) \equiv \mathbb{E}(X_\ell | 0 \leq p \cdot X \leq w)$, or

$$\bar{x}_\ell(p,w) = \frac{\displaystyle\int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_L^{\max}(x_1,...,x_{L-1})} x_\ell f(x_1,...,x_L) dx_1...dx_L}{F(p,w)}, \tag{18}$$

which includes expressions (2) and (15) as special cases.

Let $\bar{x}(p,w) \equiv (\bar{x}_1(p,w),...,\bar{x}_L(p,w))$ denote the vector of mean demands. The first observation from (17) and (18) is that *mean demands are homogeneous in* (p,w). This result is easy to verify since prices and income only influence $\bar{x}(p,w)$ through $\mathbf{X}^{\max}$ and a proportional change in p and w leave this vector unchanged, e.g., $F(\theta p, \theta w) = F(p,w)$ and $\bar{x}(\theta p, \theta w) = \bar{x}(p,w)$ for $\theta > 0$. Moreover, if commodity L is determined as residual, then $X_L = \mathbf{X}_L^{\max}(X_1,...,X_{L-1})$ for all possible realizations of $X_1,...,X_{L-1}$. In this case, *demands add up to income.*

The probability distribution function used in (18) is $\pi_{p,w}(x) \equiv F(x)/F(p,w)$. This probability can be interpreted as the "observed" probability that choices will be at least x, given (p,w). I do not characterize $\pi_{p,w}(x)$, but notice that $\pi_{p,w}(x)$ is also homogeneous in (p,w). I focus on the mean demands $\bar{x}(p,w)$.

Own price effects.– I next examine the general properties of mean demands $\bar{x}(p,w)$. I provide all derivations in Appendix A. These derivations are not difficult. They are tedious as they rely on repeated differentiations under the integral sign.

Let $dx_{(L)}$ denote the differential when dx_L is excluded and let $x_{(L)}$ denote the vector

x when x_L is excluded. For commodity ℓ, Appendix A shows that

$$\left.\frac{\partial \bar{x}_\ell(p,w)}{\partial p_\ell}\right|_{\bar{w}} = -\int_0^{\mathbf{X}_1^{\max}} \dots \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} \{x_\ell - \bar{x}_\ell(p,w)\}^2 \left(\frac{f(x_{(L)})}{F(p,w)}\frac{1}{p_L}\right) dx_{(L)}, \tag{19}$$

where $f(x_{(L)}) \equiv f(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)}))$, which is only a function of $x_{(L)}$.

The intuition provided in the previous section applies here with a slight adaptation. Consider the case of $L = 2$. Let $\bar{x}_1(p_1', p_2, w')$ denote the resulting demand once p_1 and w change to p_1' and w'. The differential in (19) implies that everything in the overlapping area that determines $\bar{x}_1(p_1', p_2, w')$ and $\bar{x}_1(p_1, p_2, w)$ cancels. Thus, only the changes along the thin boundary given by $\mathbf{X}_2^{\max}(x_1)$ remain, which explains why price changes are in general integrated using the conditional density $f(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)}))$ and the marginal response along the surface $\mathbf{X}_L^{\max}(x_{(L)})$.[11]

The own price effect can be illustrated using Figure 1 when $L = 2$. The income compensation to a change in p_1 removes the triangular area with vertices $(\mathbf{X}_1^{\max}, 0)$, $(\mathbf{X}_1^{\max\prime\prime}, 0)$, and $(\bar{x}_1, \bar{x}_2)$, and adds the triangular area with vertices $(0, w'/p_2)$, $(0, w/p_2)$, and $(\bar{x}_1, \bar{x}_2)$. This pivoting has to be properly integrated using $f(x_1, \mathbf{X}_2^{\max}(x_1))$. The important point to notice is that the area that is removed once prices increase favors high values of X_1 whereas the area that is added favors low values.

The compensated own price change (19) is proportional to the negative of the conditional variance of X_ℓ about $\bar{x}_\ell(p,w)$. Let $Var[X_\ell] \equiv \mathbb{E}[(X_\ell - \bar{x}_\ell(p,w))^2 | X_{(L)} \le \mathbf{X}_{(L)}^{\max}, X_L = \mathbf{X}_L^{\max}]$. Then,

$$\left.\frac{\partial \bar{x}_\ell(p,w)}{\partial p_\ell}\right|_{\bar{w}} = -Var[X_\ell]. \tag{20}$$

Since variances are positive, *a compensated increase in p_ℓ reduces $\bar{x}_\ell(p,w)$.*

Income effects.– Changes in income yield:

$$\frac{\partial \bar{x}_\ell(p,w)}{\partial w} = \int_0^{\mathbf{X}_1^{\max}} \dots \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} \{x_\ell - \bar{x}_\ell(p,w)\} \left(\frac{f(x_{(L)})}{F(p,w)}\frac{1}{p_L}\right) dx_{(L)}. \tag{21}$$

[11] In essence, recall that $p_L \mathbf{X}_L^{\max}(x_{(L)}) = w - p_{(L)} \cdot x_{(L)}$, which is the only maximum feasible consumption that depends on the entire price vector p and on income w. Thus, changes in economic conditions that relax or tighten the budget constraint always affect $\mathbf{X}_L^{\max}(x_{(L)})$. The function $\mathbf{X}_L^{\max}(x_{(L)})$ can be seen as the analog of having changes in prices or income always affecting the Lagrange multiplier in traditional optimization problems.

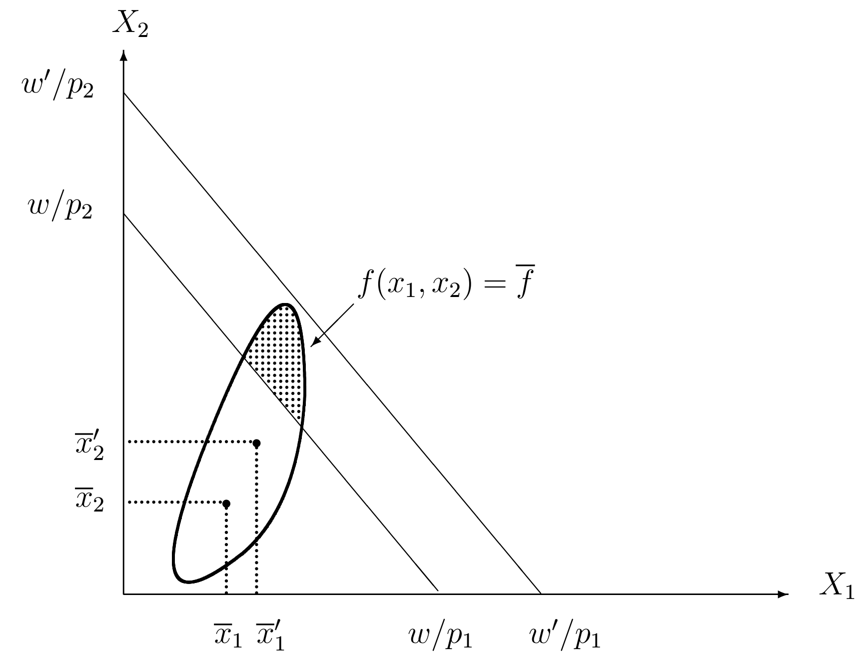

Figure 2: Positive income effects when X_1 and X_2 are "positively associated." The density $f(x_1, x_2)$ is represented by its contour at $\bar{f}$. As income increases, the shaded area is added to determine mean demands. This area favors high values of X_1.

