Unscheduled Appointments

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1 Introduction

Few academics schedule all of their office hours individually with students, preferring to schedule some hours when any students who desire may arrive. This "open" arrangement creates waiting on occasion, and may also result in low-value use of time. As a meeting strategy, it is generally defended on the principle that the office hours are used more intensely, that is, the resource of the professor's time is used more efficiently.

Open office hours involve the organizational form known as a commons. Appointments assign property rights of a specified time to a student, while open office hours do not, opening the resource freely to all. The commons organizational form is generally considered a tragedy because it lacks a mechanism to prevent overuse of resources, leading to inefficient allocation (Hardin, 1968). But if so, why is this organizational form so popular among academics, including economists? We will show here that in many settings, the open arrangement is actually more efficient than appointments.[1]

This question has widespread applicability in other commons environments. Coase (1959) argued that property rights should be developed over wireless spectrum. While much of this spectrum is closed, there are notable exceptions. Radio spectrum for cordless phones, CB radio, walkie-talkies, and wireless computer connections known as wi-fi (802.11b and g) are open — any complying use is permitted. Manufacturers just use the spectrum as much as they want for complying devices. We notice the commons problem when our computers try to connect to someone else's insecure wireless access point, or when our phone picks up a neighbor's call, but overall the arrangement works well and certainly has led to a proliferation of devices. It is not plausible that assigning property rights to the spectrum would produce higher value than the current arrangement in these applications.

In recent years a significant debate has emerged over the regulation of wireless spectrum. Benkler (1998) and Noam (1998) advocate open spectrum, in part on the grounds that new technologies enable more efficient use of the airwaves so that spectrum is no longer a scarce good. The latter proposed that, in the event of continuing scarcity at peak moments, spectrum should remain open but be priced, a call supported by Benkler (2002). With or without a price, the regime of open

[1] Heller and Eisenberg (1998) argue that the organizational form of property rights can also lead to a tragedy. If multiple parties have property rights over different aspects of a single resource, the transaction costs involved in assembling the bundle of rights necessary to utilize the resource can lead to underuse. We ignore this issue by assuming that property rights are allocated in efficiently sized bundles.

spectrum is corresponds to the case of unscheduled appointments.

The proposal for open spectrum was opposed by Hazlett (1998), who argued that spectrum is still scarce, and that a regime of open but priced access would impose prohibitive transaction costs. The argument that continuing scarcity recommends closed spectrum was seconded by Cave and Webb (2004), among others. The regime of property rights corresponds to the case of scheduled appointments.

This problem also emerges in the sale of time-dependent goods and services, such as rental cars, airline seats, and movie tickets. Ski resorts and amusement parks commonly assign rides on a first-come first-serve basis through queues, and generally do not take reservations for rides at specific times. Barro and Romer (1987) showed that this method is nearly efficient even in times of peak demand. Art museums, in contrast, regularly take reservations in periods of high demand.

Restaurant reservations tend to be an extreme example, with a high probability that a reservation is booked but not used. Thus, restaurants are a mixed method, with reservations and people appearing late to take advantage of unused reservations. Is it more efficient to assign property rights or to have people waiting for tables? If the market has it right, the correct answer is a mix of the two. In particular, it is not assignment in advance.

Scheduling appointments in advance will idle resources and may also lead to overuse of the resources, at least when the appointment is not priced. Moreover, at the time of an appointment, the value is not known and may change, resulting in a jumping the gun inefficiency of the kind detailed by Roth and Xing (1994). Open office hours, in contrast, incur higher waiting costs and may also be inefficiently utilized unless an auction is held for the slot. The important thing to understand is that both advance scheduling and ex post allocation involve inefficiencies, so that the comparison is not at all obvious and in particular the commons approach may dominate. Either can be more efficient, and this paper provides a characterization of efficiency.

We model the distribution of a single good which is costless but available only in a quantity insufficient to meet demand. Scarce natural resources such as wireless spectrum fit this description. Alternatively, our model describes the case where production is costly but cannot be conditioned on realized demand, and where undistributed units of the good lose their value. This would describe, for example, the sale of airline seats, the quantity of which normally cannot be adjusted in the short run. Individuals have single-unit demands for the good, drawn from some known distribution. Recipients of the good pay a fixed price.

There are two dates. At time zero, the individuals may call for reservations, if reservations are allowed. We study both the case where individuals know the realizations of their value at time zero as well as the case where individuals learn of their valuation at time one. At time one, the individuals choose whether to arrive at the distribution center, and incur a transportation cost if they arrive. Holders of reservations are guaranteed a unit of the good, while those without reservations are randomly selected to receive a unit of the good.

We prove several results for when the transportation cost is low. The *expected surplus* of a distribution is a function which returns, for any price, the expected consumer surplus conditional on trade at that price. For the case where individuals know the realizations of their values at the time that the reservations are made, we show that, if the good is unpriced, then a necessary and sufficient condition for reservations to be optimal is that the expected surplus is decreasing at zero. Furthermore, if prices are positive, then a sufficient condition for reservations to be optimal is that the expected surplus is decreasing at the price. For the case where individuals do not know their valuations at time zero, we show that reservations are never optimal.

