

# Equilibrium Default

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## Abstract

We model the optimal financing of a firm with limited commitment. Entrepreneurs need outside funding for projects, but can default at any moment and take some outside opportunity. The value of this opportunity is random and not observable by the lender. We show that the optimal dynamic contract may allow default along the equilibrium path. Focusing on the dynamics of default, debt and capital accumulation, we find that over the life of the project the probability of default declines, long-term debt falls and capital rises.

## Introduction

This paper derives the optimal dynamic contract that finances a firm with limited commitment. An entrepreneur has a project that needs outside funds to cover an initial fixed cost as well as a stream of capital investments. Lending is constrained by voluntary repayment: the entrepreneur can, at any moment, walk away for some outside opportunity. The value of the outside opportunity is random and privately observed by the entrepreneur. Within this environment, we study constrained-efficient long-term contracts between the outside lender and the firm's entrepreneur. That is, we place no ad hoc assumptions on contracting,

but impose the constraints arising from the entrepreneur's private information and lack of commitment.

We assume that outside opportunities are unattractive relative to the investment project, so that it is never efficient to take them. With commitment the outside opportunities are completely irrelevant. However, without commitment, the availability of the outside opportunities does constrain lending because the entrepreneur may take leave for them because when they are privately, but not socially, efficient.

When the value of the outside option is non random or is publicly observed, as in Albuquerque and Hopenhayn (2004), then it is never optimal to exercise it. Default never occurs along the equilibrium path. This property is also characteristic of other limited commitment models, such as in applications to consumption insurance and smoothing among individuals (Kocherlakota, 1996; Alvarez and Jermann, 2000), or international borrowing and lending by sovereign countries (Thomas and Worrall, 1994; Kehoe and Perri, 2002). While all these models capture the friction that ex post *potential* default introduces on ex ante lending, they are completely silent on *actual* default behavior.

In our model, in contrast, default occurs along the equilibrium path. Whenever the value of the outside opportunity is random, and its realization is not observed by the lender, we show that the optimal contract allows firms to exercise the outside option. The contract treats all firms identically, but over time some firms default and choose to exit the contract, while others choose to stay.

Our focus is on the dynamics of capital, debt and default. The optimal contract offered by the lender is forward looking, so it specifies and anticipates all these variables. We find that the probability of default is highest at the beginning of the lending relationship and declines over time. Capital rises over time as the entrepreneur pays back the long-term debt to the lender for initiating the project. The lender may play an increasing role in the financing of capital investments through short-term debt.

Our environment is closest to Albuquerque and Hopenhayn (2004) model of firm finance.

Their model features limited commitment, but all uncertainty is publicly observable by the lender. Clementi and Hopenhayn (2006), on the other hand, considers unobservable shocks to revenue, but with no commitment problems. In both papers, if liquidation is assumed valuable enough to the lender, then exit may occur after a sufficiently bad history of shocks, but there is no default. For contrast, we assume in our model that the liquidation value to the lender is zero: the interaction of private information and limited commitment is then key for our results on default.

## 1 An Investment Project in Need of Funding

We study the efficient contract between an agent and a principal. Both are infinitely lived, risk neutral and discount flows at the same rate  $r > 0$ . The agent is needed to run the project, she can exclusively operate the technology. The principal can help finance it by enduring negative flows. In contrast the agent must consume non-negatively and starts with no financial wealth.<sup>1</sup>

The project requires an initial irreversible investment of  $I_0$ . This payment initiates capital at  $k_0$  and opens access to the following stream of profits. At any point in time  $t$ , if  $k(t)$  is the current stock of capital and  $\dot{k}(t) = \frac{\partial}{\partial t}k(t)$  is its rate of change, flow returns are  $R(k(t), \dot{k}(t))$  which capture profits net of investment costs.

We assume the return function  $R(k, \dot{k})$  is concave, and strictly so in at least one of its arguments. When it is strictly concave in  $\dot{k}$  we say there are adjustment costs in capital accumulation. If, on the other hand, it is linear in  $\dot{k}$  there are no adjustment costs and we write  $R(k, \dot{k}) = R(k, 0) - \dot{k}$ . Finally, we impose a condition on the cross partial that ensures there is at most one steady-state level capital at the first-best.

**Assumption 1.** *The return function  $R(k, \dot{k})$  is twice continuously differentiable and concave, with either: (a)  $R(\cdot, \dot{k})$  strictly concave for all  $\dot{k}$ ; or (b)  $R(k, \cdot)$  strictly concave for all  $k$ . In*

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<sup>1</sup>Alternatively, we could assume the principal also consumes non-negatively but holds a large enough initial wealth position.

addition, the cross-partial derivative satisfies  $R_{21}(k, 0) \leq 0$  for all  $k$ .

A simple example satisfying these assumptions is

$$R(k, \dot{k}) = \Phi(k) - \delta k - c(\dot{k}) - \dot{k},$$

where  $\Phi$  is concave and represents revenues,  $\Psi$  is convex with  $\Psi'(0) = 0$  and represents adjustment costs, and  $\delta \geq 0$  is the rate of depreciation. In later sections we develop a few other examples.