The sign of (21) cannot be determined without additional assumptions about the statistical dependence between X_ℓ and X_L. To describe these additional assumptions, notice that in (21) one needs to determine

$$\int_0^{\mathbf{X}_1^{\max}} \dots \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} x_\ell f(x_{(L)}) dx_{(L)} \gtrless \bar{x}_\ell(p, w) \int_0^{\mathbf{X}_1^{\max}} \dots \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} f(x_{(L)}) dx_{(L)}.$$

Since the density $f(x_{(L)})$ is evaluated at the maximum feasible value of X_L, the previous expression compares $\mathbb{E}[X_\ell | X_{(L)} \leq \mathbf{X}_{(L)}^{\max}, X_L = \mathbf{X}_L^{\max}]$ with $\bar{x}_\ell(p, w) \equiv \mathbb{E}[X_\ell | X_{(L)} \leq \mathbf{X}_{(L)}^{\max}, X_L \leq \mathbf{X}_L^{\max}]$; see, e.g., (18). To sign (21) one needs to understand what happens to the mean value of X_ℓ when X_L takes its highest possible value. *If X_ℓ and X_L are "positively associated," a high value of X_L implies a high value for X_ℓ. In this case, income effects are positive. If X_ℓ and X_L are "negatively associated," a high value of X_L implies a low value for X_ℓ and this yields negative income effects.*[12]

[12] It is enough to consider "positive association" as positive likelihood dependence: $f(x_1, x_2)$ is said to be *positively likelihood ratio dependent* if $f(x_1', x_2') f(x_1, x_2) \geq f(x_1', x_2) f(x_1, x_2')$, for $x_1' > x_1$ and $x_2' > x_2$. Thus, it is more likely to observe that X_1 and X_2 take larger values together and smaller values

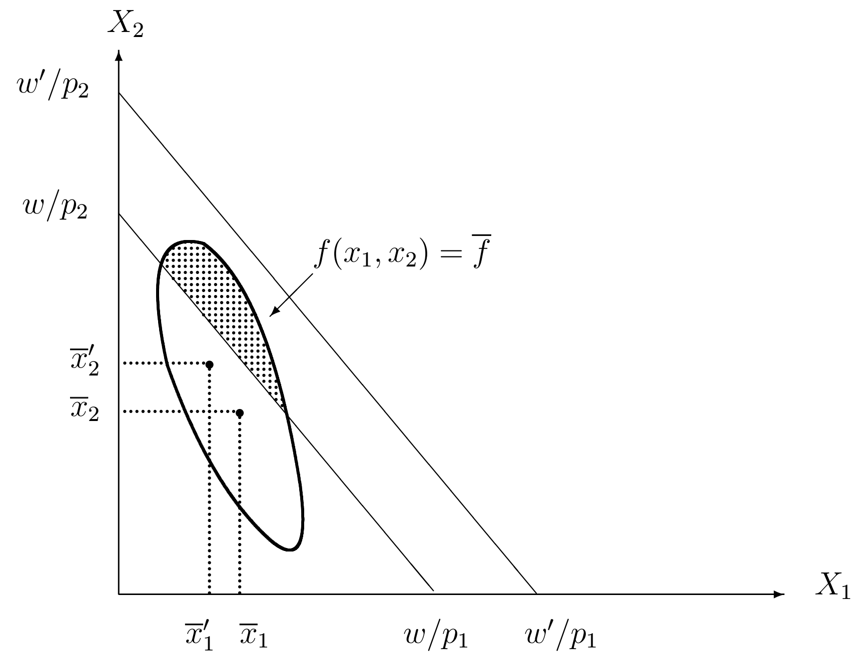

Figure 3: Negative income effects in $\bar{x}_1(p, w)$ when X_1 and X_2 are "negatively associated." The density $f(x_1, x_2)$ is represented by its contour at $\bar{f}$. As income increases, the shaded area is added to determine mean demands. This area favors low values of X_1.

Figures 2 and 3 illustrate graphically this dependence in the case of $L = 2$ and interior demands. The figures depict the contour of the density $f(x_1, x_2)$ along with the budget set at two different income levels. Figure 2 presents the case of "positively associated" variables. At income w, mean demands are $(\bar{x}_1, \bar{x}_2)$, which are determined excluding the shaded area. At $w' > w$, the shaded area becomes part of the relevant density that determines mean demands. Since this area favors high values of X_1, its mean demand increases to $\bar{x}_1'$. The case of negative income effects is depicted in Figure 3. If X_1 and X_2 are "negatively associated," as income increases, the new (shaded) area added to determine mean demands favors low values of X_1 and this lowers mean values.

The Slutsky matrix.– I next show that cross-price effects are symmetric and that the Slutsky matrix is negative semi-definite. Let $\mathcal{S}(p, w)$ denote the Slutsky matrix. A central point in this sub-section is that $\mathcal{S}(p, w)$ is closely related to a variance-covariance matrix.

together than any mixture of these. Figure 2 presents an illustration of this case. The multivariate analog requires $f(x_1, ..., x_L)$ to satisfy a positive likelihood ratio dependence for every pair (x_k, x_ℓ) when the $L - 2$ remaining variables are fixed; see, e.g., Shaked and Shanthikumar (2007).

This close relationship ensures symmetry and negative semi-definiteness.

Focus on commodity k. A compensated change in p_ℓ yields

$$\left.\frac{\partial \bar{x}_k(p,w)}{\partial p_\ell}\right|_{\bar{w}} = -\int_0^{\mathbf{X}_1^{\max}} \dots \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} \{x_\ell - \bar{x}_\ell(p,w)\}\{x_k - \bar{x}_k(p,w)\}\left(\frac{f(x_{(L)})}{F(p,w)}\frac{1}{p_L}\right) dx_{(L)}. \tag{22}$$

I derive the previous expression in Appendix A, where I also show that:

$$\left.\frac{\partial \bar{x}_k(p,w)}{\partial p_\ell}\right|_{\bar{w}} = \left.\frac{\partial \bar{x}_\ell(p,w)}{\partial p_k}\right|_{\bar{w}}. \tag{23}$$

I will restate the terms in $\mathcal{S}(p,w)$ using a more standard statistical notation based on a variance-covariance matrix. Let Σ be the *conditional variance-covariance matrix* of $(X_{(L)}, \mathbf{X}_L^{\max}(X_{(L)}))$ about $\bar{x}(p,w)$. The elements of Σ are of the form $\Sigma_{k\ell} = Cov[X_k, X_\ell] \equiv \mathbb{E}[(X_k - \bar{x}_k(p,w))(X_\ell - \bar{x}_\ell(p,w))|X_{(L)} \leq \mathbf{X}_{(L)}^{\max}, X_L = \mathbf{X}_L^{\max}]$, which are equivalent to (22). Then,

$$\mathcal{S}_{k\ell}(p,w) \equiv \left.\frac{\partial \bar{x}_k(p,w)}{\partial p_\ell}\right|_{\bar{w}} = -Cov[X_k, X_\ell], \tag{24}$$

for $k, \ell = 1, ...L-1$, and with X_L substituted by $\mathbf{X}_L^{\max}(X_{(L)})$ when $k, \ell = L$.

The Slutsky matrix can be written as a matrix of second order moments, $\mathcal{S}(p,w) = -\Sigma$. Being a variance-covariance matrix, Σ is positive semi-definite; see, e.g., Fisz (1963, pp. 89-90). Thus, *the Slutsky matrix $S(p,w)$ is negative semi-definite.* For the variance-covariance matrix to be positive definite instead of just positive semi-definite, the vector $X_{(L)} - \bar{x}_{(L)}(p,w)$ must be linearly independent from $\mathbf{X}_L^{\max}(X_{(L)}) - \bar{x}_L(p,w)$; see, e.g., Fisz (1963, Theorem 3.6.6). Thus, *if the consumption of commodity L is not residual, $S(p,w)$ is negative definite.* This distinction is important for the integrability of mean demands.

The next Theorem summarizes the key implications I have presented so far:

Theorem 1 *Suppose individuals randomly choose X subject to (13). Then, $\bar{x}(p,w)$ is linearly homogeneous in (p,w) and $\mathcal{S}(p,w)$ is negative definite and symmetric. If $X_{(L)}$ is randomly chosen and X_L is determined as a residual, $\bar{x}(p,w)$ is linearly homogeneous in (p,w), $\mathcal{S}(p,w)$ is negative semi-definite and symmetric, and $p \cdot \bar{x}(p,w) = w$.*

Homogeneity is discussed earlier in this section. Economically, Theorem 1 implies that if commodity L is determined as a residual, X_L would be redundant as its mean demand can be obtained as a linear combination of the other commodities. This redundancy is of the usual kind. In utility maximizing models, homogeneity and adding up imply that the negative semi-definiteness of the Slutsky matrix cannot be extended to negative definiteness; see, e.g., Mas-Colell et al. (1995, Proposition 2.F.3). Statistically, the linear dependence between random variables gives rise to a singular variance-covariance matrix; see, e.g., Fisz (1963, Theorem 3.6.6).

The negative semi-definiteness of $\mathcal{S}(p,w)$ establishes the compensated *Law of Demand* under general conditions, including interior demands; see, e.g., Mas-Colell et al. (1995, pp. 34-35). Symmetry in $\mathcal{S}(p,w)$ is the key implication of utility maximizing behavior; see, e.g., Mas-Colell et al. (1995, Chapters 2 and 3). The intuition behind this property under random choice behavior is transparent due to (24). Since the consumption vector X is composed of random variables, this vector obeys standard statistical properties such as symmetry in the variance-covariance matrix.