2 The model

There is a finite $N = \{1, 2, ..., n\}$ set of agents. There is a scarce indivisible good of which m units are available, $n > m > 0$. Agents have a demand for a single unit of this good, v_i, drawn from a known distribution $F(v) \sim [0, \infty)$ with density $f(v)$. Reservations may or may not be scheduled in advance. At the distribution time, agents choose to arrive (or not) at the distribution point and incur transportation cost c if they arrive. The m units of the good are then sold to the individuals who arrive at price p, with those individuals who have reservations being given first priority.

2.1 Individual values known at the time of the reservation

In the case of "unscheduled appointments" there are no reservations. Individuals who arrive are not guaranteed a unit of the good. An individual will choose to arrive if her valuation exceeds a cutoff $v^* \geq p + c$, and will be served with probability

$$\alpha_u \equiv \sum_{i=0}^{n-1} \binom{n-1}{i} (1 - F(v^*))^i \, F(v^*)^{n-1-i} \min\left\{\frac{m}{i+1}, 1\right\},$$

where v^* is defined by the equation $c = (v^* - p)\, \alpha_u$.

The probability that i people arrive is $\binom{n}{i} (1 - F(v^*))^i \, F(v^*)^{n-i}$. At most m units of the good can be distributed, thus the expected number of individuals who are served is given by

$$\kappa(v^*) \equiv \sum_{i=0}^{n} \binom{n}{i} (1 - F(v^*))^i \, F(v^*)^{n-i} \min\{m, i\}.$$

Note that $\kappa(v^*) = n\,(1 - F(v^*))\, \alpha_u$. The expected value of an individual who arrives is

$$E\left[v | v \geq v^*\right] = \int_{v^*}^{\infty} \frac{x f(x)}{1 - F(v^*)} dx.$$

The total social welfare is the expected number of people who are served, times their expected values, minus the expected number of people who arrive, $n\,(1 - F(v^*))$, times their transportation cost c, or:

$$W_U(c) = \kappa(v^*) E\left[v | v \geq v^*\right] - n\,(1 - F(v^*))\, c.$$

In the case of "scheduled appointments" reservations are allowed. Individuals know their valuations at the time of making the reservation. Calling for a reservation is costless, so all individuals whose valuations exceed $p + c$ will call for a reservation. The first m callers will be awarded reservations and will be guaranteed a unit of the good. The total social welfare is the expected number of people who arrive times their expected values less their transportation cost, or

$$W_A(c) = \kappa(p + c)\left(E\left[v|v \geq p + c\right] - c\right).$$

Note that if $c = 0$, $v^* = p$, and thus social welfare is the same under both scheduled and unscheduled appointments.

An individual's expected surplus is the amount by which the individual's expected value exceeds the price p, conditional on the value being higher than p, or

$$\Gamma(p) = E[v|v \geq p] - p.$$

We derive a necessary and sufficient condition for when the transportation cost c is small and the good is unpriced. In this case, unscheduled appointments dominate scheduled appointments if and only if the expected surplus is increasing at zero; that is, if $\Gamma'(0) \geq 0$.

Theorem 2.1. *For sufficiently small c, if the good is unpriced,* $W_U(c) \geq W_A(c)$ *if and only if* $f(0)\int_0^\infty xf(x)dx \geq 1$.

For the uniform distribution, scheduled dominates unscheduled. For the exponential distribution with a zero base, the expected surplus is constant, and thus this is a transition case. Finally, if $F(x) = x^a$, for $a < 1$, unscheduled dominates scheduled for sufficientl small costs c. Roughly speaking, unscheduled appointments dominate scheduled appointments if the frequence of low values is very high, more than one over the mean.

Many theoretical studies assume that hazard rates are non-decreasing, or, equivalently, that the probability distribution is log-concave. This condition is sufficient to imply that the expected surplus is non-increasing.[2] In such a case, scheduling dominates unscheduled for any price, not just the zero price contemplated in Theorem 2.1, as we now show.

Theorem 2.2. *For sufficiently small c, if the prices are positive,* $W_A(c) > W_U(c)$ *for every distribution such that* $\Gamma'(p) \leq 0$.

2.2 Individual valuations not known at the time of the reservation

In some settings, it may be more realistic to assume that agents do not know their valuations at the time that the reservation is made. In this case, all agents call for reservations which are awarded to the first m callers. Agents with reservations arrive to purchase the unit of the good if their valuations exceed $p + c$. Agents without reservations may choose to arrive hoping to purchase one of the expected $mF(p + c)$ units remaining. These individuals will arrive if they have valuations exceeding $\hat{v} \geq p + c$, and will be served with probability

$$\alpha_s \equiv \sum_{i=0}^{m}\binom{m}{i}\left(1 - F(p + c)\right)^i F(p{+}c)^{m-i}\sum_{j=0}^{n-m-1}\binom{n-m-1}{j}\left(1 - F(\hat{v})\right)^j F(\hat{v})^{n-m-1-j}\min\left\{\frac{m-i}{j+1}, 1\right\},$$

[2] For more on the relationship between these assumptions, see Bagnoli and Bergstrom (2005). They refer to the expected surplus as the Mean Residual Lifetime Function.

where $\hat{v}$ is defined by the equation $c = (\hat{v} - p)\,\alpha_s$.

