Optimal Allocation of an Indivisible Good[*]

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Abstract

In this paper we study the allocation of an indivisible good between two agents. We consider the mechanism introduced in Zhou (2007) that gives agent one the good conditionally and gives agent two an option that allows him to buy the good from agent one at a fixed price. We show that this mechanism is optimal among all feasible strategy-proof mechanisms.

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1. Introduction

The problem of allocation of an indivisible good has been with us since the ancient time. When agents have different preferences for the good, and when such information is private, it is difficult to always achieve the efficient allocation. In this paper we consider the problem of optimal mechanism design in the simple model of allocating an indivisible good between two agents when each agent's valuation of the good is drawn from [0,1]. We are focused on mechanisms that are robust and practical. More specifically, we require that mechanisms are strategy-proof so that it is a dominant strategy for each agent to reveal the private information truthfully. In addition, we require that mechanisms are feasible so that there is no need for injection of outside money. Our goal is to identify the most efficient mechanism among all feasible strategy-proof mechanisms.

Up until now the best known feasible strategy-proof mechanism is perhaps the V-C-G pivotal mechanism (Vickery, 1961, Clarke, 1971, Groves, 1973). Though it always gives the good to the agent with the higher valuation, it occasionally needs to impose hefty taxes on both agents to ensure that they do not want to lie (Green and Laffont, 1977). On average its advantage of always assigning the good to the right agent is negated by the adverse effect of the money outflow under its tax scheme. In a recent paper Zhou (2007) introduces a mechanism that gives agent one the good conditionally and gives agent two an option that allows him to buy the good from agent one at a fixed price. He shows that his mechanism is more efficient than not only the pivotal mechanism, but also all feasible Groves mechanisms. In this paper we strengthen Zhou's result by showing that his mechanism is actually more efficient than all feasible strategy-proof mechanisms.

Our result makes some significant contributions to the literature on strategy-proof mechanisms. First, although options have been extensively used in finance as instruments to control risks for investors or as means to provide incentives for managers, they have not appeared in the literature on fair allocation. It is for the first time that an option is derived as a part of an optimal mechanism for allocation of an indivisible good. Second, while the optimal mechanism approach has been standard for the study of Bayesian mechanisms since the pioneering work of Myerson (1980), it has rarely been used by anyone to study strategy-proof mechanisms.[1] After Green and Laffont demonstrate the discord between strategy-proof-ness and full efficiency in general quasi-linear models, many scholars have tried to resolve this discord by weakening strategy-proof-ness to Bayesian incentive compatible mechanisms (d'Aspremont and Gerard-Varet, 1979). The majority of work on strategy-proof mechanisms simply drops efficiency and instead relies on alternative axioms to derive characterizations of various strategy-proof mechanisms.[2] While these axiomatic models might have shed lights on other aspects of strategy-proof mechanisms, they are silent on the trade-off between incentives and efficiency. The optimal mechanism design approach allows us to address this trade-off properly as a typical economic problem: we try to find the most efficient mechanism that satisfies both the feasibility and incentive constraints.

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[1] Myerson's work on optimal auction starts as an optimal mechanism design problem with Bayesian mechanisms. It happens that the resulting optimal mechanism is actually strategy-proof. Another exception is the optimal mechanism design problem with one agent.

[2] A few recent papers that deal with models similar to ours include: Ohseto (1999, 2006), Svensson and Larsson (2002).

2. The Main Result

Consider a model with two agents and one indivisible good, which can be consumed by one agent only. Each agent has a quasi-linear utility function for the good and money,

$$v_i(x_i, t_i; \theta_i) = \theta_i \, x_i + t_i \,,$$

in which $x_i \in \{0, 1\}$. Agents' types, θ_1 and θ_2, are drawn from $[0, 1]$. The exact value of θ_i is known to agent *i* only.

