Bayesian Nash Equilibrium

Goals

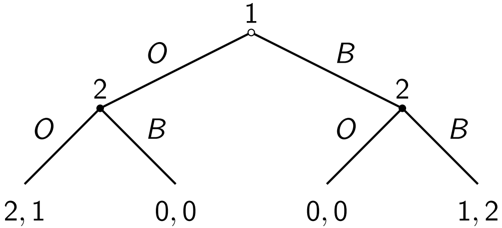
In the final two weeks:

- Understand what a game of incomplete information (Bayesian game) is
- Understand how to model static Bayesian games
- Be able to apply Bayes Nash equilibrium to make predictions in static Bayesian games
- Understand how to model sequential Bayesian games
- Be able to apply perfect Bayesian equilibrium to make predictions in sequential Bayesian games
- Experience a sampling of the diverse applications to which these concepts can be applied

We only have 2 weeks, so let's make the most of it!

Private vs. Public Information

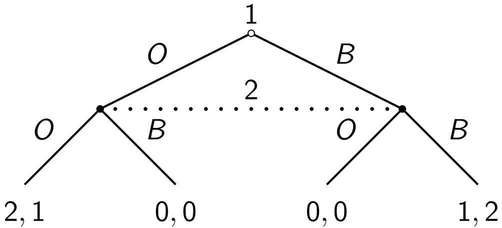
In Sequential BoS, all information is public, meaning everyone can see all the same information:



. . . but in many strategic situations, one player might know something that other players do not know.

Private vs. Public Information

Example: In this extensive-form representation of regular BoS, Player 2 cannot observe the action chosen by Player 1.



When one player knows something that others do not, we call this *private* or *asymmetric* information.

Imperfect Information

- There are different forms of asymmetric information
- When one player does not observe another player's actions, we call this *imperfect information*
- We can treat all simultaneous move games as games with *imperfect information*

Game of imperfect information are important, but in many strategic settings there is private information about other things as well.

Recall: BoS variant

Player 1 is unsure whether Player 2 wants to go out with her or avoid her, and thinks that these two possibilities are equally likely. Player 2 knows Player 1's preferences. So Player 1 thinks that with probability 1/2 she is playing the game on the left and with probability 1/2 she is playing the game on the right.

	O	B
O	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

2 wishes to meet

	O	B
O	$2, 0$	$0, 2$
B	$0, 1$	$1, 0$

2 wishes to avoid

This is an example of a game in which one player does not know the payoffs of the other.

Private Information

More examples:

- Bargaining over a surplus and you aren't sure of the size
- Buying a car of unsure quality
- Job market: candidate is of unsure quality
- Juries: unsure whether defendant is guilty
- Auctions: sellers, buyers unsure of other buyers' valuations

When some players do not know the payoffs of the others, a game is said to have *incomplete information*. It's also known as a *Bayesian game*.

In-class game 1: First-price auction

Let's see what it's like to play a game with *incomplete information*

- I have a copy of the Mona Lisa that I want to sell for cash
- Each of you has a *private* valuation for the painting, only known to you
- I will auction it off to the highest bidder
- Everyone submits a bid (sealed $\rightarrow$ simultaneous)
- Highest bidder wins the painting, pays *their bid*
- If tie, I'll flip a coin

Note: Thanks to everyone who participated in the Moblab trials.
Results + $$$ promised in a future class.

In-class game 2: Second-price auction

- I have one copy of the Mona Lisa that I want to sell for cash
- Each of you has a *private* valuation for the painting, only known to you
- I will auction it off to the highest bidder
- Everyone submits a bid (sealed $\rightarrow$ simultaneous)
- Highest bidder wins the painting, pays *the second-highest bid*
- If tie, I'll flip a coin

How do we model Bayesian games?

One approach:

- Random events are considered an act of nature (that determine game structure)
- Treat nature as another (non-strategic) player
- Draw nature's decision nodes in extensive form

Treat game as extensive form game with *imperfect info*: players may/may not observe *nature's* action

Recall: BoS variant

Player 1 is unsure whether Player 2 wants to go out with her or avoid her, and thinks that these two possibilities are equally likely. Player 2 knows Player 1's preferences. So Player 1 thinks that with probability 1/2 she is playing the game on the left and with probability 1/2 she is playing the game on the right.

	O	B
O	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

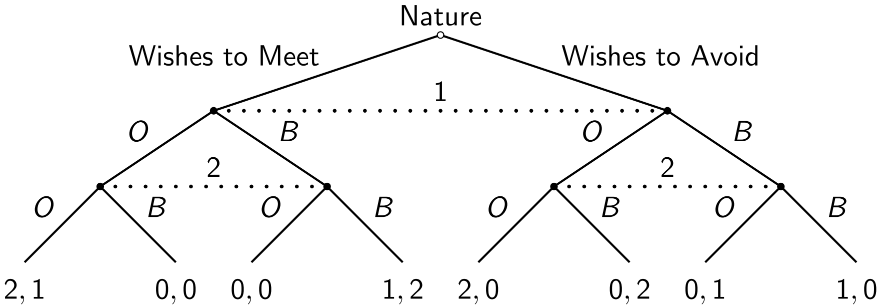
2 wishes to meet

	O	B
O	$2, 0$	$0, 2$
B	$0, 1$	$1, 0$

2 wishes to avoid

Let's put this into extensive form

BoS variant in extensive form



How do we model Bayesian games?