Integrability. As a consequence of Theorem 1, $\bar{x}(p,w)$ can be rationalized as being the result of the maximization of some utility function by a "representative agent":

Theorem 2 *There exists a continuous, non-decreasing, and quasi-concave utility function that rationalizes $\bar{x}(p,w)$ as an incomplete system of demands. If $X_{(L)}$ is randomly chosen and X_L is determined as a residual, the utility function $u:\Omega\to\mathbb{R}$ is such that $\bar{x}(p,w)$ is the unique solution to $\max_{x\geq 0}\{u(x): p\cdot x = w\}$.*

Theorem 2 establishes the integrability of interior demands as an incomplete system of demands; see Epstein (1982) for integrability results of this kind. I present additional remarks for this case in Appendix A. The fact that integrability holds for L commodities is important. As noted by Katzner (1970, pp. 67-68), integrability in the two-commodity case can be resolved easily. Notice also that integrability here holds for random choices drawn from a general probability distribution function.

Discussion. (i) Theorems 1 and 2 have important practical implications. These results suggest that tests of the compensated *Law of Demand* or test of symmetry may not be powerful tests for "rational behavior." Both properties are equally consistent with "irrational behavior" and cannot be used to discriminate between these alternatives. Empirical analyses of demand integrability may also have low power to distinguish situations in which individuals, on average, randomly choose their consumptions or rationally maximize an underlying utility function.[13]

(ii) The utility maximizing representation for the demand system $\bar{x}(p, w)$ makes it possible to measure consumer welfare. Under "irrational behavior," welfare, e.g., the *consumer's surplus*, have no significance. Neither the observed choices nor the process by which these choices are revealed provides a basis for normative statements.

Finally, (iii) for the class of utility functions that rationalize $\bar{x}(p, w)$, Theorem 2 establishes a new dual representation. The demand system $\bar{x}(p, w)$ can be seen as arising from the constrained maximization of $u(x)$ or from random choices drawn according to $F(x)$. This dual representation played a central role in Becker (1962), where remarks on the *as if* justification of rational behavior are provided.[14]

4 Examples of random choice models

I have not imposed any particular restriction on the distribution function used to draw random choices. Particular restrictions beyond those implied by probability theory are not necessary for Theorems 1 and 2. An advantage of such generality is that the previous framework can be used to study some aspects of random choice models that result from particular functional forms or behavioral restrictions on choice probabilities.

This section lists some examples of "random choice" models formulated for individual decision-making. Behavioral aspects need to be brought into the discussion. I treat these aspects as primitive and invariant characteristics of the individuals. Some of these models can be stochastically rationalized as "random preference" models. Rationality (whether regular or stochastic) is not needed here.

[13] Blundell et al. (2003, p. 211) mentioned this point in their empirical study of demand integrability.

[14] Appendix B provides several remarks about this dual representation. Appendix B also derives a market supply curve using similar arguments as those discussed here and shows, through numerical simulations, that the results obtained here do not require a large number of market agents.

I consider continuous probability distributions. Continuous choice models can be derived as the infinitesimal limit of discrete choice models; see, e.g., Ben-Akiva and Watanatada (1981, p. 327) and McFadden (1976, pp. 311-312). Recall that the consumption set is Ω. I assume that Ω is bounded. Since decision theorists focus on changes in the set of alternatives, I make the dependence on the choice set Ω explicit.

Bounded rationality. The following preferential choice models are consistent with the fact that people make different choices in nearly identical situations.

Luce's model.– Let $F(A|\Omega)$ denote the probability that an alternative from A is chosen when the alternatives available are Ω. In Luce (1959), choice probabilities satisfy the *independence from irrelevant alternatives*. For $A \subseteq \Omega$ and $B \subseteq \Omega$ such that $A \subseteq B$, $F(A|\Omega) = F(A|B)F(B|\Omega)$. This assumption yields a continuous logit formulation

$$F(dx|\Omega) \equiv \Pr\{X \in dx|\Omega\} = \frac{\exp\{v(x)/\phi\}dx}{\displaystyle\int_\Omega \exp\{v(x)/\phi\}dx}, \tag{25}$$

where $v(x)$ is a *direct utility function.* $v(x)$ is direct in the sense that it represents (reflexive, transitive, and complete) preferences over consumption goods. In contrast to an *indirect utility function*, $v(x)$ is independent of (p, w). That is, $F(x|\Omega)$ is formulated for individual decisions and *not* for the "observed" choice probabilities at the market level. "Observed" choice probabilities depend on (p, w); see (27) below.

Luce's model (25) accommodates extreme behaviors in the rationality spectrum through the parameter ϕ. The larger ϕ is, the greater the degree of "irrationality." That is,

$$\lim_{\phi\to\infty} F(dx|\Omega) = \frac{dx}{\displaystyle\int_\Omega dx},$$

which yields the uniform distribution used by Becker (1962). When $\phi \to 0$, the choice problem becomes a deterministic utility maximization problem. That is, $\lim_{\phi\to 0} F(x|\Omega) = 1$ if $v(x) \geq \max_{y\in\Omega}\{v(y)\}$ and zero otherwise; see Anderson et al. (1992, p. 42).

Tversky's elimination model.– Tversky (1972) considered a choice process in which alternatives are eliminated sequentially. Individuals first choose a subset $A \subseteq \Omega$ with probability $\int_A Q(\Omega, s)ds$ and then choose alternatives within this subset, e.g.,

$$F(x|\Omega) = \int_\Omega F(x|s)Q(\Omega, s)ds, \tag{26}$$

where $F(x|s)$ represents the probability of selecting x given s, and $Q(\Omega, s)$ is a weighting scheme that determines the probability of choosing s in Ω, with $\int_\Omega Q(\Omega, s)ds = 1$. Different assumptions on $Q(\Omega, s)$ yield particular elimination methods. Tversky (1972) can accommodate complex patterns of substitutability not possible in Luce (1959), e.g., (25) is a special case of (26); see Sattath and Tversky (1976, p. 83).

Luce (1959) and Tversky (1972) yield a probability distribution on each choice subset in Ω. Once the feasibility condition (13) is taken into account, mean demands $\bar{x}(p, w)$ can be obtained as in the previous section, e.g., as in (18). Moreover, the "observed" choice probabilities at the market level, which necessarily depend on (p, w), are

$$\pi_{p,w}(x|\Omega) \equiv \frac{F(x|\Omega)}{F(p, w|\Omega)}, \tag{27}$$

where $F(p, w|\Omega)$ is defined as in (17).

Random choices in Luce (1959) and Tversky (1972) can be seen as being driven by "random preferences"; see, e.g., Anderson et al. (1992, Chapter 2). Further, both models satisfy principles of stochastic regularity and transitivity. For example, let $x \in A, B \subseteq \Omega$, and suppose that A and B are feasible at (p, w). Then, for all x in $A \subseteq B$, $\pi_{p,w}(x|A) \geq \pi_{p,w}(x|B)$ and $\min[\pi_{p,w}(x|A), \pi_{p,w}(x|B)] \geq \pi_{p,w}(x|A \cup B) \geq \pi_{p,w}(x|A)\pi_{p,w}(x|B)$. Sattath and Tversky (1976) demonstrated these inequalities for discrete choice probabilities analogous to $F(x|\Omega)$ in (25) and (26). Since $\pi_{p,w}(x|\Omega)$ is a re-scaling of these probabilities, and since (p, w) are not changing, their analysis applies to $\pi_{p,w}(x|\Omega)$.[15] These inequalities provide testable predictions examined by the empirical literature.[16]

[15] Since these inequalities apply to discrete choices, one can partition Ω into "alternative groups" $\hat{x}_1, ..., \hat{x}_N$, with $A = \hat{x}_1$ and $B = \hat{x}_1 \cup ... \cup \hat{x}_j$. The mean value theorem can be used to re-express the "observed" choice probabilities. This representation is fairly standard; see, e.g., McFadden (1976, pp. 311-313).

[16] Rieskamp et al. (2006) discussed the empirical relevance of these models. Empirical tests suggest *similarity effects* that violate the independence of irrelevant alternatives and *attraction effects* that violate the regularity condition; see also Roe et al. (2001, pp. 371-372).

Difference models and decision field theory.– Gonzalez-Vallejo (2002) and Roe et al. (2001) are recent probabilistic choice models consistent with violations of stochastic regularity and stochastic transitivity.

In Roe et al. (2001), the utility function is $v(x) = \sum_{\ell}^{L} v^{\ell}(x_{\ell})$. Each utility component $v^{\ell}(x_{\ell})$ is perturbed by a normally distributed random variable with an overall variance-covariance matrix σ. Individual decisions are based on

$$F(x|\Omega) = \Phi(v(x) \geq \max_{y \in \Omega}\{v(y)\}; \sigma). \tag{28}$$

In Gonzalez-Vallejo (2002), $v(x)$ is the proportional difference between alternative consumption bundles. Tversky (1969) also studied proportional difference models. Thurstone's classical preferential choice model is a particular case of (28). In this class of models, changes in the covariance matrix across choice sets lead to violations of the independence of irrelevant alternatives and violations of stochastic regularity. As before, (28) is formulated for individual decision-making. Mean demands and "observed" choice probabilities are functions of (p, w).