Although it is clear that the efficient allocation is to give the good to agent *i* with a higher value of θ_i, it is not always possible to identify and execute the efficient allocation given that both θ_1 and θ_2 are private information. In this paper we consider direct mechanisms that ask agents to report their types and use their reported types to determine the allocation as well as transfers between the agents.[3]

A direct mechanism *M* consists of four functions: $x_1(\theta_1,\theta_2),\, x_2(\theta_1,\theta_2),\, t_1(\theta_1,\theta_2),$ $t_2(\theta_1,\theta_2)$, in which $x_1(\theta_1,\theta_2)$ and $x_2(\theta_1,\theta_2)$ specify the allocation of the good with

$$x_i(\theta_1,\theta_2) = 0 \text{ or } 1, \text{ and } x_1(\theta_1,\theta_2) + x_2(\theta_1,\theta_2) = 1, \quad \forall\, \theta_1, \theta_2 \,,$$

and $t_1(\theta_1,\theta_2)$ and $t_2(\theta_1,\theta_2)$ are transfers between the two agents. For a mechanism to be practical, we require that no money comes from outside (although we allow agents to "burn" money): we say that a mechanism is feasible if

$$t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2) \le 0, \quad \forall\, \theta_1, \theta_2 \,.$$

For a mechanism to be effective, it should provide incentives for agents to report their types truthfully: we say that a mechanism is *M* strategy-proof (or truthful, or straight-forward) if

$$\begin{aligned} \theta_1 x_1(\theta_1,\theta_2) + t_1(\theta_1,\theta_2) \; &\ge \theta_1 x_1(\tilde{\theta}_1,\theta_2) + t_1(\tilde{\theta}_1,\theta_2), \quad \forall\, \theta_1, \tilde{\theta}_1, \theta_2 \\ \theta_2 x_2(\theta_1,\theta_2) + t_2(\theta_1,\theta_2) \; &\ge \theta_2 x_1(\theta_1,\tilde{\theta}_2) + t_1(\theta_1,\tilde{\theta}_2), \quad \forall\, \theta_1, \theta_2, \tilde{\theta}_2 \,. \end{aligned}$$

We are focused on mechanisms that are feasible and strategy-proof. The necessary condition for a pair of allocation functions $x_1(\theta_1,\theta_2)$ and $x_2(\theta_1,\theta_2)$ to be part of a strategy-proof mechanism *M* is that each $x_i(\theta_1,\theta_2)$ is increasing in θ_i. As long as each $x_i(\theta_1,\theta_2)$ is increasing in θ_i, we can always find transfers $t_1(\theta_1,\theta_2)$ and $t_2(\theta_1,\theta_2)$ so that together they form a strategy-proof mechanism. However, the feasibility constraint put limit on the sum of $t_1(\theta_1,\theta_2)$ and $t_2(\theta_1,\theta_2)$: the requirement $t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2) \le 0$ for

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[3] By the revelation principle, our analysis extends to all indirect mechanisms in which both agents have dominant strategies at every type profile.

all θ_1, θ_2 may actually imply $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) < 0$ for some θ_1, θ_2. In particular, the efficient allocation rule that assigns the good to the agent of higher type can be supported by many feasible transfer functions $t_1(\theta_1, \theta_2)$ and $t_2(\theta_1, \theta_2)$, but for all feasible transfers $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) < 0$ must hold at some θ_1, θ_2 (Green and Laffont, 1977).

In light of the result by Green and Laffont, there exists no feasible strategy-proof mechanism that is more efficient than all other feasible strategy-proof mechanisms at all type profiles (θ_1, θ_2). The natural question is then: how can we find a feasible strategy-proof mechanism that is more efficient than others on average? To answer this question, we need to impose some distributional structure on (θ_1, θ_2). This distributional structure is used only for the evaluation of the average efficiency of each feasible strategy-proof mechanism. It does not change the set of mechanisms under consideration. We assume that θ_1 and θ_2 are independently and identically distributed on [0, 1]. For simplicity, we can even assume they are uniformly distributed on [0, 1].

