Perfect Bayesian Equilibrium

Goals

In the final two weeks:

- Understand what a game of incomplete information (Bayesian game) is
- Understand how to model static Bayesian games
- Be able to apply Bayes Nash equilibrium to make predictions in static Bayesian games
- Understand how to model sequential Bayesian games
- Be able to apply perfect Bayesian equilibrium to make predictions in sequential Bayesian games
- Experience a sampling of the diverse applications to which these concepts can be applied

We only have 2 weeks, so let's make the most of it!

Why did we need SPNE?

This game has two NE, (I, A) and (O, F), but (O, F) is not sequentially rational:

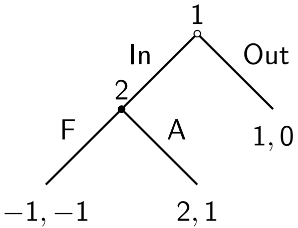
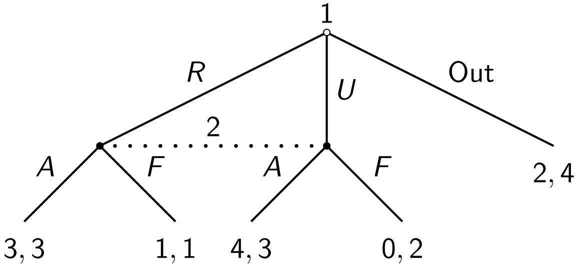


Figure: The entry game

SPNE is useful because it eliminates NE that involve subgame behavior that is not rational.

A Challenge for SPNE

Now consider this variation of the entry game:



Now 1 can not only choose whether or not to enter, but also whether or not to prepare for a fight. Preparation is costly, but it reduces the cost of a fight.

A Challenge for SPNE

How to make predictions in this game?

- Convert to normal form → two NE: (U, A) and (O, F)
- In original game, (O, F) was implausible and not subgame perfect. What about here?
- Intuitively, should be the same: once 2's information set is reached, F is never optimal
- (O, F) "shouldn't" be subgame perfect... but it is!
- Why? 2's information set does not start a subgame, so according to the precise definition, both NE are subgame perfect

Our definition is failing us: (O, F) is just as implausible here, but SPNE definition does not exclude it here, where there is imperfect information.

A Challenge for SPNE

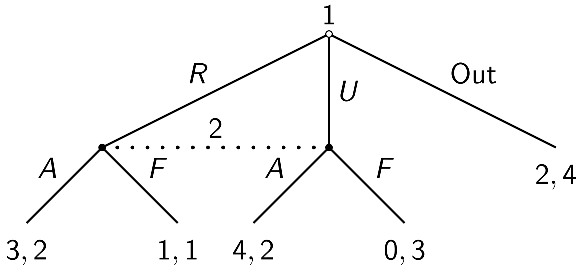
We need a new concept

- Should maintain intuition of SPNE:
  - Everyone should be acting optimally
  - Even off the equilibrium path
- But can be applied to *all* games of imperfect info
- Everyone is acting optimally *at all information sets*

This is not so straightforward

A Challenge for SPNE

Now suppose 2 prefers to fight if challenger is unready, but not if she is ready:



- Normal form: only NE is (O, F)
- But is F optimal at Player 2's information set?
- Depends on what 2 believes 1 chose

Solution should consider players' beliefs about what others chose

To develop new concept: go back to basics

Original concept, NE, requires:

- Players choose an optimal strategy given their beliefs about what others are doing and
- These beliefs are correct

New concept keeps these requirements, *plus* they must hold true at every point at which you make a choice

- Kind of like SPNE
- In simultaneous games $\rightarrow$ your beliefs and others' strategies are exactly identical

Here, however, the other person's strategy does not completely pin down beliefs

Incorporating beliefs into equilibrium

New concept, *perfect Bayesian equilibrium*, is defined for pairs of strategy profiles and beliefs. What is a belief?

Definition

A *belief system* in an extensive game is a function that assigns to each information set a probability distribution over the histories in that information set.

- In other words, a beliefs system reflects how people think they arrived at each of their information sets.
- Example: in entry game variant, one beliefs (system) for 2 at her information set is $\Pr(R) = 0.2$
- One strategy profile may be an equilibrium for one set of beliefs, but not another

We call a strategy profile combined with a beliefs system in an extensive game an *assessment*

Perfect Bayesian Equilibrium defined

Definition

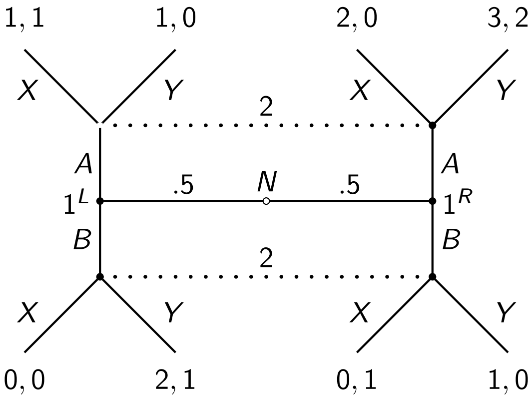
An assessment in an extensive game is a *perfect Bayesian equlibrium* if

- **Sequential Rationality**: Each player's strategy specifies optimal actions, given her beliefs and the strategies of the other players and
- **Consistency of Beliefs**: Each player's belief is consistent with the strategy profile (following Bayes rule, where appropriate).