Satisficing behavior.– The previous examples rely on maximizing behavior. Herbert Simon advocated *satisficing* as a principle of individual choice. In Simon (1957, p. 252), a bundle x is satisfactory provided that $v(x) \geq k$, where $k \in \Omega$ is a (given) feasible aspiration level. Under random choice, individual decisions would be similar to (28), but the stochastic comparison would not rely on the best alternative, as in (28). Instead, individuals would be satisfied with a bundle whose utility value exceeds k, e.g., $F(x|\Omega) = \Phi(v(x) \geq k; \sigma)$. Mean demands $\bar{x}(p, w)$ and "observed" choice probabilities $\pi_{p,w}(x|\Omega)$ can be obtained as in the previous examples.

Irrational behavior. The previous examples rely on a well-defined utility function $v(x)$. Random choices often result from the absence of utility functions, e.g., from *incomplete preferences.* Individuals sometimes act on impulse or are guided by instincts and emotions better described as irrational. These instances were the main motivation in Becker (1962).[17] Several examples of situations where rational decision-making is of

[17] Becker (1962) considered "impulsive good deciders" and Chant (1963) "impulsive money deciders"

limited value have been discussed by Elster (1989).

5 Concluding remarks

This paper characterized mean (or market) demands for a general random choice model that does not require the use of preferences or maximizing behavior. As shown in this paper, well-behaved demand functions can be rigorously derived for a much wider range of individual choice behaviors than those typically consider in economics. In the random choice model formulated here, mean demands are observationally equivalent to those based on the behavior of a "representative, rational agent." The intuition, which confirms and extends the view advanced by Gary Becker, is that the predictive capacity of demand theory lies primarily in the understanding of how *opportunity sets* change and not exclusively on individual decision-making assumptions.

Empirical studies of choice behavior show systematic and often predictable violations of rational behavior. To accommodate the fact that people make different choices in nearly identical situations, economists and psychologists have formulated descriptive boundedly rational random choice models. Under some conditions, the new comparative statics results presented here can be directly applied to this fairly large class of models of non-rational behavior. The market implications of these alternative decision-making assumptions turned out to be the same as those of standard demand theory. The interpretation of these results depends on whether one views the glass as half full or half empty. The findings are reassuring since they show that demand theory is robust to non-rational behavior. However, they also suggest that alternative (and potentially more realistic) assumptions about individual decision-making do not necessarily yield a better understanding of aggregate behavior.

but both of these modes of decision are closely related. Implusive behavior is pervasive but a real-world example may bring out the idea more clearly. In an ethnographic study, McGrath (1989, pp. 433) observed male shoppers in gift shops on Christmas Eve tended "to make large, rapid, spontaneous, and often random purchases." Impulse and irrational considerations have also been discussed by Kagel et al. (1995) in their experimental studies of demand behavior in non-human subjects.

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6 Appendix A: Omitted proofs and derivations

Benchmark example. Consider first the Slutsky equation for X_1. This equation follows from the definition of the Slutsky compensation (4). For completeness, notice that

$$\left.\frac{\partial \mathbf{X}_1^{\max}}{\partial p_1}\right|_{\bar{w}} = \frac{\partial \mathbf{X}_1^{\max}}{\partial p_1} + \frac{\bar{x}_1(p,w)}{p_1}.$$

This and expressions (6) and (11) yield (10).

Next consider the mean demand for X_2. Some steps in this case are analogous to the proof of Proposition 1. Consider first the own-price effect. Using equation (3), one has:

$$\left.\frac{\partial \bar{x}_2(p,w)}{\partial p_2}\right|_{\bar{w}} = \left(\left.\frac{\partial (w/p_2)}{\partial p_2}\right|_{\bar{w}}\right) - \left.\frac{\partial \bar{x}_1(p,w)}{\partial p_2}\right|_{\bar{w}} \frac{p_1}{p_2} - \frac{\bar{x}_1(p,w)p_1}{p_2^2},$$

with $\left(\left.\frac{\partial (w/p_2)}{\partial p_2}\right|_{\bar{w}}\right) = \frac{\bar{x}_1(p,w)p_1}{p_2^2}$, $\left.\frac{\partial \bar{x}_1(p,w)}{\partial p_2}\right|_{\bar{w}} = \frac{\partial \mathbb{E}\left[X_1 \middle| 0 \le X_1 \le \mathbf{X}_1^{\max}\right]}{\partial \mathbf{X}_1^{\max}} \left(\frac{\bar{x}_2(p,w)}{p_1}\right)$,
and

$$\left.\frac{\partial \bar{x}_2(p,w)}{\partial p_2}\right|_{\bar{w}} = -\frac{\partial \mathbb{E}\left[X_1 \middle| 0 \le X_1 \le \mathbf{X}_1^{\max}\right]}{\partial \mathbf{X}_1^{\max}} \left(\frac{\bar{x}_2(p,w)}{p_2}\right) < 0.$$

The Slutsky equation for X_2 follows from simple rearrangements. For instance, notice that the uncompensated own price effect is $\partial \bar{x}_2(p,w)/\partial p_2 = -\bar{x}_2(p,w)/p_2$ which rules out Giffen goods. Income effects, however, can be negative for $\bar{x}_2(p,w)$. That is,

$$\frac{\partial \bar{x}_2(p,w)}{\partial w} = \left(1 - \frac{\partial \mathbb{E}\left[X_1 \middle| 0 \le X_1 \le \mathbf{X}_1^{\max}\right]}{\partial \mathbf{X}_1^{\max}}\right) \frac{1}{p_2}. \tag{A1}$$

Corollary 1 *Under the conditions of Proposition 1, suppose $F(x_1)$ is a log-concave distribution. Then, $\bar{x}_2(p,w)$ is a normal good.*

Proof. The proof relies on (A1) and the following fact about log-concave distributions:

$$0 \le \frac{\partial \mathbb{E}\left[X_1 \middle| 0 \le X_1 \le \mathbf{X}_1^{\max}\right]}{\partial \mathbf{X}_1^{\max}} \le 1,$$

see Goldberger (1983, Appendix A). ■

Since the uniform distribution is log-concave, this corollary is implicit in Becker (1962) and in models that rely on the uniform case. Further, $|\mathcal{S}(p,w)|$ can be written as

$$|\mathcal{S}(p,w)| = \bar{x}_2(p,w)\left(\frac{\partial \mathbb{E}\left[X_1 | \, 0 \le X_1 \le \mathbf{X}_1^{\max}\right]}{\partial \mathbf{X}_1^{\max}}\right)\begin{vmatrix} -p_2/p_1^2 & 1/p_1 \\ 1/p_1 & -1/p_2 \end{vmatrix},$$

with $|\mathcal{S}(p,w)| = 0$. Symmetry and negative semi-definiteness always hold in the special case of two-commodities.

General theorems. In order to simplify the derivations in this section, I next present an auxiliary Lemma that I use repeatedly. The Lemma serves to simplify the calculations of price and income effects.[18] The central implication of the following Lemma is that in order to determine the response in mean demands to changes in prices and income, one only needs to evaluate the changes in the most interior integral. Economically, this result makes sense. Recall that $p_L \mathbf{X}_L^{\max}(x_{(L)}) = w - p_{(L)} \cdot x_{(L)}$, which is the only maximum feasible consumption that depends on the entire price vector p and on income w.

Lemma 1 *Let* $G(p,w) \equiv \int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_L^{\max}(x_{(L)})} g(x)dx$*, where* $\mathbf{X}^{\max}$ *is given in (16). Then,*

$$\begin{aligned} \left.\frac{\partial G(p,w)}{\partial p_\ell}\right|_{\bar{w}} &= \int_0^{\mathbf{X}_1^{\max}} ... \frac{\partial}{\partial p_\ell}\left[\int_0^{\mathbf{X}_L^{\max}(x_{(L)})} g(x)dx_L\right]\Bigg|_{\bar{w}} dx_{(L)}, \\ &= \int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} g(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)}))\left(\left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_\ell}\right|_{\bar{w}}\right) dx_{(L)}, \end{aligned}$$

for $\ell = 1, ..., L$.

Proof of Lemma 1. The proof is a repeated application of Leibnitz's rule for differentiation under the integral sign. The case of $L = 2$ and $p_1 = p_2 = 1$ is available in Khuri (2003, pp. 307-308).