Let $\mathcal{F}$ denote the class of all feasible strategy-proof mechanisms. We define the efficiency coefficient of each $M \in \mathcal{F}$ as the average total utilities of both agents under *M*:

$$EC(M) = \int_0^1 \int_0^1 \left(\theta_1 x_1(\theta_1, \theta_2) + \theta_2 x_2(\theta_1, \theta_2) + t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2)\right) \, d\theta_1 d\theta_2 \; .$$

Our goal is to find a mechanism $M^* \in \mathcal{F}$ with the highest efficiency coefficient

$$EC(M^*) = \mathrm{Max}_{M \in \mathcal{F}} \; EC(M) \; .$$

The well-known pivotal mechanism has an efficiency coefficient of $\frac{1}{3}$ only. It is not even the most efficient one among all feasible Groves mechanisms. Zhou shows that the best Groves mechanism has an efficiency coefficient of $\frac{7}{12}$ (Zhou, 2007). Then, he proposes the next mechanism M^*. It gives the good to agent one conditionally and, at the same time, issues a call option to agent two that allows agent two to buy the good from agent one at the price of one half. Obviously, agent two will exercise the option if and only if $\theta_2 > 0.5$. Formally, M^* can be described as follows,

$$\begin{cases} x_1(\theta_1, \theta_2) = 1, \\ t_1(\theta_1, \theta_2) = 0, \\ x_2(\theta_1, \theta_2) = 0, \\ t_2(\theta_1, \theta_2) = 0, \end{cases} \quad \text{when } \theta_2 \le 0.5 \,; \;\text{ and } \begin{cases} x_1(\theta_1, \theta_2) = 0, \\ t_1(\theta_1, \theta_2) = 0.5, \\ x_2(\theta_1, \theta_2) = 1, \\ t_2(\theta_1, \theta_2) = -0.5, \end{cases} \quad \text{when } \theta_2 > 0.5 \,.$$

It is straightforward to verify $EC(M^*) = \frac{5}{8}$. Hence, M^* is more efficient than all feasible Groves mechanisms. The main goal of this paper is to show that M^* is actually more efficient than all feasible strategy-proof mechanisms.

Theorem $EC(M^*) \geq EC(M)$ for all $M \in \mathcal{F}$.

Before we prove the theorem, we briefly discuss some other positive features of M^*.

Individual Rationality Under mechanism M^*, neither agent suffers any utility loss at any situation: agent one either has the good or is paid by agent two if he loses the good, and agent two does not have to exercise his option to buy the good unless its value exceeds the price he has to pay. Hence, M^* respects individually rationality.

Budget-Balanced-ness The mechanism M^* is not only feasible, it actually never creates any aggregate deficit. This is very nice for anyone who wants to use M^* in practical use since one will never face the dilemma of disposing a positive sum of money after the good is allocated.

It is surprising that M^*, one of the optimal mechanisms, actually satisfies budget-balanced-ness. Since every mechanism has two parts – the allocation of the indivisible good and the transfers between agents, the loss of efficiency may come from either source: either the good is not given to the right agent, or there is a net money outflow when the sum of transfers is negative. When we view all mechanisms from this angle, Groves mechanisms and budget-balanced mechanisms are two extremes: any Groves mechanism eliminates inefficiency of the first type, whereas any budget-balanced mechanism eliminates inefficiency of the second type. One would have conjectured that the average inefficiency should be minimized by some mechanism that would admit some degrees of inefficiencies of both types. However, our result shows that the average inefficiency is minimized when the second type of inefficiency is completely removed.

Distributional Equity As agent one is the designated receiver of the good under the mechanism M^*, agent one enjoys more utility gain than agent two. By simple calculation, for any value of θ, the interim utility of agent one of type θ is $EU_1(\theta) = \frac{\theta}{2} + \frac{1}{4}$, and the interim utility of agent two of type θ is $EU_2(\theta) = \max\left\{\theta - \frac{1}{2}, 0\right\}$. Hence,

$$EU_1(\theta) > EU_2(\theta), \quad \forall \theta \in [0,1] \ .$$

However, this distributional inequity can be easily resolved when we randomize between M^* and M^{**}, which is the same as M^* except that agent two is the designated receiver of the good.