The total number of individuals without reservations who are served is given by

$$\lambda(\hat{v}, p{+}c) \equiv \sum_{i=0}^{m} \binom{m}{i} (1 - F(p + c))^i \, F(p{+}c)^{m-i} \sum_{j=0}^{n-m} \binom{n-m}{j} (1 - F(\hat{v}))^j \, F(\hat{v})^{n-m-j} \min\{m - i, j\}\,.$$

Note that $\lambda(\hat{v}, p + c) = (n - m)\,(1 - F(\hat{v}))\,\alpha_s$. Total social welfare is given by:

$$W_S(p) = m\,(1 - F(p + c))\,(E\,[v|v \geq p + c] - c)\; + \lambda(\hat{v}, p + c) E\,[v|v \geq \hat{v}] - (n - m)\,(1 - F(\hat{v}))\,c.$$

In this case it is less obvious, but equally true, that, when $c = 0$, social welfare is the same under both scheduled and unscheduled appointments.

Lemma 2.3. *When $c = 0$, $W_U(p) = W_S(p)$.*

When the agents do not know their valuations in advance, and the transportation cost c is sufficiently small, unscheduled appointments dominates scheduled appointments regardless of the price chosen and regardless of the distribution of the valuations. If prices are strictly positive, unscheduled appointments strictly dominates scheduled appointments. This theorem does not depend on the assumption that the expected surplus is non-increasing.

Theorem 2.4. *For sufficiently small c, $W_U(p) \geq W_S(p)$ for every distribution F. Furthermore, if $p > 0$, $W_U(p) > W_S(p)$.*

3 Conclusion

We model the allocation of scarce resources and prove several results about the optimality of scheduling reservations when transportation costs are low. First, if values are known at the time that reservations are made, and if the item is unpriced, then scheduling is optimal if and only if the expected surplus is decreasing at zero. Second, if values are known and the item is priced, then scheduling dominates if the expected surplus is decreasing at the price. Third, if values are not known at the time that reservations are made, scheduling is never optimal. This last result is true even though the distribution of values is known at the time reservations are made.

Appendices

A Proofs

There are a few basic equalities that will be used throughout the proofs. First, let $\beta = (\beta_0, \beta_1, ..., \beta_n)$ be an arbitrary $n + 1$ dimensional vector. Then:

$$\sum_{i=0}^{n} \binom{n}{i} (1 - F(x))^i \, F(x)^{n-i} \beta_i = \sum_{i=0}^{m} \sum_{j=0}^{n-m} \binom{m}{i} \binom{n-m}{j} (1 - F(x))^{i+j} \, F(x)^{n-i-j} \beta_{i+j} \qquad (1)$$

Proof. By Vandermonde's identity, $\binom{n}{i} = \sum_{j=0}^{i} \binom{m}{i-j}\binom{n-m}{j}$.

Thus the left hand side of (1) is equal to $\sum_{i=0}^{n} \sum_{j=0}^{i} \binom{m}{i-j}\binom{n-m}{j} (1-F(x))^{i} F(x)^{n-i} \beta_i$.

If we replace i with $i+j$, then this becomes $\sum_{i+j=0}^{n} \sum_{j=0}^{i+j} \binom{m}{i}\binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \beta_{i+j}$.

We can rewrite this expression as $\sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{m}{i}\binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \beta_{i+j}$.

Because $\binom{a}{b} = 0$ for $a < b$, this is equivalent to $\sum_{i=0}^{m} \sum_{j=0}^{n-m} \binom{m}{i}\binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \beta_{i+j}$. □

$$\kappa(x) = \lambda(x,x) + m(1-F(x)) \tag{2}$$

Proof. Recall that $\kappa(x) = \sum_{i=0}^{n} \binom{n}{i} (1-F(x))^{i} F(x)^{n-i} \min\{m,i\}$.

By expression (1), this equals $\sum_{i=0}^{m} \sum_{j=0}^{n-m} \binom{m}{i}\binom{n-m}{j} (1-F(x))^{i+j} F(x)^{n-i-j} \min\{m,i+j\}$.