[18] I do not examine the median demands or demands based on other measures of central tendency. There is no analogous way to define the median for the bivariate or multivariate cases. One approach is to generalize the notion of univariate conditional median using marginal distributions, as in (12).

For the general proof, it is enough to consider changes in p_1 since this price enters in all limits of integration. All other price changes can be seen as special cases.

$$\left.\frac{\partial G(p,w)}{\partial p_1}\right|_{\bar{w}} = \left[\int_0^{\mathbf{X}_2^{\max}(x_1)} \cdots \int_0^{\mathbf{X}_M^{\max}(x_1,\ldots,x_{L-1})} g(x_1,\ldots,x_L)dx_1\ldots dx_L\right]_{x_1=\mathbf{X}_1^{\max}} \left.\frac{\partial \mathbf{X}_1^{\max}}{\partial p_1}\right|_{\bar{w}}$$
$$+\int_0^{\mathbf{X}_1^{\max}} \frac{\partial}{\partial p_1}\left.\left[\int_0^{\mathbf{X}_2^{\max}(x_1)} \cdots \int_0^{\mathbf{X}_L^{\max}(x_1,\ldots,x_{L-1})} g(x_1,\ldots,x_L)dx_2\ldots dx_L\right]\right|_{\bar{w}} dx_1.$$

The first term is zero since $\mathbf{X}_2^{\max}(\mathbf{X}_1^{\max}) = (w - p_1(w/p_1))/p_2 = 0$. Further, as noted in the text, since $\mathbf{X}_2^{\max}(\mathbf{X}_1^{\max}) = 0$, then $\mathbf{X}_\ell^{\max} = 0$ for $\ell \geq 2$ also as there would be no income left for the consumption of these commodities. The second term becomes

$$\int_0^{\mathbf{X}_1^{\max}} \left\{\left[\int_0^{\mathbf{X}_3^{\max}(x_1,x_2)} \cdots \int_0^{\mathbf{X}_L^{\max}(x_1,\ldots,x_{L-1})} g(x)dx_2\ldots dx_L\right]_{x_2=\mathbf{X}_2^{\max}(x_1)} \left.\frac{\partial \mathbf{X}_2^{\max}(x_1)}{\partial p_1}\right|_{\bar{w}}\right\} dx_1$$
$$+\int_0^{\mathbf{X}_1^{\max}} \int_0^{\mathbf{X}_2^{\max}(x_1)} \frac{\partial}{\partial p_1}\left.\left[\int_0^{\mathbf{X}_3^{\max}(x_1,x_2)} \cdots \int_0^{\mathbf{X}_L^{\max}(x_1,\ldots,x_{L-1})} g(x)dx_3\ldots dx_L\right]\right|_{\bar{w}} dx_1dx_2,$$

whose first component is also zero as $p_3\mathbf{X}_3^{\max}(x_1, \mathbf{X}_2^{\max}(x_1)) = p_2(\mathbf{X}_2^{\max}(x_1) - \mathbf{X}_2^{\max}(x_1)) = 0$. Thus, as before, the first component evaluates a definite integral over a degenerate interval and this equals zero. The only relevant component is the second one which also needs to be evaluated using Leibnitz's rule.

By the way the limits of integration are defined, the derivative operator moves toward high values of ℓ. For instance, the ℓ step of the sequence of integrals is given by

$$\left.\frac{\partial G(p,w)}{\partial p_1}\right|_{\bar{w}} = 0 + \ldots + 0 \ (\ell - 1 \text{ times})$$
$$+\int_0^{\mathbf{X}_1^{\max}} \cdots \frac{\partial}{\partial p_1}\left.\left[\int_0^{\mathbf{X}_\ell^{\max}(x_1,\ldots,x_{\ell-1})} \cdots \int_0^{\mathbf{X}_L^{\max}(x_1,\ldots,x_{L-1})} g(x)dx_\ell\ldots dx_L\right]\right|_{\bar{w}} dx_1\ldots dx_{\ell-1},$$

where the evaluation of the integral with the upper limit of integration $\mathbf{X}_{\ell+1}^{\max}(x_1,\ldots,x_\ell)$ evaluated at $x_\ell = \mathbf{X}_\ell^{\max}(x_1,\ldots,x_{\ell-1})$ will also equal zero. Moreover, $\mathbf{X}_k^{\max} = 0$ for all $k \geq \ell$.

The last term in the sequence is

$$\left.\frac{\partial G(p,w)}{\partial p_1}\right|_{\bar{w}} = \int_0^{\mathbf{X}_1^{\max}} \ldots \left.\frac{\partial}{\partial p_1}\left[\int_0^{\mathbf{X}_L^{\max}(x_1,\ldots,x_{L-1})} g(x)dx_L\right]\right|_{\bar{w}} dx_1 \ldots dx_{L-1},$$

which under Leibnitz's rule simply becomes

$$\left.\frac{\partial G(p,w)}{\partial p_1}\right|_{\bar{w}} = \int_0^{\mathbf{X}_1^{\max}} \ldots \int_0^{\mathbf{X}_{L-1}^{\max}(x_1,\ldots,x_{L-2})} g(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)})) \left(\left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_1}\right|_{\bar{w}} \right) dx_{(L)}.$$

■

Proof of Theorem 1. The proof was started in the text and is completed here. To obtain (19), consider (18) and write this expression as $\bar{x}_\ell(p,w) = N_\ell(p,w)/F(p,w)$, with

$$N_\ell(p,w) \equiv \int_0^{\mathbf{X}_1^{\max}} \ldots \int_0^{\mathbf{X}_L^{\max}(x_{(L)})} x_\ell f(x)dx, \text{ and}$$

$$F(p,w) \equiv \int_0^{\mathbf{X}_1^{\max}} \ldots \int_0^{\mathbf{X}_L^{\max}(x_{(L)})} f(x)dx,$$

as the numerator and the denominator respectively. Using Lemma 1, notice that

$$\left.\frac{\partial N_\ell(p,w)}{\partial p_\ell}\right|_{\bar{w}} = \int_0^{\mathbf{X}_1^{\max}} \ldots \int_0^{\mathbf{X}_L^{\max}(x_{(L)})} x_\ell f(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)})) \left(\left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_\ell}\right|_{\bar{w}} \right) dx_{(L)},$$

$$\left.\frac{\partial F(p,w)}{\partial p_\ell}\right|_{\bar{w}} = \int_0^{\mathbf{X}_1^{\max}} \ldots \int_0^{\mathbf{X}_L^{\max}(x_{(L)})} f(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)})) \left(\left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_\ell}\right|_{\bar{w}} \right) dx_{(L)}.$$

Further, notice that the quotient rule implies

$$\left.\frac{\partial \bar{x}_\ell(p,w)}{\partial p_\ell}\right|_{\bar{w}} = \left.\frac{\partial N_\ell(p,w)}{\partial p_\ell}\right|_{\bar{w}} \left(\frac{1}{F(p,w)}\right) - \left.\frac{\partial F(p,w)}{\partial p_\ell}\right|_{\bar{w}} \left(\frac{N_\ell(p,w)}{F(p,w)^2}\right),$$

which can be written in simple terms as

$$\left.\frac{\partial \bar{x}_\ell(p,w)}{\partial p_\ell}\right|_{\bar{w}} = \left(\frac{1}{F(p,w)}\right) \left[\left.\frac{\partial N_\ell(p,w)}{\partial p_\ell}\right|_{\bar{w}} - \bar{x}_\ell(p,w) \left.\frac{\partial F(p,w)}{\partial p_\ell}\right|_{\bar{w}} \right].$$

This last expression, upon substitution, implies

$$\left.\frac{\partial \bar{x}_\ell(p,w)}{\partial p_\ell}\right|_{\bar{w}} = \int_0^{\mathbf{X}_1^{\max}} \dots \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} \{x_\ell - \bar{x}_\ell(p,w)\} \left(\frac{f(x_{(L)})}{F(p,w)} \left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_\ell}\right|_{\bar{w}} \right) dx_{(L)}. \tag{A2}$$

where $f(x_{(L)}) \equiv f(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)}))$.

Separately, and as in the previous section, a compensated change in p_ℓ only needs to be evaluated in terms of its effect on $\mathbf{X}_L^{\max}(x_{(L)})$. This result implies that price effects are channeled through changes in the maximum feasible consumption. In particular,

$$\left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_\ell}\right|_{\bar{w}} = \frac{\bar{x}_\ell(p,w) - x_\ell}{p_L}. \tag{A3}$$

Finally, substitution of (A3) into (A2) yields (19).