Let's apply this to the games we've seen

Practice with PBE

We found two NE in this game: (AA, YX) and (BA, YY)



What can we say about the PBE for this game?

Signaling games

This type of game is called a *signaling game*: 1's action sends a signal about her type

- Note: two categories of potential perfect Bayesian equilibria
- **Pooling**: (AA, YX)
  - Both types of player 1 send the same signal
  - Observing A is uninformative, beliefs about who chose A ($\Pr(L|A)$) constrained by prior probabilities
  - B is a zero-probability event, $\Pr(L|B)$ can be anything
- **Separating**: (BA, YY)
  - Two types of player 1 send different signals
  - Signals are perfectly informative
  - $\Pr(L|A) = 0$ and $\Pr(L|B) = 1$

The information conveyed by the signal is part of/determined in equilibrium, must be consistent with actions

Signaling games

In our example, let's check for pooling, separating PBE?

- What beliefs can go with (AA, YX)?
  - $\Pr(L|A) = 0.5$
  - $\Pr(L|B)$ can be anything, but...
  - For X to be a BR to B, we need $\Pr(L|B) \leq 1/2$
  - A range of PBE: $[(AA, YX), \Pr(L|A) = 0.5, \Pr(L|B) \leq 1/2]$
  - Same behavior, but range of possible beliefs
- What beliefs can go with (BA, YY)?
  - Only $\Pr(L|A) = 0$ and $\Pr(L|B) = 1$

This game has equilibria in which both types pool on A and a separating equilibrium in which L type chooses B

Signaling games

In general, follow these steps to find PBE in a signaling game:

- Start with a strategy for 1 (2 ways to pool, 2 ways to separate)
- Write down what beliefs have to be following each signal (can be anything for signal never supposed to be sent)
- Given beliefs, calculate 2's best response to each signal
- Check whether 1's strategy is optimal, given how 2 will respond.

Let's try this

Application: Westley and Humperdinck

- Westley is weak or strong, equally likely, known only to him
- Westley chooses to get out of bed (O) or stay in bed (B)
- Humperdinck observes Westley's action, but not type
- Humperdinck must choose to fight (F) or surrender (S)
- H gets 0 for S, 1 from F if W is weak, -2 if strong
- Weak W (only) pays cost c to get out of bed
- W gets 1 from S, 0 from F if strong, -1 if weak

Application: Westley and Humperdinck

- Draw extensive form

- For what values of c is there a separating equilibrium?
- Describe such an equilibrium
- For what values of c is there a pooling equilibrium in which W always gets out of bed?
- Describe such an equilibrium
- What does the conclusion of the scene tell you about the value of c?

Can you describe a pooling equilibrium in which both types choose B? What is weird/unattractive about this equilibrium?

Application: Westley and Humperdicnk

First examine OO as part of a potential pooling equilibrium:

- Beliefs: $\Pr(w|O) = 0.5$ and $\Pr(w|B) =???$
- H's response to his beliefs:
  - Given O, $EU(F) = -0.5$ and $EU(S) = 0$ so choose S
  - Given B, choose S if $\Pr(w|B) \leq 0.5$, F if $\Pr(w|B) \geq 0.5$
- Is W's OO optimal?
  - Given SS: **NO** because weak W would rather stay in bed
  - Given SF: O is optimal for strong W, but, if weak, only if $c \leq 2$

Only for $c \leq 2$ is there an OO pooling equilibrium:
$[(OO, SF), \Pr(w|O) = 0.5, \Pr(w|B) \geq 0.5]$

Application: Westley and Humperdicnk

Next examine BO as a potential separating equilibria:

- Beliefs: $Pr(w|O) = 0$ and $\Pr(w|B) = 1$
- H's response to his beliefs
    - Given O, S
    - Given B, F
- is W's BO optimal given SF?
    - O is optimal for strong W, but, if weak, B only optimal if $c \geq 2$

Only for $c \geq 2$ is there an BO separating equilibrium:
$[(BO, SF), \Pr(w|O) = 0, \Pr(w|B) = 1]$