Rearranging terms, thus becomes:

$$\sum_{i=0}^{m} \binom{m}{i} (1-F(x))^{i} F(x)^{m-i} \sum_{j=0}^{n-m} \binom{n-m}{j} (1-F(x))^{j} F(x)^{n-m-j} (\min\{m-i,j\}+i).$$

This last expression is equivalent to $\lambda(x,x) + m(1-F(x))$. □

$$\frac{d}{dx} E[v|v \geq x] = \frac{f(x)}{1-F(x)} (E[v|v \geq x] - x) \tag{3}$$

Proof. Recall that $E[v|v \geq x] = \int_x^\infty \frac{vf(v)}{1-F(x)} dv$, or $\frac{1}{1-F(x)} \int_x^\infty vf(v)dv$. Using the chain rule,

$$\frac{d}{dx} E[v|v \geq x] = \frac{f(x)}{(1-F(x))^2} \int_x^\infty vf(v)dv - \frac{1}{1-F(x)} xf(x), \text{ or } \frac{f(x)}{1-F(x)} (E[v|v \geq x] - x).$$

□

The proofs of the following thre statements are straightforward and left to readers.

$$\kappa'(x) = \sum_{i=0}^{n} \binom{n}{i} (1-F(x))^{i} F(x)^{n-i} \left(\frac{n-i}{F(x)} - \frac{i}{1-F(x)} \right) f(x) \min\{m,i\} \tag{4}$$

$$\left.\frac{dv^*}{dc}\right|_{c=0} = \frac{n(1-F(p))}{\kappa(p)} \tag{5}$$

$$\left.\frac{d\hat{v}}{dc}\right|_{c=0} = \frac{(n-m)(1-F(p))}{\lambda(p,p)} \tag{6}$$

A.1 Proof of Theorems 2.1 and 2.2.

If we take the derivative of $W_A(c)$ with respect to the transportation cost, c, we get:

$$W_A'(c) = \kappa'(p+c)\left(E\left[v|v \geq p+c\right] - c\right) + \kappa(p+c)\left(\left[\frac{d}{d(p+c)}E\left[v|v \geq p+c\right]\right] - 1\right).$$

Evaluated at $c = 0$, this becomes:

$$W_A'(0) = \kappa'(p)E\left[v|v \geq p\right] + \kappa(p)\left[\frac{d}{dp}E\left[v|v \geq p\right]\right] - \kappa(p).$$

If we take the derivative of $W_U(c)$ with respect to the transportation cost, c, we get:

$$W_U'(c) = \kappa'(v^*)\frac{dv^*}{dc}E\left[v|v \geq v^*\right] + \kappa(v^*)\left[\frac{d}{dv^*}E\left[v|v \geq v^*\right]\right]\frac{dv^*}{dc} - n\left(1 - F(v^*)\right) + nf(v^*)c\frac{dv^*}{dc}.$$

Evaluated at $c = 0$, this becomes:

$$W_U'(0) = \kappa'(p)\left.\frac{dv^*}{dc}\right|_{c=0} E\left[v|v \geq p\right] + \kappa(p)\left[\frac{d}{dp}E\left[v|v \geq p\right]\right]\left.\frac{dv^*}{dc}\right|_{c=0} - n\left(1 - F(v^*)\right).$$

Using expression (5) and simplifying, we have:

$$W_U'(0) = \left(\kappa'(p)E\left[v|v \geq p\right] + \kappa(p)\left[\frac{d}{dp}E\left[v|v \geq p\right]\right] - \kappa(p)\right)\left.\frac{dv^*}{dc}\right|_{c=0}.$$

Or, $W_U'(0) = W_A'(0)\left.\frac{dv^*}{dc}\right|_{c=0}$. Expression (5) is greater than one, thus

$$W_A'(0) \geq W_U'(0) \text{ if and only if } W_A'(0) \leq 0.$$

This is equivalent to: $\kappa(p)\frac{f(p)}{1-F(p)}\left(E\left[v|v \geq p\right] - p\right) \leq \kappa(p) - \kappa'(p)E\left[v|v \geq p\right]$.

Evaluated at $p = 0$ this is: $\kappa(0)f(0)E\left[v|v \geq 0\right] \leq \kappa(0)$.

This is true if and only if: $f(0)\int_0^\infty xf(x)dx \leq 1$, which proves Theorem 2.1.

By assumption, $\Gamma'(p) \leq 0$, which implies that $\frac{f(p)}{1-F(p)}\left(E\left[v|v \geq p\right] - p\right) \leq 1$.

This implies that $\kappa(p)\frac{f(p)}{1-F(p)}\left(E\left[v|v \geq p\right] - p\right) \leq \kappa(p)$.

It follows from the fact that $\kappa'(p) \leq 0$ and $E\left[v|v \geq p\right] > 0$ that $W_A'(0) \geq W_U'(0)$ for all prices p.

Furthermore, because $\kappa'(p) < 0$ for all $p > 0$, it follows that $W_A'(0) > W_U'(0)$ for all prices $p > 0$.

At $p = 0$, the fact that $\kappa'(0) = 0$ implies that $W_A'(0) > W_U'(0)$ if and only if

$$\kappa(p)\frac{f(p)}{1-F(p)}\left(E\left[v|v \geq p\right] - p\right) < \kappa(p), \text{ which is true if and only if } \Gamma'(0) < 0.$$

This proves Theorem 2.2.

A.2 Proof of Lemma 2.3.

If $c = 0$, $\hat{v} = p$, and thus $W_S(0) = m\,(1 - F(p))\,E[v|v \geq p] + \lambda(p,p)E[v|v \geq p]$.

By expression (2), this equals $\kappa(p)E[v|v \geq p] = W_U(0)$.

A.3 Proof of Theorem 2.4.