The principal may commit to any contractual agreement, but the agent lacks such commitment. Before describing these contracts and the constraints that arise from this lack of commitment, it is useful to study the first-best benchmark allocation that maximizes the present value of profits without any such constraints.

## 1.1 First best

Denote by  $W(k)$  the value of the optimal accumulation problem given some initial capital  $k_0$ :

$$W(k_0) \equiv \sup_{\{k(t)\}} \int_0^\infty e^{-rt} R(k(t), \dot{k}(t)) dt \quad \text{s.t.} \quad k(0) = k_0.$$

Let  $k^*(t)$  for  $t \in [0, \infty)$  denote an optimal trajectory. Since the problem is convex, the value function  $W$  is increasing, continuously differentiable and concave. Its Bellman equation is given by

$$rW(k) = \sup_{\dot{k}} \left( R(k, \dot{k}) + W'(k)\dot{k} \right). \quad (1)$$

Consider the case with adjustment costs, so that  $R(k, \cdot)$  strictly concave. Let  $g(k)$  denote the policy function for capital accumulation  $\dot{k}$  that maximizes (1). Then the optimal trajectory  $k^*(t)$  solves the ordinary differential equation

$$\dot{k}^*(t) = g(k^*(t)) \quad \text{with} \quad k^*(0) = k_0.$$

There exists at most one finite steady state  $k_{ss}$  with  $g(k_{ss}) = 0$ . To see this, combine the first order condition  $R_2(k_{ss}, 0) + W'(k_{ss}) = 0$ , with the envelope condition,  $rW'(k_{ss}) = R_1(k_{ss}, 0)$ , to obtain the steady-state condition

$$\frac{1}{r}R_1(k_{ss}, 0) + R_2(k_{ss}, 0) = 0.$$

Assumption 1 then ensures that the left-hand side is locally decreasing, so there exists at most one value  $k_{ss}$  satisfying this equation. It also ensures the global stability of any such steady state:  $\dot{k} = g(k) > 0$  for  $k < k_{ss}$  and  $\dot{k} = g(k) < 0$  for  $k > k_{ss}$ .

We assume that the project starts with capital below its steady-state level. If no steady state exists, so that  $g(k) > 0$  for all  $k$  then we say that  $k_{ss} = \infty$ .

**Assumption 2.**  $k_0 \leq k_{ss}$ .

Things are particularly simple without adjustment costs, when  $R(k, \cdot)$  is linear. Integrating by parts gives

$$\int_0^\infty e^{-rt} R(k(t), \dot{k}(t)) dt = k_0 + \int_0^\infty e^{-rt} (R(k(t), 0) - rk(t)) dt.$$

Thus, the supremum is attained by a path that is discontinuous at  $t = 0$  and sets capital  $k^*(t)$  to the constant level  $k_{ss}$  that maximizes flow profits net of capital rental,  $R(k, 0) - rk$ .

The value function is linear in initial capital:  $W(k) = k_0 + \frac{1}{r} \max_k \{R(k, 0) - rk\}$ .

In either case, the project is worthwhile investing if and only if  $W(k_0) - I_0 > 0$ . To get things going, we assume this is the case throughout.

**Assumption 3.**  $W(k_0) - I_0 > 0$ .

## 1.2 Outside option

The agent receives outside options which, if taken, leave a residual value of zero to the lender. (This could be modified and a positive liquidation value included.) The outside value

$s$  arrives with Poisson intensity  $\lambda$  and is distributed according to  $F(s, k)$ , which increases in first stochastic order with  $k$ . Notice that aside from the dependence on  $k$  there is no persistence in the process for the shocks. This simplifies the analysis, but is not essential.<sup>2</sup>

When the outside option is publicly observable, this problem has the same structure as Thomas-Worrall and has been studied in Albuquerque-Hopenhayn in a discrete time setting. Our main objective is to depart from the previous literature by assuming that this outside value is privately observed. For contrast, however, we will first consider the observable case.

To make things stark, we assume that the outside option is never good enough to warrant separation: within the support of  $F(\cdot, k)$  it is never efficient to so, for all values of  $k$ .

**Assumption 4.** *If  $F(s, k) \in (0, 1)$  then  $s \leq W(k)$ .*

As we show below, this implies that there will be no liquidation when the outside option is observable. Separations will be optimal, however, when outside options are privately observed by the agent, since avoiding them may be too costly. This emphasizes a novel aspects of our model: default, as inefficient separations, occurs in equilibrium.

### 1.3 Two Interpretations

We offer two economic interpretations of the model.

In the first, investment takes place in physical capital and the contract describes the lending relationship between an entrepreneur/firm/borrower and an outside lender. The value of the outside option may depend on capital to the extent that the entrepreneur is able to walk away with some of the firm’s resources.

In the second, investment takes place in human capital and the contract describes the optimal employment relationship between a worker and her employer. The outside option may be interpreted as a new employment opportunity, the value of which may depend on capital to the extent that human capital is at least partly “general” and is not entirely “firm specific”.