For commodity L, (19) implies that the own price effect is given by

$$\left.\frac{\partial \bar{x}_L(p,w)}{\partial p_L}\right|_{\bar{w}} = -\int_0^{\mathbf{X}_1^{\max}} \dots \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} \left\{\mathbf{X}_L^{\max}(x_{(L)}) - \bar{x}_L(p,w)\right\}^2 \left(\frac{f(x_{(L)})}{F(p,w)} \frac{1}{p_L} \right) dx_{(L)}.$$

The main difference is that X_L has been substituted by $\mathbf{X}_L^{\max}(x_{(L)})$, which is a (hyper) surface along $x_{(L)}$. Notice that in the benchmark example, the own price effect (6) can be written as $\partial \bar{x}_1(p,w)/\partial p_1|_{\bar{w}} = -\{\mathbf{X}_1^{\max} - \bar{x}_1(p,w)\}^2 f(\mathbf{X}_1^{\max})/F(\mathbf{X}_1^{\max})p_1$, which is the analog of the result just presented.

To obtain income effects (21), one simply needs to evaluate the appropriate derivative of $\mathbf{X}_L^{\max}(x_{(L)})$. For instance

$$\frac{\partial N_\ell(p,w)}{\partial w} = \int_0^{\mathbf{X}_1^{\max}} \dots \int_0^{\mathbf{X}_L^{\max}(x_{(L)})} x_\ell f(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)})) \left(\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial w} \right) dx_{(L)},$$

and similarly for the denominator. Once these expressions are substituted back into the quotient rule result, one obtains (21).

Finally, the cross-partial effects (A4) and (A5) also follow using the appropriate derivative of $\mathbf{X}_L^{\max}(x_{(L)})$. For instance

$$\left.\frac{\partial N_\ell(p,w)}{\partial p_k}\right|_{\bar{w}} = \int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_L^{\max}(x_{(L)})} x_\ell f(x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)})) \left(\left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_k}\right|_{\bar{w}} \right) dx_{(L)},$$

and similarly for the denominator. These calculations yield

$$\left.\frac{\partial \bar{x}_k(p,w)}{\partial p_\ell}\right|_{\bar{w}} = \int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} \{x_k - \bar{x}_k(p,w)\} \left(\frac{f(x_{(L)})}{F(p,w)} \left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_\ell}\right|_{\bar{w}} \right) dx_{(L)}, \tag{A4}$$

where the compensated change in $\mathbf{X}_L^{\max}(x_{(L)})$ is the same as the one obtained in (A3). Next consider the response of $\bar{x}_\ell(p,w)$ to a compensated change in p_k. Following Lemma 1, and repeating the steps just taken, one can show that:

$$\left.\frac{\partial \bar{x}_\ell(p,w)}{\partial p_k}\right|_{\bar{w}} = \int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} \{x_\ell - \bar{x}_\ell(p,w)\} \left(\frac{f(x_{(L)})}{F(p,w)} \left.\frac{\partial \mathbf{X}_L^{\max}(x_{(L)})}{\partial p_k}\right|_{\bar{w}} \right) dx_{(L)}, \tag{A5}$$

where $\partial \mathbf{X}_L^{\max}(x_{(L)})/\partial p_k\big|_{\bar{w}} = (\bar{x}_k(p,w) - x_k)/p_L$. Simple substitutions establish symmetry, (23).

The only property that deserves further comment is the negative semi-definiteness of $\mathcal{S}(p,w)$, which is analogous to the positive semi-definiteness of Σ. I mentioned before that $\mathcal{S}_{k\ell}(p,w)$ can be written as a covariance term given by

$$Cov[X_k, X_\ell] = \int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} \{x_k - \bar{x}_k(p,w)\} \{x_\ell - \bar{x}_\ell(p,w)\} \tilde{f}(x_{(L)}) dx_{(L)},$$

where $\tilde{f}(x_{(L)})$ is a simple correction since the density $f(x_{(L)})$ does not integrate to one. That is, the normalization $\tilde{f}(x_{(L)}) \equiv f(x_{(L)})\varphi(p,w)$ with

$$\varphi(p,w) \equiv \frac{1}{F(p,w)p_L} \left(\int_0^{\mathbf{X}_1^{\max}} ... \int_0^{\mathbf{X}_{L-1}^{\max}(x_{(L-1,L)})} f\left(x_{(L)}\right) dx_{(L)} \right),$$

serves to treat $\tilde{f}(x_{(L)})$ as a proper density in order to define the matrix of second moments. Notice that the term $\varphi(p,w)$ is a positive constant and hence it does not influence any of the conclusions of the analysis.

Let $\tilde{x} \equiv (x_{(L)}, \mathbf{X}_L^{\max}(x_{(L)}))$ denote a vector of possible demands. To complete the proof, consider a vector υ and the non-negative quadratic form $[\upsilon \cdot (\tilde{X} - \bar{x}(p,w))]^2 = \upsilon \cdot (\tilde{X} - \bar{x}(p,w))(\tilde{X} - \bar{x}(p,w)) \cdot \upsilon$, so that the conditional expectation described in (24) satisfies $\mathbb{E}[\upsilon \cdot (\tilde{X} - \bar{x}(p,w))]^2 = \upsilon \cdot \Sigma\upsilon$, which only takes non-negative values. Thus, Σ is positive semi-definite, which is a well-known property of variance-covariance matrices, Fisz (1963, pp. 89-90).

To verify the conditions for positive definiteness, notice that one only needs to check the last term in $\tilde{x}$ because all other variables have a joint density, i.e., their distribution is non-degenerate; see Fisz (1963, p. 90). Notice that $\mathbf{X}_L^{\max}(X_{(L)})$ is a linear function of $X_{(L)}$. That is, for a given realization of demands $X_{(L)}$, the maximum feasible consumption $\mathbf{X}_L^{\max}(X_{(L)})$ is linear in $X_{(L)}$. More specifically, $p_L\mathbf{X}_L^{\max}(X_{(L)}) = w - p_{(L)} \cdot X_{(L)}$. This result is a consequence of the way the maximum feasible consumption has been defined; see (16). However, $\bar{x}_L(p,w)$ is not always linearly related to $\bar{x}_{(L)}(p,w)$. In particular, $\bar{x}_L(p,w)$ will be linearly related to $\bar{x}_{(L)}(p,w)$ when X_L is selected as a residual and $X_L = \mathbf{X}_L^{\max}(X_{(L)})$ for all possible realizations of $X_{(L)}$. In this case, it is straightforward to show that $p_L\bar{x}_L(p,w) = w - p_{(L)} \cdot \bar{x}_{(L)}(p,w)$. In other words, Σ is positive semi-definite but not positive definite since $\upsilon \cdot \Sigma\upsilon = 0$ holds for some $\upsilon \neq 0$. That is, if $X_L = \mathbf{X}_L^{\max}(X_{(L)})$ for all possible realizations of $X_{(L)}$.[19] ■

Integrability. The next theorem studies the integrability of mean demands when income is not exhausted. The results complement Theorem 2.

Theorem 3 *Consider a system of mean demands $\bar{x}(p,w)$ and suppose X_L is chosen randomly. Then, there exists a continuous, non-decreasing, and quasi-concave utility function $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that $\bar{x}(p,w)$ and $\bar{x}_0(p,w) \equiv w - p \cdot \bar{x}(p,w)$ are the unique solution to $\max_{x,x_0 \geq 0} \{u(x,x_0) : p \cdot x + x_0 = w\}$.*

Proof of Theorems 2 and 3. Katzner (1970, Chapter 4) discusses in detail the conditions for integrability in a complete demand system. Briefly, let $e(p,u)$ denote

[19] Consider the case of $L = 2$. Then, $\mathbf{X}_2^{\max}(x_1) - \bar{x}_2(p_1,p_2,w) = (w - p_1x_1)/p_2 - \bar{x}_2(p_1,p_2,w)$. Assume $X_2 = \mathbf{X}_2^{\max}(X_1)$ for all possible realizations of X_1. In this case, X_2 is determined as a residual. This implies that $\bar{x}_2(p_1,p_2,w) = (w - p_1\bar{x}_1(p_1,p_2,w))/p_2$; see, e.g., (3). Then $\mathbf{X}_2^{\max}(X_1) - \bar{x}_2(p_1,p_2,w)$ becomes $-(p_1/p_2)(X_1 - \bar{x}_1(p_1,p_2,w))$ which is a linear function of $(X_1 - \bar{x}_1(p_1,p_2,w))$. In this case the variance-covariance matrix will be positive semi-definite but not positive definite.

the *expenditure function*: $e(p,u) \equiv \min_{x \geq 0}\{p \cdot x : u(x) \geq u\}$. Thus, $\partial e(p,u)/\partial p_\ell = h_\ell(p,u)$, with $h_\ell(p,u)$ as the *Hicksian demand* for ℓ which satisfies $h_\ell(p,u) = \bar{x}_\ell(p, e(p,u))$. Symmetry in $\mathcal{S}(p,w)$ is necessary and sufficient for the existence of a solution for the previous partial differential equation system. Further, a negative semi-definite $\mathcal{S}(p,w)$ is necessary and sufficient for the solution of the previous system to be concave in p. As Theorem 1 noted, these requirements are met if X_L is determined as a residual. The utility function can then be recovered from the expenditure function.