If we take the derivative of $W_S(c)$ with respect to the transportation cost, c, we get:

$$W_S'(c) = -mf(p+c)\left(E\left[v|v \geq p+c\right] - c\right)$$

$$+m\left(1 - F(p+c)\right)\left(\tfrac{f(p+c)}{1-F(p+c)}\left(E\left[v|v \geq p+c\right] - p - c\right) - 1\right) + \tfrac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}E\left[v|v \geq \hat{v}\right]$$

$$+\lambda\left\{\hat{v}, p+c\right\}\tfrac{f(\hat{v})}{1-F(\hat{v})}\left(E\left[v|v \geq \hat{v}\right] - \hat{v}\right)\tfrac{d\hat{v}}{dc} - (n-m)\left(1 - F(\hat{v})\right) + (n-m)f(\hat{v})\tfrac{d\hat{v}}{dc}c.$$

After simplifying:

$$W_S'(c) = -mpf(p+c) - m\left(1 - F(p+c)\right) - (n-m)\left(1 - F(\hat{v})\right) + (n-m)f(\hat{v})\tfrac{d\hat{v}}{dc}c$$

$$-\lambda(\hat{v}, p+c)\tfrac{\hat{v}f(\hat{v})}{1-F(\hat{v})}\tfrac{d\hat{v}}{dc} + E\left[v|v \geq \hat{v}\right]\left[\tfrac{\lambda\{\hat{v},p+c\}f(\hat{v})}{1-F(\hat{v})}\tfrac{d\hat{v}}{dc} + \tfrac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\right].$$

At $c = 0$, if we substitute expression (6), this becomes:

$$W_S'(0) = E\left[v|v \geq p\right]\left[(n-m)f(p) + \tfrac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\Big|_{c=0}\right] - npf(p) - n\left(1 - F(p)\right).$$

From the proof of Theorem 2.1 and substituting expression (5), we get:

$$W_U'(0) = n\left(f(p) + \tfrac{(1-F(p))\kappa'(p)}{\kappa(p)}\right)E\left[v|v \geq p\right] - npf(p) - n\left(1 - F(p)\right).$$

Thus, $W_U'(0) \geq W_S'(0)$ if and only if $\frac{n(1-F(p))\kappa'(p)}{\kappa(p)} \geq \frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\Big|_{c=0} - mf(p)$.

Note that $\frac{d}{dc}\lambda\left\{\hat{v}, p+c\right\} = \sum_{i=0}^{m}\binom{m}{i}\left(1 - F(p+c)\right)^i F(p+c)^{m-i}\left(\frac{m-i}{F(p+c)} - \frac{i}{1-F(p+c)}\right)f(p+c)$

$$\sum_{j=0}^{n-m}\binom{n-m}{j}\left(1 - F(\hat{v})\right)^j F(\hat{v})^{n-m-j}\min\{m-i, j\} + \sum_{i=0}^{m}\binom{m}{i}\left(1 - F(p+c)\right)^i F(p+c)^{m-i}$$

$$\sum_{j=0}^{n-m}\binom{n-m}{j}\left(1 - F(\hat{v})\right)^j F(\hat{v})^{n-m-j}\left(\tfrac{n-m-j}{F(\hat{v}} - \tfrac{j}{1-F(\hat{v})}\right)\tfrac{d\hat{v}}{dc}f(\hat{v})\min\{m-i, j\}.$$

Evaluated at $c = 0$, and using expression (6), this becomes:

$$\tfrac{d}{dc}\lambda\left\{\hat{v}, p+c\right\}\Big|_{c=0} = f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1 - F(p)\right)^{i+j}F(p)^{n-i-j}$$

$$\left(\left(\tfrac{m-i}{F(p)} - \tfrac{i}{1-F(p)}\right) + \left(\tfrac{n-m-j}{F(p)} - \tfrac{j}{1-F(p)}\right)\tfrac{(n-m)(1-F(p))}{\lambda(p,p)}\right)\min\{m-i, j\}$$

This simplifies to:

$$\frac{d}{dc}\lambda\left\{\hat{v},p+c\right\}\Big|_{c=0} = f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}$$

$$\left(\left(\frac{m-i}{F(p)}-\frac{i}{1-F(p)}\right)+\left(\frac{n-m-j}{F(p)}-\frac{j}{1-F(p)}\right)\frac{(n-m)(1-F(p))}{\lambda(p,p)}\right)\min\{m-i,j\}$$

Thus $W_U'(0) \geq W_S'(0)$ if and only if:

$$\frac{n(1-F(p))\kappa'(p)}{\kappa(p)} \geq f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}$$

$$\left(\left(\frac{m-i}{F(p)}-\frac{i}{1-F(p)}\right)+\left(\frac{n-m-j}{F(p)}-\frac{j}{1-F(p)}\right)\frac{(n-m)(1-F(p))}{\lambda(p,p)}\right)\min\{m-i,j\}-mf(p)$$