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<sup>2</sup>We also studied a version of the model with persistent shocks and obtained similar results.

## 2 Observable Outside Options: No Default

When the outside option is observable, the principal can always retain the agent by ensuring that she is better off within the relationship. This may require the contract to match outside options as they arise. Assumption 4 ensures that doing so is preferable to breaking the relationship. Thus, default, defined here as an inefficient separation, never occurs in equilibrium.

The problem can be solved recursively using the continuation utility of the borrower  $V$  as a state variable together with the stock  $k$ . Let  $B(V, k)$  denote the value to the lender, which must satisfy the Hamilton-Jacobi-Bellman equation

$$rB(V, k) = \max_{\dot{k}, V(s), \dot{V}, w} \left( R(k, \dot{k}) - w + \lambda \int (B(V(s), k) - B(V, k)) F(ds, k) + B_1(V, k) \dot{V} + B_2(V, k) \dot{k} \right) \quad (2)$$

subject to

$$w \geq 0, \quad (3)$$

$$rV = w + \lambda \int (V(s) - V) F(ds, k) + \dot{V}, \quad (4)$$

$$V(s) \geq s \quad \forall s. \quad (5)$$

Here  $V(s) - V$  represents the promised jumps in continuation utility when an outside offer  $s$  occurs.

For high enough values of  $V$  the first best is attainable as follows. Define  $V_{\min}(k)$  to be the smallest value that can be delivered to the agent assuming  $k$  follows the optimal path  $\dot{k} = g(k)$ . This value satisfies the following Bellman equation with  $w = 0$ :

$$rV_{\min}(k) = \lambda \int_{V_{\min}(k)} (s - V_{\min}(k)) F(ds, k) + V'_{\min}(k) g(k). \quad (6)$$

For any value  $V \geq V_{\min}(k)$  define the dividend schedule  $d(V, k)$  as follows:

$$rV = d(V, k) + \lambda \int_V (s - V)F(ds, k). \quad (7)$$

This schedule will be decreasing in  $k$  and will reach zero precisely at the point where  $V_{\min}(k(t))$  catches up to  $V$ .

**Theorem 2.1.** *For any  $(V, k)$  where  $V \geq V_{\min}(k)$ ,  $B(V, k) = W(k) - V$  so the corresponding investment policy coincides with the optimal one.*

*Proof.* Use the dividend schedule defined above. We show now recursively that  $B(V, k) = W(k) - V$ . Substituting on the right hand side of equation (2)

$$\begin{aligned} rB(V, k) &= \max_{\dot{k}} R(k, \dot{k}) - w + \lambda \int_V (W(k) - s - (W(k) - V(V(s), k)))F(ds, k) - \dot{V} + W'(k)k \\ &= \max_{\dot{k}} R(k, \dot{k}) - [d(V, k) + \lambda \int_V (s - V)F(ds, k)] + \dot{V} + W'(k)k \\ &= r(W(k) - V). \end{aligned}$$

□

**Corollary 2.2.** *The optimal contract is financially feasible if and only if  $W(k_0) - I_0 \geq V_{\min}(k_0)$ .*

Note that at this particular point with  $V_0 = V_{\min}(k_0)$  the value of debt  $B(V, k_0)$  is decreasing in  $V$ . This implies that the value to the lender is maximized at some  $V < V_{\min}(k_0)$ . Under the condition given in the above Corollary, competitive bidding of lenders would lead to an efficient investment policy while monopoly power would lead to underinvestment. If the condition of the corollary is not satisfied, competitive lending would also lead to  $V_0 < V_{\min}(k_0)$ , where  $B(V_0, k_0) = I_0$ . Underinvestment will occur until a sufficiently high outside option occurs that sets the continuation value of the borrower  $V(t) \geq V_{\min}(k(t))$ .

### 3 Unobservable Outside Options: Equilibrium Default

We turn now to the case where the outside value  $s$  is privately observed by the borrower. Assume for now that there is no recall with respect to this outside option, so it is lost if not taken when the arrival occurs. As before, we consider the a recursive representation for the lenders problem with state variables  $(V, k)$ . The key difference to the case of observable outside option is that continuation values for the agent cannot be made contingent on  $s$ , for otherwise all agents would claim having the outside option that maximizes the continuation  $V(\tilde{s})$ . Letting  $\dot{V}$  represent the rate of change of  $V$ , the agent's value evolves according to:

$$rV = w + \lambda \int_V (s - V)F(ds, k) + \dot{V}. \quad (8)$$

Setting  $w = 0$ , which is optimal, defines a function  $\dot{V} = m(V, k) = rV - \lambda \int_V (s - V)F(ds, k)$  which is increasing in  $V$  and decreasing in  $k$ . Let  $h(V, k) \equiv \lambda(1 - F(V, k))$  denote the hazard rate of separation, which is increasing in  $k$  and decreasing in  $V$ . Note that a high current value of capital  $k$  increases the current hazard rate directly, and has a positive indirect effect on future hazard rates through a lower rate of change  $\dot{V}$ .