The integrability of incomplete demand systems is discussed in Epstein (1982). Interior demands can be rationalized if one treats the demand system as incomplete and assumes that one "residual" commodity completes the system. Three observations are important: first, the symmetry properties $\mathcal{S}(p,w)$ yield a solution of the system of partial differential equations $\partial e(p,u)/\partial p_\ell = \bar{x}_\ell(p, e(p,u))$ for $\ell = 1, ..., L$. Second, integrability in incomplete systems requires a *negative definite* Slutsky matrix. This requirement is stronger than in the case of complete systems and it cannot be weakened; see Epstein (1982, Example 1). As Theorem 1 noted, interior demands satisfy this condition here.

Finally, the constant of integration in the previous system cannot be uniquely determined and one must make assumptions about how to complete the system. Assuming the existence of x_0 accomplishes this. In general, let $\bar{x}_0(p, p_0, w)$ denote the vector of demands for N commodities that complete the demand system and let p_0 represent their corresponding vector of prices. If $\bar{x}(p, p_0, w) = \bar{x}(p,w)$ and $\bar{x}_0(p, p_0, w) = \bar{x}_0(p_0, w)$, then there is a utility function $u : \Omega \times \mathbb{R}^N_+ \to \mathbb{R}$ such that $\bar{x}(p,w)$ and $\bar{x}_0(p_0,w)$ are the solution to $\max_{x \geq 0, x_0 \geq 0}\{u(x, \Psi(x_0)) : p \cdot x + p_0 \cdot x_0 = w\}$, with $\Psi(x_0)$ linearly homogeneous; see Epstein (1982). ■

7 Appendix B: NOT FOR PUBLICATION

Random ("dual") representations. Theorems 2 and 3 show that mean demands that arise from individuals randomly choosing their consumptions can be represented as demands that arise from the maximization of some utility function $u(x)$,

$$\mathbb{E}(X|0 \leq p \cdot X \leq w) = \text{argmax}_{x \geq 0}\{u(x) : p \cdot x = w\}.$$

This sub-section provides a partial examination of the "converse" problem and provides two random or "irrational" *representations* of well-behaved demands. Suppose there is a given demand function $h_\ell(p, w)$ for commodity ℓ and let $h(p, w) \equiv (h_1(p, w), ..., h_L(p, w))$. Assume $h(p, w)$ is homogeneous in (p, w) and $p \cdot h(p, w) = w$.

The first representation assumes that "irrational" demands are on average feasible whereas the second is individually feasible but it restricts the support of the distribution. This feasibility condition is the same as the one used by Katzner (1970) to discuss errors and shocks to individual demand functions. In particular, as in Katzner (1970, p. 161), "if [the consumer] were to choose from the same budget set many times,"on average" he would choose the utility maximizing bundle."

Lemma 2 *Let $\mathbf{z}_\ell \equiv w/p_\ell$ and $\mathbf{z} \equiv (\mathbf{z}_1, ..., \mathbf{z}_L)$ denote the set of extreme points of the budget set (13). Then, $h(p, w) = h(\mathbf{z})$.*

Proof. Notice that demands satisfy $h_\ell(p_1, ..., p_L, w) = h_\ell\left(p_1/w, ..., p_L/w, 1\right)$. This is just a consequence of homogeneity. ■

The goal in the first representation is to find a distribution function $F^*(x)$ such that

$$h_\ell(\mathbf{z}) = \frac{\int_0^{\mathbf{z}_1} ... \int_0^{\mathbf{z}_L} x_\ell f^*(x) dx}{F^*(\mathbf{z})}, \tag{B1}$$

for all $\ell = 1, ..., L$ where $F^*(\mathbf{z}) \equiv F^*(\mathbf{z}_1, ..., \mathbf{z}_L)$ and $f^*(x)$ is the joint density of $F^*(x)$. Notice that (B1) integrates random individual demands over a rectangular area instead of over the triangular area given by (13). The derivations needed to construct the first representation are straightforward.[20]

A second representation can be constructed using results from *convex analysis*. Every point in a convex and compact set of finite dimension can be written as a convex combination of the extreme points of the set; see, e.g., Phelps (2001, p. 1). Let C be a compact convex subset of $\mathbb{R}^L$ and let $\mathbf{z}$ denote the extreme points of C. Every point $x \in C$ can be written as a convex combination of $\mathbf{z}$:

$$x = \sum_{i=1}^{L} \alpha_i \mathbf{z}_i,$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{L} \alpha_i = 1$. The previous representation has a probabilistic interpretation:

Theorem 4 (Minkowski's Integral Representation) *Let C be a compact convex subset of $\mathbb{R}^L$. Then, for every point $x \in C$, there exists a probability measure μ concentrated in the extreme points of C such that $x = \int_C z d\mu(z)$. If C is a simplex, μ is unique.*

Proof. Let $\delta_{\mathbf{z}_i}$ be the Dirac measure on the point $\mathbf{z}_i$, i.e., for every Borel set B, $\delta_{\mathbf{z}_i}(B) = 1$ if $\mathbf{z}_i \in B$ and zero otherwise. Then, $\mu(C) = \sum_{i=1}^{L} \alpha_i = 1$ and μ is a measure

[20] The derivations rely on an inversion formula for the right-truncated mean. Write (B1) as

$$h_\ell(\mathbf{z})F^*(\mathbf{z}) = \int_0^{\mathbf{z}_\ell} x_\ell \left(\frac{\partial F^*(\mathbf{z}_1, .., \mathbf{z}_{\ell-1}, x_\ell, \mathbf{z}_{\ell+1}, ..., \mathbf{z}_L)}{\partial x_\ell} \right) dx_\ell,$$

which on differentiation with respect to $\mathbf{z}_\ell$ yields $(\partial h_\ell(\mathbf{z})/\partial \mathbf{z}_\ell)F^*(\mathbf{z}) + h_\ell(\mathbf{z})(\partial F^*(\mathbf{z})/\partial x_\ell) = \mathbf{z}_\ell(\partial F^*(\mathbf{z})/\partial x_\ell)$. Let $\eta_\ell(\mathbf{z}) \equiv \partial h_\ell(\mathbf{z})/\partial \mathbf{z}_\ell$ denote the derivative function of $h_\ell(\mathbf{z})$. A convenient way to write the previous expression is $\partial \ln F^*(\mathbf{z})/\partial x_\ell = \eta_\ell(\mathbf{z})/[\mathbf{z}_\ell - h_\ell(\mathbf{z})]$. Integration yields

$$F^*(\mathbf{z}_{(\ell)}, x_\ell) = \Phi_\ell(\mathbf{z}_{(\ell)}) \exp \left\{ - \int_{x_\ell}^{+\infty} \left(\frac{\eta_\ell(\mathbf{z}_{(\ell)}, u)}{u - h_\ell(\mathbf{z}_{(\ell)}, u)} \right) du \right\}, \tag{B2}$$

where $F^*(\mathbf{z}_{(\ell)}, x_\ell) \equiv F^*(\mathbf{z}_1, .., \mathbf{z}_{\ell-1}, x_\ell, \mathbf{z}_{\ell+1}, ..., \mathbf{z}_L)$ and similarly for $h_\ell(\mathbf{z}_{(\ell)}, u)$ and $\eta_\ell(\mathbf{z}_{(\ell)}, u)$, and with $\Phi_\ell(\mathbf{z}_{(\ell)}) = \Phi_\ell(\mathbf{z}_1, .., \mathbf{z}_{\ell-1}, \mathbf{z}_{\ell+1}, ..., \mathbf{z}_L)$ determined such that $F^*(x)$ is a distribution function. This distribution $F^*(x)$ can be found by solving the L simultaneous equations (B2).

with support $\{\mathbf{z}_1, ..., \mathbf{z}_L\}$. Therefore,

$$\int_C z d\mu(z) = \sum_{i=1}^{L} \alpha_i \int_C z d\delta_{\mathbf{z}_i}(z) = \sum_{i=1}^{L} \alpha_i \mathbf{z}_i = x.$$

Uniqueness can be established generally for the case of a simplex; see Phelps (2001, Chapter 10). ■

To apply the previous representation it is enough to use Lemma 2 and to notice that the budget set (13) is convex and compact; see Mas-Colell et al. (1995, p. 22).[21]

Supply and market equilibrium. In this sub-section, I present a simple partial equilibrium analysis in competitive markets to outline the nature of results that random choice may yield in market situations. The purpose of this section is to show that the nice properties of random choice also apply to the analysis of supply curves.[22]

Consider the market of commodity X_1. The demand side is determined by Proposition 1 and the supply side can be derived based on maximizing behavior or as the outcome of randomization. Let q denote the amount of output produced of this commodity. For a given price p, profits are $pq - K - C_v(q)$, where K and $C_v(q)$ denote the fixed cost and the variable cost function respectively. Factor prices are held constant and costs are well-behaved. Let $\mathbf{Q}^{\min}(p)$ denote the *minimum scale* (i.e., the "shut down" point) determined by $p = C_v(\mathbf{Q}^{\min}(p))/\mathbf{Q}^{\min}(p)$. Notice that $\partial \mathbf{Q}^{\min}(p)/\partial p$ is positive along increasing average costs.