Multiplying each side by $\lambda(p,p)\kappa(p)$:

$$\lambda(p,p)n\left(1-F(p)\right)\kappa'(p) \geq f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$$

$$\left(\left(\frac{m-i}{F(p)}-\frac{i}{1-F(p)}\right)\lambda(p,p)\kappa(p)+\left(\frac{n-m-j}{F(p)}-\frac{j+\frac{m}{n-m}}{1-F(p)}\right)\kappa(p)(n-m)\left(1-F(p)\right)\right)$$

Combining statements (4) and (1):

$$\kappa'(p)=\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\left(\frac{n-i-j}{F(p)}-\frac{i+j}{1-F(p)}\right)f(p)\min\left\{m,i+j\right\}.$$

Rearranging terms and applying statement (1):

$$\kappa'(p)=f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\left(\frac{n-i-j}{F(p)}-\frac{i+j}{1-F(p)}\right)\min\left\{m-i,j\right\}+$$

$$mf(p)\left(1-F(p)\right)\sum_{i=0}^{n-1}\binom{n-1}{i}\left(1-F(p)\right)^{i}F(p)^{n-1-i}\left(\frac{n-1-i}{F(p)}-\frac{i+1}{1-F(p)}\right).$$

Note that $\sum_{i=0}^{n-1}\binom{n-1}{i}\left(1-F(p)\right)^{i}F(p)^{n-1-i}\left(\frac{n-1-i}{F(p)}-\frac{i+1}{1-F(p)}\right)=\frac{-1}{1-F(p)}$. Thus:

$$\kappa'(p)=f(p)\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\left(\frac{n-i-j}{F(p)}-\frac{i+j}{1-F(p)}\right)\min\left\{m-i,j\right\}-mf(p).$$

Substituting for $\kappa'(p)$ and dividing each side by $f(p)$, it follows that $W_U'(p) \geq W_S'(p)$ if and only if:

$$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\left\{m-i,j\right\}$$

$$\left(\left(\frac{n-i-j}{F(p)}-\frac{i+j}{1-F(p)}\right)\lambda(p,p)-m\right)n\left(1-F(p)\right)\geq$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$$

$$\left(\left(\frac{m-i}{F(p)}-\frac{i}{1-F(p)}\right)\lambda(p,p)\kappa(p)+\left(\frac{n-m-j}{F(p)}-\frac{j+\frac{m}{n-m}}{1-F(p)}\right)\kappa(p)(n-m)\left(1-F(p)\right)\right)$$

Multiplying each side by $F(p)$:

$$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\left\{m-i,j\right\}$$

$\left(n^2\left(1-F(p)\right)\lambda(p,p)-n(i+j)\lambda(p,p)-mn\left(1-F(p)\right)F(p)\right)\geq$

$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$

$\left(m\lambda(p,p)-\frac{i\lambda(p,p)}{1-F(p)}+(n-m)^2\left(1-F(p)\right)-(n-m)j-mF(p)\right)\kappa(p)$

Rearranging terms:

$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\left\{m-i,j\right\}$

$\left(n^2\left(1-F(p)\right)\lambda(p,p)-mn\left(1-F(p)\right)F(p)+\left(mF(p)-m\lambda(p,p)-(n-m)^2\left(1-F(p)\right)\right)\kappa(p)\right)\geq$

$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$

$\left(n(i+j)\lambda(p,p)-\frac{i\lambda(p,p)\kappa(p)}{1-F(p)}-(n-m)j\kappa(p)\right)$

Using the substitution in statement (2) and cancelling terms:

$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\left\{m-i,j\right\}$

$\left[-m\left(n\left(1-F(p)\right)-\kappa(p)\right)^2-mF(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]\geq$

$\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\{m-i,j\}$

$\left(n(i+j)\left(\kappa(p)-m\left(1-F(p)\right)\right)-\frac{i(\kappa(p)-m(1-F(p)))\kappa(p)}{1-F(p)}-(n-m)j\kappa(p)\right)$

Note that $\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}\min\left\{m-i,j\right\}X_{ij}$

$=\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}(m-i)X_{ij}$

$-\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}(m-i-j)X_{ij}.$

It follows that $W_U'(p)\geq W_S'(p)$ if and only if:

$-m\left[\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+F(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$

$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}\left(1-F(p)\right)^{i}F(p)^{n-i}(m-i)\right]\geq$

$m\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}$

$\left((i+j)n\left(\kappa(p)-m\left(1-F(p)\right)\right)-i\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)-j(n-m)\kappa(p)\right)$

$-\sum_{i=0}^{m}\sum_{j=0}^{n-m}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}i$

$$\left((i+j)n\left(\kappa(p)-m\left(1-F(p)\right)\right)-i\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)-j(n-m)\kappa(p)\right)$$

$$-\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}(m-i-j)(i+j)n\left(\kappa(p)-m\left(1-F(p)\right)\right)+$$

$$\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}(m-i-j)i\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)+$$

$$\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m}{i}\binom{n-m}{j}\left(1-F(p)\right)^{i+j}F(p)^{n-i-j}(m-i-j)j(n-m)\kappa(p)$$