The value to the lender  $B(V, k)$  satisfies the following Bellman equation:

$$rB(V, k) = \max_{\dot{k}, w} \left( R(k, \dot{k}) - w - h(V, k)B(V, k) + B_1(V, k)m(V, k) + B_2(V, k)\dot{k} \right), \quad (9)$$

giving first order conditions for capital accumulation:

$$R_2(k, \dot{k}) + B_2(V, k) = 0. \quad (10)$$

This equation implicitly defines the policy function  $\dot{k} = g(V, k)$ .

It would be useful to know whether the value function is concave in  $V$  for given  $k$ . Although we cannot answer this question in general, we can relate the local curvature of  $V$  to the dynamics of  $V$  and  $k$ . Note that  $m_1(V, k) = r + h(V, k)$ . Using this and applying

the envelope theorem to (9) the above is obtained. The value function  $B(V, k)$  satisfies the following equation:

$$-h_1(V, k)B(V, k) + B_{11}(V, k)m(V, k) + B_{21}(V, k)\dot{k} = 0 \quad (11)$$

The first order condition implies that  $B_{21}(V, k) = -R_{22}(V, k)\frac{\partial k}{\partial V}$ , which is nonnegative if  $\dot{k}$  is increasing in  $V$  or if  $R$  is linear in  $\dot{k}$ , e.g. no adjustment cost. If in turn  $\dot{V} \geq 0$  equation (11) implies that  $B_{11} \leq 0$ .

## 4 Two Cases Solved

This section considers two special cases. In the first scenario, the distribution of outside options  $F$  is independent of  $k$ . For example, perhaps installed physical capital cannot be taken by the borrower upon default. Or in the human-capital accumulation case, human capital may be entirely firm specific. This case provides a stark contrast between the situation where the outside option is publicly observable and the one where it is not.

In the second scenario, returns and outside values are linear in  $k$ . Homogeneity allows us to reduce the state variables of the contracting problem, which delivers a sharper characterization.

### 4.1 Outside Value Independent of Capital

When  $F$  is independent of  $k$  and outside options are observable, the lack of commitment plays no role: either the project implemented with first-best capital accumulation, or not at all, and this choice is fully efficient. This follows because  $V_{\min}(k)$  is independent of  $k$ . If, in addition, there are no adjustment costs, then there are no transitional dynamics. Capital is started immediately at its steady state level.

This is no longer the case when the outside option is privately observed by the agent. When combined with the agent's lack of commitment, inefficient separations or default then

arise. This then affects capital accumulation as well as the financing decision. It is no longer crucial that outside options depend on capital  $k$  for the lack of commitment to create losses in efficiency. Finally, even without adjustment costs, there are now nontrivial transitional dynamics towards the steady state.

All this is in sharp contrast to the case with a publicly observable outside option, so this case helps highlights the role of our assumption that outside options are private information.

Equation (8) specializes to:

$$rV = w + \lambda \int_V (s - V)F(ds) + \dot{V} \quad (12)$$

which integrating by parts can be written as:

$$rV = w + \lambda \int_V h(s)ds + \dot{V}. \quad (13)$$

where  $h(s) \equiv \lambda(1 - F(s))$ . As in Albuquerque-Hopenhayn it is optimal to set  $w = 0$  until  $V$  reaches  $\bar{s}$ . (At that point a constant wage  $w = r\bar{s}$  can be used and the first-best is attained). The lower bound  $V_{\min}$  for  $V$  is obtained setting  $\dot{V} = 0$  and it is the same described above. For any initial value  $V_0 > V_{\min}$  the above differential equation with  $w = 0$  determines an increasing path  $V(t)$  that converges to  $\bar{s}$ . The rise in  $V$  determines a decreasing path  $h(t)$  for the hazard rate of separation. Let  $S(t) = e^{-\int_0^t h(\tau)d\tau}$  denote the corresponding survival function. The optimal accumulation path from  $(k_0, V_0)$  then solves:

$$\max_{k(t)} \int e^{-rt} S(t) R(k(t), \dot{k}(t)) dt$$

This is a fairly standard accumulation problem with a risk adjusted discount factor. It can be shown that the optimal accumulation rule satisfies:

$$(r + h(t))R_2(k, \dot{k}) = -R_1(k, \dot{k}) + \frac{d}{dt}R_2(k, \dot{k}) \quad (14)$$

Intuitively, delaying investment saves on the interest and hazard of loss of the corresponding adjustment cost. This is weighted against the loss of marginal revenue and changes in the adjustment cost. Note that in the special case with no adjustment cost where  $R(k, \dot{k}) = f(k) - \dot{k}$  this condition reads  $(r + h(t)) = f'(k)$ , a standard investment equation relating the stock of capital to the risk adjusted interest. For that case, the stock of capital rises over time to the stationary optimal level as the risk of default decreases to zero.