How do we make predictions? Think back to basics of extensive form:

- Convert to (Bayesian) normal form
- Note: Nature's choice determines what type of person Player 2 is $\rightarrow$ 2's strategy specifies action of both
- Note: we have to write down the *expected* payoffs if there is uncertainty
- Try for Player 1 in BoS variant

How do we model Bayesian games?

The Bayesian normal form of our variant of BoS with incomplete information:

	OO	OB	BO	BB
O	2	1	1	0
B	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Figure: The expected payoffs of Player 1

How do we model Bayesian games?

How do we make predictions? Think back to basics of extensive form:

- Convert to (Bayesian) normal form
- Note: Nature's choice determines what type of person Player 2 is $\rightarrow$ 2's strategy specifies action of both
- Note: we have to write down the *expected* payoffs if there is uncertainty
- Try for Player 1 in BoS variant
- Now do the same for Player 2, but. . .
- Fill in payoff for each type, put in ordered pair in each cell

How do we model Bayesian games?

The Bayesian normal form of our variant of BoS with incomplete information:

	OO	OB	BO	BB
O	$2, (1, 0)$	$1, (1, 2)$	$1, (0, 0)$	$0, (0, 2)$
B	$0, (0, 1)$	$\frac{1}{2}, (0, 0)$	$\frac{1}{2}, (2, 1)$	$1, (2, 0)$

Figure: The expected payoffs of Player 1 and the payoffs for each type of Player 2

Bayesian Nash equilibrium

- Bayesian Nash equilibrium = straightforward extension of NE:
- Each *type of* player chooses a strategy that maximizes *expected* utility given the actions of all *types of* other players *and that player's beliefs about others' types*
- In our BoS variant:
  - Action of Player 1 is optimal (maximizes expected utility) given the actions of the two types of Player 2 (and Player 1s beliefs about Player 2s type)
  - Action of each type of Player 2 is optimal, given the action of Player 1

Claim: (O, OB) is a Bayesian Nash equilibrium

Warm up I: BoS or PD?

You and a friend are playing a 2×2 matrix game, but you're not sure if it's BoS or PD. Both are equally likely.

	A	B
A	$2, 2$	$0, 3$
B	$3, 0$	$1, 1$

PD

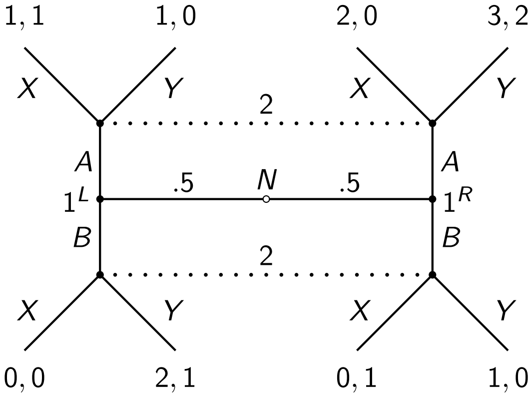
	A	B
A	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

BoS

Clicker vote: (2nd to last digit of perm) even = Player 1; odd = Player 2

Warm up II

Can you put this game into Bayesian normal form?



Second price auction with private values

- Suppose v_i is distributed uniformly over $[0, 1]$
- Key insight: that bidding one's value is a weakly dominant strategy
  - Suppose that the highest other bid, $\bar{b}$, is above your valuation.
  - Suppose that $\bar{b}$ is below your valuation.
  - What about ties?
- So there is a Nash equilibrium in which everyone bids her valuation

This auction format is successful (in theory) because it identifies the buyer with the highest valuation and awards the object to her. However, she does not pay her full valuation for it

First price auction with private values

How can the seller squeeze a bit more revenue out of the winning buyer?

- Perhaps she should make the winner pay her own bid?
- That is what happens in a first price auction
- As a bidder, you have two considerations:
  - you want to win
  - conditional on winning, you want to pay as little as possible

First price auction with private values

- You never would want to bid more than your valuation, because then if you win, you actually lose money
- It turns out that there is an equilibrium in which everyone bids some fraction of their valuation, q
- The question is, what is the appropriate fraction?
- Suppose that for all i, $b_i = qv_i$.
- It must be the case that for the appropriate value of q, this strategy maximizes each player's utility, given that everyone else is using this strategy.
- So what is player i's utility?

First price auction with private values

$$U(b_i|v_i) = (v_i - b_i)\,\Pr(\text{win}|b) = (v_i - b_i)\,\Pr(b_i > qv_j \text{ for all } j \neq i)$$

$$= (v_i - b_i)\,\Pr(b_i > qv_j)^{n-1} = (v_i - b_i)\,\Pr(v_j < \frac{b_i}{q})^{n-1} = (v_i - b_i)(b_i/q)^{n-1}$$

$$\frac{1}{q^{n-1}}[vb_i^{n-1} - b_i^n]$$

Differentiating and setting equal to zero yields

$$b = \frac{n-1}{n} v_i,$$

so $q = \frac{n-1}{n}$. Thus everyone bids less than her value, trading off the probability of winning with the possibility of making a profit if she wins. The more people there are, the closer she bids to her value.