Or:

$$-m\left[\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+F(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}\left(1-F(p)\right)^{i}F(p)^{n-i}(m-i)\right]\geq$$

$$-m^2\left(n\left(1-F(p)\right)-\kappa(p)\right)^2$$

$$-m\left(1-F(p)\right)\sum_{i=1}^{m}\sum_{j=0}^{n-m}\binom{m-1}{i-1}\binom{n-m}{j}\left(1-F(p)\right)^{i-1+j}F(p)^{n-i-j}$$

$$\left((i+j)n\left(\kappa(p)-m\left(1-F(p)\right)\right)-i\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)-j(n-m)\kappa(p)\right)-$$

$$\left(1-F(p)\right)\sum_{k=1}^{m}\binom{n-1}{k-1}\left(1-F(p)\right)^{k-1}F(p)^{n-k}(m-k)n^2\left(\kappa(p)-m\left(1-F(p)\right)\right)+$$

$$\left(1-F(p)\right)\sum_{i=1}^{m}\sum_{j=0}^{m-i}\binom{m-1}{i-1}\binom{n-m}{j}\left(1-F(p)\right)^{i-1+j}F(p)^{n-i-j}(m-i-j)m\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)+$$

$$\left(1-F(p)\right)\sum_{i=0}^{m}\sum_{j=1}^{m-i}\binom{m}{i}\binom{n-m-1}{j-1}\left(1-F(p)\right)^{i+j-1}F(p)^{n-i-j}(m-i-j)(n-m)^2\kappa(p)$$

Or:

$$-m\left[\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+F(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}\left(1-F(p)\right)^{i}F(p)^{n-i}(m-i)\right]\geq$$

$$-m^2\left(n\left(1-F(p)\right)-\kappa(p)\right)^2$$

$$-m\left(1-F(p)\right)\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}\left(1-F(p)\right)^{k+j}F(p)^{n-1-k-j}$$

$$\left((k+j+1)n\left(\kappa(p)-m\left(1-F(p)\right)\right)-(k+1)\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)-j(n-m)\kappa(p)\right)$$

$$-\left(1-F(p)\right)\sum_{i=1}^{m}\binom{n-1}{i-1}\left(1-F(p)\right)^{i-1}F(p)^{n-i}(m-i)n^2\left(\kappa(p)-m\left(1-F(p)\right)\right)$$

$$+\left(1-F(p)\right)\sum_{i=1}^{m}\binom{n-1}{i-1}\left(1-F(p)\right)^{i-1}F(p)^{n-i}(m-i)m\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+\left(1-F(p)\right)\sum_{i=1}^{m}\binom{n-1}{i-1}\left(1-F(p)\right)^{i-1}F(p)^{n-i}(m-i)(n-m)^2\kappa(p)$$

Or:

$$-m\left[\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+F(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}\left(1-F(p)\right)^{i}F(p)^{n-i}(m-i)\right]\geq$$

$$-m^2\left(n\left(1-F(p)\right)-\kappa(p)\right)^2$$

$$-m\left(1-F(p)\right)\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}\left(1-F(p)\right)^{k+j}F(p)^{n-1-k-j}(k+j)n\left(\kappa(p)-m\left(1-F(p)\right)\right)$$

$$-m\left(1-F(p)\right)\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}\left(1-F(p)\right)^{k+j}F(p)^{n-1-k-j}n\left(\kappa(p)-m\left(1-F(p)\right)\right)$$

$$+m\left(1-F(p)\right)\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}\left(1-F(p)\right)^{k+j}F(p)^{n-1-k-j}k\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+m\left(1-F(p)\right)\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}\left(1-F(p)\right)^{k+j}F(p)^{n-1-k-j}\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+m\left(1-F(p)\right)\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-1}{k}\binom{n-m}{j}\left(1-F(p)\right)^{k+j}F(p)^{n-1-k-j}j(n-m)\kappa(p)$$

$$+m\left(n\left(1-F(p)\right)-\kappa(p)\right)^2\sum_{i=1}^{m}\binom{n-1}{i-1}\left(1-F(p)\right)^{i-1}F(p)^{n-i}(m-i)$$

Dividing each side by m:

$$-\left[\left(n\left(1-F(p)\right)-\kappa(p)\right)^2+F(p)\left(n\left(1-F(p)\right)-\kappa(p)\right)\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}\left(1-F(p)\right)^{i}F(p)^{n-i}(m-i)\right]\geq$$

$$-m\left(n\left(1-F(p)\right)-\kappa(p)\right)^2$$

$$-(n-1)\left(1-F(p)\right)^2\sum_{k=1}^{n-1}\binom{n-2}{k-1}\left(1-F(p)\right)^{k-1}F(p)^{n-1-k}n\left(\kappa(p)-m\left(1-F(p)\right)\right)$$

$$-\left(1-F(p)\right)n\left(\kappa(p)-m\left(1-F(p)\right)\right)$$

$$+(m-1)\left(1-F(p)\right)^2\sum_{k=0}^{m-1}\sum_{j=0}^{n-m}\binom{m-2}{k-1}\binom{n-m}{j}\left(1-F(p)\right)^{k+j}F(p)^{n-1-k-j}\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+\left(1-F(p)\right)\left(\frac{\kappa(p)^2}{1-F(p)}-m\kappa(p)\right)$$

$$+(n-m)\left(1-F(p)\right)^2\sum_{k=0}^{m-1}\sum_{j=1}^{n-m-1}\binom{m-1}{k}\binom{n-m-1}{j-1}\left(1-F(p)\right)^{k+j}F(p)^{n-1-k-j}(n-m)\kappa(p)$$

$$+\left(n\left(1-F(p)\right)-\kappa(p)\right)^2\sum_{i=1}^{m}\binom{n-1}{i-1}\left(1-F(p)\right)^{i-1}F(p)^{n-i}(m-i)$$