## 4.2 Homogeneous Returns and Outside Options

In this section we consider a specification where the outside value is affected by  $k$  in a linear fashion and where the revenue function  $R$  is homogeneous of degree one in  $(k, \dot{k})$ . That is, the outside option has the form  $sk$ , where  $s$  has c.d.f.  $F(ds)$ . Note that with this specification, the hazard rate of separation  $h(V, k) = \lambda(1 - F(\frac{V}{k}))$  is homogenous of degree zero in  $(V, k)$  and the function  $m(V, k)$  is homogenous of degree one. The value and policy functions inherits this property.

**Proposition 4.1.** *The value function  $B(V, k)$  and the policy function  $g(V, k)$  are homogeneous of degree one in  $(V, k)$ .*

We can exploit the homogeneity to rewrite the Bellman equation as a function of the ratio  $v = V/k$ :

$$rb(v) = \max_{\dot{k}} R(1, \dot{k}) - w - h(v, 1)b(v) + b'(v)m(v, 1) + (b(v) - b'(v)v)\dot{k}$$

where  $b(v) = B(v, 1)$ . The optimal choice in the rate of growth  $\dot{k} = g(v)$  solves:

$$R_2(1, \dot{k}) + b(v) - b'(v)v = 0. \tag{15}$$

$$R_{22}(1, \dot{k}) \frac{\partial \dot{k}}{\partial v} = b''(v) \tag{16}$$

so  $\frac{\partial \dot{k}}{\partial v}$  has the opposite sign of  $b''$ . Finally, the envelope condition for this problem is

$$-h_1(v, 1)b(v) + b''(v)(m(v, 1) - vk) = 0. \quad (17)$$

Note that

$$\dot{v} = \frac{\dot{V}}{k} - \frac{V \dot{k}}{k^2} = \frac{1}{k}(m(v, 1) - vk) \quad (18)$$

so the envelope condition can be rewritten as:

$$-h_1(v, 1)b(v) + b''(v)\dot{v} = 0, \quad (19)$$

showing that the sign of  $b''$  is the opposite of  $\dot{v}$ . This proves the following.

**Proposition 4.2.** *If  $B(v)$  is concave, both  $v$  and  $\dot{k}$  increase over time and the hazard rate decreases.*

### 4.3 Simulations

This section presents numerical computations of the two special cases considered above. The first set of simulations corresponds to the case of no outside options and the second to the linear case.

#### 4.3.1 Outside option independent of $k$

The environment is as follows. The return function  $R(k, \dot{k}) = f(k) - \delta k - \dot{k}$ , where  $f(k) = k^{0.5}$  and  $\delta$  corresponds to the depreciation rate. Letting  $k^*$  denote the optimal steady state capital stock, the optimal value in absence of constraints is

$$W = \frac{f(k^*) - (r + \delta)k^*}{r},$$

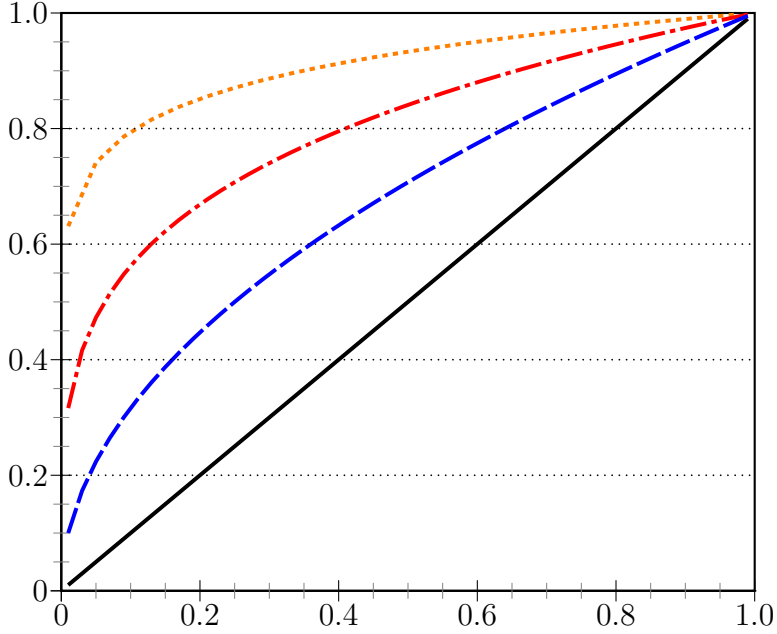


Figure 1: Distributions of outside options:

where  $f'(k^*) = r + \delta$ . The interest rate used is  $r = 5\%$  and depreciation rate  $\delta = 5\%$ . Outside values are given by a distribution function  $F(s) = (s/\bar{s})^\gamma$ , where  $\bar{s}$  corresponds to the upper bound of the support that we chose equal to  $W$ , which equals 50 with the above parameter values. This is the highest possible value consistent with inefficient liquidation in the first best (or in the case where the contract is contingent on the outside option.) The alternative scenarios we consider differ in the parameter  $\gamma$  of this distribution taking values  $\gamma \in \{0.1, 0.25, 0.5, 1\}$ . Figure 1 below presents the c.d.f. corresponding to different values of  $\gamma$ , where lower values correspond to higher functions. For higher  $\gamma$  outside options are more attractive, so default and borrowing constraint should be more prevalent.