Suppose that firms randomly select production levels Q using a density function $f(q)$. Each randomly selected production plan must satisfy $Q \geq \mathbf{Q}^{\min}(p)$. Let $\bar{q}(p)$ denote the

[21] The *Choquet's Integral Representation Theorem*, extensively discussed in Phelps (2001), examines the general representation of elements of a compact convex set as integral averages over the extreme points of the set. This integral representation is valid in infinite dimensional spaces and it has a probabilistic interpretation analogous to the probabilistic interpretation of Theorem 4.

[22] These discussions are motivated in part by Kirzner's (1962) earlier criticisms to models of irrational behavior. Essentially, Kirzner (1962) suggested that leaving out the supply side of the market considerably weakens any conclusion one can draw about random choices.

mean supply curve. This curve results from the truncation of unfeasible production plans:

$$\bar{q}(p) \equiv \mathbb{E}[Q|Q \geq \mathbf{Q}^{\min}(p)] = \frac{\displaystyle\int_{\mathbf{Q}^{\min}(p)}^{+\infty} qf(q)dq}{1 - F(\mathbf{Q}^{\min}(p))}, \tag{B3}$$

where $1-F(\mathbf{Q}^{\min}(p)) \equiv \displaystyle\int_{\mathbf{Q}^{\min}(p)}^{+\infty} f(q)dq$. Thus, any production plan in $[\mathbf{Q}^{\min}(p), +\infty)$ may be selected depending on $f(q)$ but any production plan in $[0, \mathbf{Q}^{\min}(p))$ must be discarded.

The comparative statics properties with respect to prices satisfy:

$$\frac{\partial \bar{q}(p)}{\partial p} = [\bar{q}(p) - \mathbf{Q}^{\min}(p)] \left(\frac{f(\mathbf{Q}^{\min}(p))}{1 - F(\mathbf{Q}^{\min}(p))} \right) \frac{\partial \mathbf{Q}^{\min}(p)}{\partial p} > 0. \tag{B4}$$

The first two terms in (B4) are positive due to the statistical properties of the *left-truncated mean* formula and have interpretations that are analogous to those given in Section 2. The last term contains all the economic information needed to determine the slope of the supply curve under randomization. As I noted before, this term is positive if average costs are increasing and so in this case the mean supply curve will be positively sloped even in the absence of maximizing behavior.[23]

Proposition 1 and (B4) also imply that there is a single equilibrium in the market for commodity X_1 which can be identified as the intersection point of the mean demand and supply curves. This finding confronts the criticism of Kirzner (1962) and it implies that changes in underlying market conditions affect equilibrium outcomes in the expected way even under "irrational behavior."

A simulation exercise. A key feature in all previous derivations is that mean demand curves are defined from the aggregation of individual choices. In here, I explore a simple simulation exercise whose purpose is to determine how 'large' the economy needs to be in order to observe consistent results. Assume two goods and non-interior choices. Assume also that p_2 and w are constant throughout. At a fixed price level p_1 let X_1^j, $j = 1, ..., n$ be an i.i.d. sequence of uniform random variables on $[0, w/p_1]$. Each j

[23] Under the assumption of profit maximization, firms will choose to produce at a level $q^*(p)$ such that $p = \partial C_v(q^*)/\partial q$ as long as $q^* \geq \mathbf{Q}^{\min}(p)$, see, e.g., Mas-Colell et al. (1995, Section 5.D).

represents an individual realization of demand for good X_1. The mean demand is:

$$\bar{x}_1^n(p,w) = \frac{1}{\#n}\sum\nolimits_{j=1}^{n} X_1^j(p,w),$$

which, by the *strong law of large numbers*, satisfies $\bar{x}_1^n(p,w) \rightarrow \bar{x}_1(p,w)$. Moreover, when prices p_1 change, one can trace an uncompensated demand curve whose elasticity should be $|\varepsilon| = 1$.

In the following simulations $w = 1$ and p_1 varies from 1 to 2. The results consider two incremental steps. The first is 0.005 and the second 0.05. This means that each individual j has 200 realizations of demand in the first case and 20 realizations in the second case. The average demand is computed for different values of n that range from $n = 1$ to $n = 1000$. In each sample, and for each value of n, a log-log linear regression estimates the elasticity of the demand curve. The number of cross-samples is 500.

Table B1. Simulation results.

	Number of individuals aggregated							
	n=1	n=5	n=10	n=25	n=50	n=100	n=500	n=1000
	A. Grid size for p_1 of 200 sample points							
$\lvert\hat{\varepsilon}\rvert$	1.0086	1.0095	1.0061	1.0005	1.0011	1.0015	1.0007	1.0001
std.err.	(0.349)	(0.100)	(0.067)	(0041)	(0.029)	(0.020)	(0.009)	(0.006)
std.dev.	[0.345]	[0.102]	[0.067]	[0.040]	[0.029]	[0.020]	[0.008]	[0.006]
R^2	0.0396	0.3343	0.5242	0.7423	0.8538	0.9219	0.9834	0.9917
	[0.027]	[0.054]	[0.043]	[0.026]	[0.014]	[0.008]	[0.001]	[0.0008]
	B. Grid size for p_1 of 20 sample points							
$\lvert\hat{\varepsilon}\rvert$	1.101	0.9901	0.9966	0.9977	0.9992	1.0013	0.9998	1.0002
std.err.	(0.954)	(0.276)	(0.188)	(0.118)	(0.083)	(0.058)	(0.026)	(0.018)
std.dev.	[1.090]	[0.297]	[0.208]	[0.125]	[0.085]	[0.061]	[0.027]	[0.018]
R^2	0.0562	0.3586	0.5497	0.7633	0.8672	0.9316	0.9857	0.9928
	[0.109]	[0.160]	[0.146]	[0.075]	[0.043]	[0.022]	[0.004]	[0.002]

Note: The number of cross-samples is 500. The elasticities are estimated using a linear fit to $\log[\bar{x}_1^n(p,w)]$ and $\log[p_1]$. The average value of the standard errors across samples is in parentheses. The cross-sample standard deviation for the estimate of the elasticity of demand and the R^2 is in brackets.

Table B1 shows that 'individual' demands are on average negatively sloped with an elasticity consistent with the predicted pattern. The estimates of the elasticity, however, are unreliable when only one individual realization is considered and the sample variation in prices is small. In fact, the elasticity cannot be statistically distinguished from zero in this case. Further, the standard deviation across and between samples is 1.090 and 0.954 respectively. Both estimates suggest that some individuals have a positively-sloped demand curve. Finally, the goodness of fit from the R^2 is, on average, about 5 percent. Thus, overall, individual demands are not consistently determined.

Consider the two cases in which $n = 5$. In these cases, the goodness of fit increase to over 30 percent and the estimates of the elasticity of demand become (statistically) close to $|\hat{\varepsilon}| = 1$. When the sample variation in prices is 200, and $n = 10$, the across sample standard deviation for the elasticity of demand is about 0.06. The goodness of fit and the statistical significance of the estimates also suggest that average demands are precisely estimated. When only 20 sample points for p_1 are considered, a similar conclusion follows if the aggregation takes place for 50 individuals. As expected, with $n = 500$ or $n = 1000$, the goodness of fit is well over 98 percent and the elasticity is precisely estimated up to three digits.

In conclusion, Table B1 shows that with enough price variation, the number of individuals needed to observe a consistent mean demand curve is small, $n = 10$. With fewer price variations, this number is larger. A market size of $n = 50$ individuals, however, do not seem excessive. That is, while the previous analyses rely on a large number of individuals, numerical approximations with fewer individuals still give support to the main theoretical conclusions of the analysis.

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