Or:

$$-\left[(n(1-F(p))-\kappa(p))^2+F(p)(n(1-F(p))-\kappa(p))\right]$$

$$\left[mF(p)-\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^i F(p)^{n-i}(m-i)\right] \geq$$

$$-m(n(1-F(p))-\kappa(p))^2$$

$$-(n^2-n)(1-F(p))^2\kappa(p)+m(n^2-n)(1-F(p))^3$$

$$-(1-F(p))n\kappa(p)+nm(1-F(p))^2$$

$$+(m-1)(1-F(p))\kappa(p)^2-(m^2-m)(1-F(p))^2\kappa(p)$$

$$+\kappa(p)^2-m(1-F(p))\kappa(p)$$

$$+(n-m)^2(1-F(p))^2\kappa(p)$$

$$+(n(1-F(p))-\kappa(p))^2\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)$$

Or:

$$-\left[mF(p)(n(1-F(p))-\kappa(p))^2+mF(p)^2(n(1-F(p))-\kappa(p))\right]$$

$$+\left[(n(1-F(p))-\kappa(p))^2+F(p)(n(1-F(p))-\kappa(p))\right]$$

$$\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^i F(p)^{n-i}(m-i) \geq$$

$$-m(n(1-F(p))-\kappa(p))^2F(p)$$

$$+(m(1-F(p))-\kappa(p))(n(1-F(p))-\kappa(p))F(p)$$

$$+(n(1-F(p))-\kappa(p))^2\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)$$

Or:

$$\left[(n(1-F(p))-\kappa(p))^2+F(p)(n(1-F(p))-\kappa(p))\right]$$

$$\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^i F(p)^{n-i}(m-i) \geq$$

$$(m-\kappa(p))(n(1-F(p))-\kappa(p))F(p)$$

$$+(n(1-F(p))-\kappa(p))^2\sum_{i=1}^{m}\binom{n-1}{i-1}(1-F(p))^{i-1}F(p)^{n-i}(m-i)$$

Note that $m-\kappa(p)=m-\lambda(p,p)-m(1-F(p))=\sum_{i=0}^{m}\binom{n}{i}(1-F(p))^i F(p)^{n-i}(m-i)$. Thus:

$$(n\,(1-F(p))-\kappa(p))^2 \sum_{i=0}^{m} \binom{n}{i} (1-F(p))^i\, F(p)^{n-i}(m-i) \geq$$

$$(n\,(1-F(p))-\kappa(p))^2 + (n\,(1-F(p))-\kappa(p))^2 \sum_{i=1}^{m} \binom{n-1}{i-1} (1-F(p))^{i-1}\, F(p)^{n-i}(m-i)$$

It follows that $W_U'(p) \geq W_S'(p)$ if and only if:

$$\sum_{i=0}^{m} \binom{n}{i} (1-F(p))^i\, F(p)^{n-i}(m-i) - \sum_{i=1}^{m} \binom{n-1}{i-1} (1-F(p))^{i-1}\, F(p)^{n-i}(m-i) \geq 0$$

Using the identity $\binom{n-1}{i} = \binom{n}{i} - \binom{n-1}{i-1}$, this equation becomes:

$$\sum_{i=0}^{m} \binom{n-1}{i} (1-F(p))^i\, F(p)^{n-i}(m-i)$$

$$+\sum_{i=0}^{m} \binom{n-1}{i-1} (1-F(p))^i\, F(p)^{n-i}(m-i) - \sum_{i=0}^{m} \binom{n-1}{i-1} (1-F(p))^{i-1}\, F(p)^{n-i}(m-i) \geq 0$$

This reduces to:

$$\sum_{i=0}^{m} \binom{n-1}{i} (1-F(p))^i\, F(p)^{n-i}(m-i) - \sum_{i=0}^{m} \binom{n-1}{i-1} (1-F(p))^{i-1}\, F(p)^{n-i+1}(m-i) \geq 0$$

Because $\binom{n-1}{-1} = 0$, and substituting j for $i-1$, we get:

$$\sum_{i=0}^{m} \binom{n-1}{i} (1-F(p))^i\, F(p)^{n-i}(m-i) - \sum_{j=0}^{m-1} \binom{n-1}{j} (1-F(p))^j\, F(p)^{n-j}(m-j-1)$$

$$= \sum_{i=0}^{m} \binom{n-1}{i} (1-F(p))^i\, F(p)^{n-i} \geq 0.$$

This last statement is clearly true, and the inequality holds strictly if and only if $p > 0$.

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