For illustrative purposes, Figure 2 shows the value functions  $B(V)$  corresponding to  $\gamma = 0.1$  and  $0.25$ . The upper value function corresponds to  $\gamma = 0.1$  and the lower one to  $\gamma = 0.25$ . Each is plotted starting at the corresponding value  $V_{\min}$ . Note that there is a section over which  $B$  is increasing, so that Pareto improvements are obtained by increasing  $V$ . Competition would drive the initial value to the area where  $B$  is decreasing. The maximum debt is given at the peak and is larger in the lower  $\gamma$  scenario. In absence of own financing of

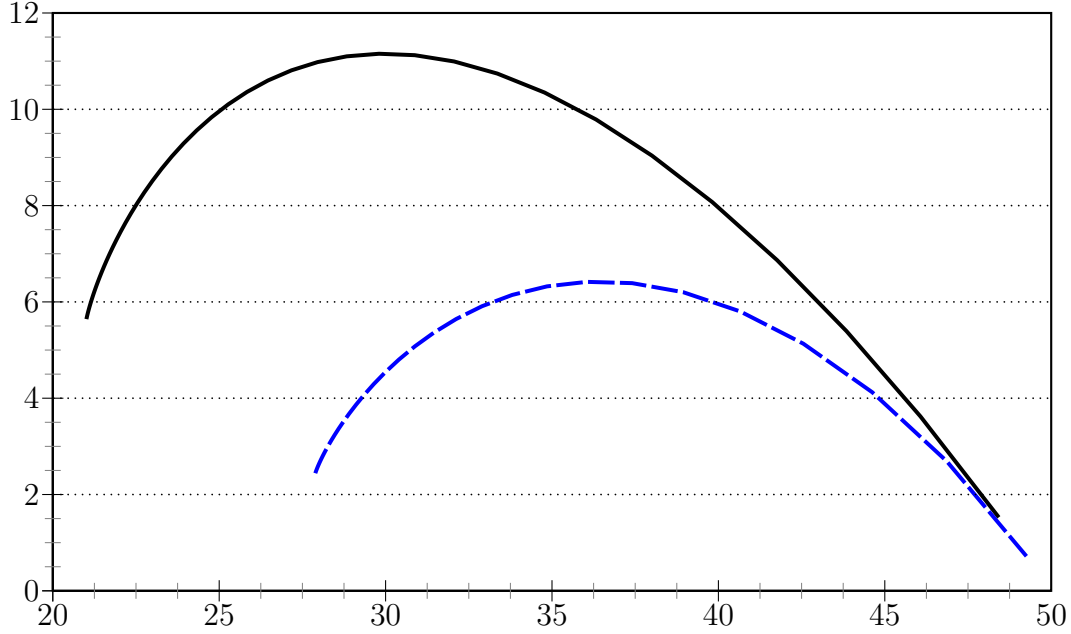


Figure 2: Value functions

the entrepreneur, the project will be financially feasible if and only if the initial investment does not exceed this maximum debt value.

Table 1 gives the evolution of the value of a firm starting in period zero at the point where debt is maximized:

It takes 7 periods (years) to reach the unconstrained steady state (because there are no adjustment costs, it would be reached immediately in the absence of informational constraints.) The value to the borrower rises as debt decreases, resulting in an increase in total surplus  $W$ . The hazard rate  $h$  decreases monotonically, starting from an initial value of 7.6%.

$t$	$V$	$B$	$W = B + V$	$h$	Survival	$k$	risk adjusted $r$
0	36.5	6.4	42.9	7.6	1.00	8.1	12.6%
1	37.9	6.3	44.2	6.7	0.93	9.0	11.7%
2	39.5	6.1	45.6	5.7	0.87	10.1	10.7%
3	41.2	5.6	46.9	4.7	0.83	11.6	9.7%
4	43.2	4.8	48.0	3.6	0.79	13.5	8.6%
5	45.3	3.7	49.0	2.4	0.77	16.2	7.4%
6	47.6	2.1	49.7	1.2	0.76	19.9	6.2%
7	50.0	0.0	50.0	0	0.75	25.0	5.0%

Table 1: Evolution of Optimal Debt with  $\gamma = 0.25$ .

$\gamma$	$V_0$	$B_0$	$W_0$	$k_0$	Leverage	
					excl. $k$	incl. $k$
0.1	30.0	11.2	41.1	11.1	27%	42.6%
0.25	36.5	6.4	42.9	8.1	15%	28.5%
0.5	40.3	4.0	44.3	6.1	9%	20.1%
1	43.2	2.4	45.6	4.5	5%	13.8%

Table 2: Summary statistics: maximum debt.

There is a 75% probability of surviving to the unconstrained stage. The last column gives the adjusted interest rates that support these allocations and values.

Table 2 gives some summary values that are useful to compare across these scenarios. The upper section gives the starting values in each scenario at which debt value is maximized. Observing the third column labelled  $B_0$  shows how stringent constraints to total debt become as  $\gamma$  increases, going from a maximum leverage of 27% for the case of  $\gamma = 0.1$  to slightly above 5% when  $\gamma = 1$ . Correspondingly, risk adjusted interest rates are higher in the latter scenarios and the initial capital financed  $k_0$  much lower.

In order to compare properties of the contract, the Table 3 provides the evolution in the cases  $\gamma = 0.1, 0.25$  and  $0.5$ , starting from the same level of debt (equal to the maximum value for  $\gamma = 0.5$ ). One way of motivating this exercise is if all these projects had the same initial cost  $I_0$  equal to this initial debt  $B_0$ . As  $\gamma$  increases, the initial risk of default increases considerably thus decreasing the initial capital  $k_0$ , decreasing the total survival probability and increasing the length of time required to reach the unconstrained state. Correspondingly, the total initial surplus  $W_0$  decreases. This decrease is not that dramatic since as  $\gamma$  increases, default is typically associated with higher outside options.

$\gamma$	$V_0$	$W_0$	$k_0$	Risk adjusted $r_0$	Years to efficient	Survival
0.1	45.6	49.6	21.0	5.9%	1.9	99%
0.25	44.8	48.8	15.5	7.7%	2.2	97%
0.5	40.3	44.3	6.1	15.2%	4.7	77%

Table 3: Summary statistics: starting from maximum  $B_0$  for  $\gamma = 0.5$ .

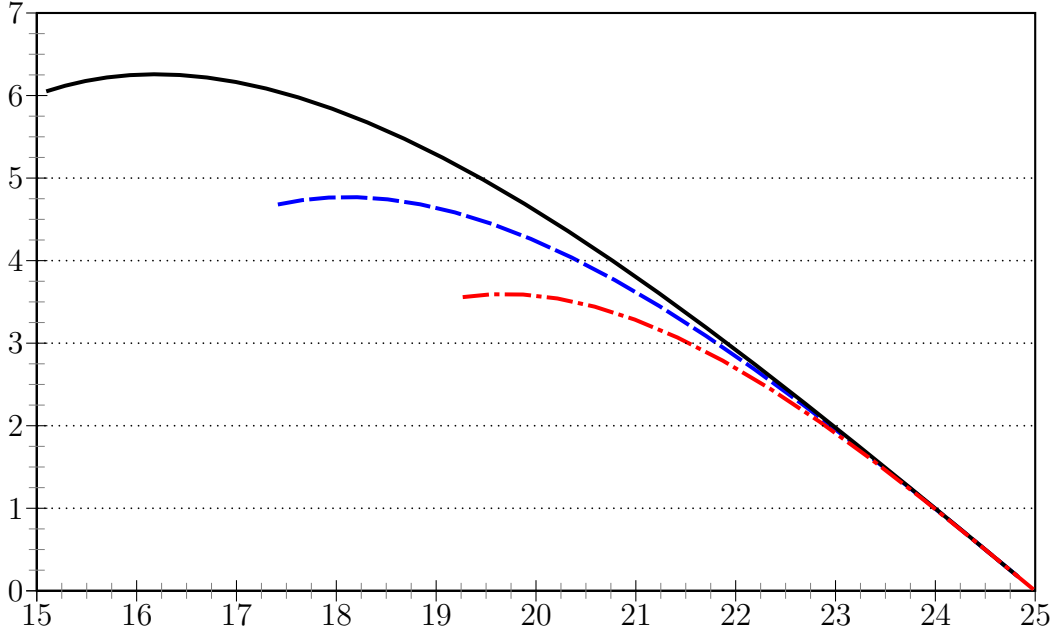


Figure 3: Value functions

#### 4.3.2 Linear case

We now provide some results for the linear case where as described the value function is homogenous of degree one in  $(V, k)$ . As discussed above, it is convenient to define  $v = V/k$  as state variable. This economy grows at a constant rate in the first best, giving total value  $wk$ . The technology for production is linear  $Ak$ , and the adjustment cost quadratic:  $C(\frac{\dot{k}}{k}) = \frac{1}{2}(\frac{\dot{k}}{k})^2$ . Outside opportunity is of the form  $sk$ , where  $s$  is distributed on  $[0, \bar{s}]$ , where following similar arguments as above  $\bar{s}$  is chosen equal to  $w$ . The distribution for  $s$  is identical as in the previous case, with values of  $\gamma = 0.25, 0.5$  and  $1$ . The statistics we present parallel those discussed above.

The value functions for the three scenarios are presented in Figure 3, where as before higher value functions  $B(v)$  correspond to lower  $\gamma$  and is associated with higher maximum debt.

Table 4 gives the paths starting at time zero from the point of maximum debt for  $\gamma = 1$ . It takes over 10 years to reach the unconstrained point, which happens with only 30% probability. Initial hazard rates are very high, reflecting good outside opportunities and

$t$	$V$	$B$	$h$	Survival	$k$ growth	risk adjusted $r$
0	19.7	3.6	21.3	1	0.4%	26.3%
1	20.1	3.6	19.8	0.81	0.6%	24.8%
2	20.5	3.5	18.2	0.67	0.9%	23.2%
3	20.9	3.3	16.4	0.57	1.2%	21.4%
4	21.4	3.1	14.5	0.48	1.5%	19.5%
5	21.9	2.8	12.5	0.42	1.7%	17.5%
6	22.4	2.4	10.4	0.38	2.0%	15.4%
7	22.9	2.0	8.2	0.34	2.2%	13.2%
8	23.5	1.4	5.9	0.32	2.3%	10.9%
9	24.1	0.9	3.5	0.31	2.4%	8.5%
10	24.7	0.3	1.1	0.30	2.5%	6.1%
10.4	25.0	0.0	0.0	0.30	2.5%	5.0%

Table 4: Evolution of Optimal Debt with  $\gamma = 1$ .

correspondingly risk adjusted interest rates are very high and capital accumulation very low (0.5% as opposed to 2.5% in the unconstrained case.). The following table provides summary statistics to compare across different  $\gamma$ 's.

$\gamma$	$V_0$	$B_0$	$W_0$	$k$ growth	Risk adjusted $r$	Leverage $B_0/W_0$
0.25	16.2	6.	3 22.4	0.6%	15.3%	27.9%
0.5	18.1	4.8	22.9	0.5%	19.9%	20.9%
1	19.7	3.6	23.3	0.4%	26.3%	15.4%

Table 5: Summary statistics: maximum debt.

As in the case analyzed in the previous section, higher  $\gamma$  considerably decreases maximum leverage, from 27.9% when  $\gamma = 0.25$  to 15.4% for  $\gamma = 1$ . Table 4 provides a comparison across scenarios starting all from a point where the value of debt is 3.6, the upper bound for the  $\gamma = 1$  case. As before, higher  $\gamma$ 's are associated to much lower total survival and increased time to the unconstrained state, a much higher initial adjusted interest rate and

$\gamma$	$V_0$	$W_0$	$k$ growth	Risk adjusted $r_0$	Years to efficient	Survival
0.25	21.2	24.8	2.2%	9.0%	6.5	88%
0.5	21.0	24.6	1.9%	13.3%	7.0	74%
1	19.7	23.3	0.4%	26.3%	10.4	30%

Table 6: Summary statistics: starting from maximum  $B_0$  for  $\gamma = 0.5$ .

corresponding much lower growth rate (0.4% when  $\gamma = 1$  as opposed to 2.2% when  $\gamma = 0.25$ ). As before and for the same reasons, the differences in default separation rates do not translate into large differences in initial total value  $W_0$ .

## 5 Implementation with Risk Adjusted Loans

In this section we ask whether the optimal contract defined above can be implemented by a risk adjusted interest schedule. To make the analysis more transparent assume the function  $B(V, k)$  is concave in  $V$  for all  $k$  and the optimal contract operates in a region where  $B$  is decreasing in  $V$ . As a consequence, we may invert for each  $k$  this function expressing the utility of the borrower as a function  $V(B, k)$  of debt  $B$  and assets  $k$ .

Consider the following capital accumulation problem for the borrower. Every period, as function of the outstanding debt  $B$  and  $k$ , the agent makes a flow payment  $r(B, k)$  to the lender, which is to be determined below. Faced with this schedule, the agent solves the following:

$$rV(B, k) = \max_{\dot{k}, c} \left( c + V_2(B, k)\dot{k} + V_1(B, k)(r(B, k) - R(k, \dot{k})) + \int_{V(B, k)} h(s, k) ds \right). \quad (20)$$

The borrower pays to the lender  $R(k, \dot{k})$ . If it exceeds  $r(B, k)$  the debt falls and otherwise it increases.

The first order condition for the agent's choice of  $\dot{k}$  is:

$$V_2(B, k) - V_1(B, k)R_2(k, \dot{k}) = 0 \quad (21)$$

If  $V$  were indeed the inverse function of  $B$ , then

$$\frac{V_2(B, k)}{V_1(B, k)} = B_2(V(B, k), k) \quad (22)$$

so the choice of  $\dot{k}$  is the same. We will now determine the function  $r(B, k)$  that will generate the same path for  $B$  as  $B(V(t), k(t))$ . First note that

$$\dot{B} = B_2(V, k)\dot{k} + B_1(V, k)V = r(B, k) - R(k, \dot{k}) \quad (23)$$

Moreover, the value to the lender must satisfy the following:

$$rB = R(k, \dot{k}) - h(V, k)B + \dot{B} \quad (24)$$

which after substitution of  $\dot{B}$  gives

$$r(B, k) = (r + h(V(B, k), k))B \quad (25)$$

Note that this payment as a ratio of  $B$  is the risk adjusted interest rate  $r + h$ .

**Proposition 5.1.** *With  $r(B, k)$  given by equation 25, the solution to the borrower's problem 20 yields a trajectory  $k(t)$  that coincides with the solution to the lender's problem (9). The function  $V(\cdot, k)$  is the inverse of  $B(\cdot, K)$ , for all  $k$ .*

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