Proof of the Theorem Since $EC(M^*) = \frac{5}{8}$, we only need to show $EC(M) \le \frac{5}{8}$ for all $M \in \mathcal{F}$. Let us begin with a more detailed description of the structure of all mechanisms $M \in \mathcal{F}$. Since *M* is strategy-proof, $x_1(\theta_1, \theta_2)$ is (weakly) increasing in θ_1 and $x_2(\theta_1, \theta_2)$ is (weakly) increasing in θ_2. As we define

$$\varphi_1(\theta_2) = \inf \left\{ b \mid x_1\left(b, \theta_2\right) = 1 \right\}, \text{ and } \varphi_2(\theta_1) = \inf \left\{ a \mid x_2\left(\theta_1, a\right) = 1 \right\},$$

both $\varphi_1(.)$ and $\varphi_2(.)$ are (weakly) increasing functions from [0, 1] to [0, 1]. Since they are increasing, they are almost continuous everywhere on [0, 1]. They are virtually inverse functions to each other since

$$\theta_1 = \varphi_1(\theta_2) \text{ if and only if } \theta_2 = \varphi_2(\theta_1)$$

at every profile (θ_1, θ_2) where both $\varphi_1(.)$ and $\varphi_2(.)$ are continuous. When we plot the graphs of these two functions on the unit square $\left[0,1\right] \times \left[0,1\right]$, they separate the region in which agent one gets the good from the region in which agent two gets the good. These two functions also play important roles in the determination of transfers to both agents. In fact, we can use them to (virtually) pin down the mechanism *M* since

$$x_1(\theta_1, \theta_2) = \begin{cases} 0, & \theta_1 < \varphi_1(\theta_2) \\ 1, & \theta_1 > \varphi_1(\theta_2) \end{cases}, \quad t_1(\theta_1, \theta_2) = \begin{cases} h_1(\theta_2), & \theta_1 < \varphi_1(\theta_2) \\ h_1(\theta_2) - \varphi_1(\theta_2), & \theta_1 > \varphi_1(\theta_2) \end{cases},$$

$$x_2(\theta_1, \theta_2) = \begin{cases} 0, & \theta_2 < \varphi_2(\theta_1) \\ 1, & \theta_2 > \varphi_2(\theta_1) \end{cases}, \quad t_2(\theta_1, \theta_2) = \begin{cases} h_2(\theta_1), & \theta_2 < \varphi_2(\theta_1) \\ h_2(\theta_1) - \varphi_2(\theta_1), & \theta_2 > \varphi_2(\theta_1) \end{cases},$$

in which $h_1(\theta_2)$ and $h_2(\theta_1)$ are two arbitrary functions as long as the feasibility condition is not violated.[4] For any fixed θ_i, agent *j* receives a transfer $h_j(\theta_i)$; he does not get the good when his type is lower than $\varphi_j(\theta_i)$, and he gets the good when his type is higher than $\varphi_j(\theta_i)$, but sees the transfer $h_j(\theta_i)$ reduced by $\varphi_j(\theta_i)$.

We divide the proof into three parts. First, we show that $EC(M) \le \frac{5}{8}$ for all mechanisms in which both $h_1(\theta_2)$ and $h_2(\theta_1)$ are increasing. Second, we prove that the same upper-bound works for mechanisms in which both $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ are step functions. We show that any such mechanism can be improved by another mechanism in which both $h_1(\theta_2)$ and $h_2(\theta_1)$ are increasing. Finally, we show the validity of the upper-

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[4] We have not specified if the good is assigned to agent one or two when $\theta_1 = \varphi_1(\theta_2)$ or $\theta_2 = \varphi_2(\theta_1)$. How the tie is broken can be arbitrary and will have no effect on the computation of $EC(M)$.

bound for any general mechanism since any general mechanism can be approximated by a sequence of mechanisms in which both $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ are step functions.

Part 1 We start with mechanisms in which $h_1(\theta_2)$ and $h_2(\theta_1)$ are increasing. First, mimicking Myerson's analysis of Bayesian mechanisms (Myerson, 1978), we can express transfers in terms of allocation rules,

$$t_1(\theta_1, \theta_2) = -\theta_1 x_1(\theta_1, \theta_2) + \int_0^{\theta_1} x_1(\tau, \theta_2)\, d\tau + h_1(\theta_2), \text{ and}$$

$$t_2(\theta_1, \theta_2) = -\theta_2 x_2(\theta_1, \theta_2) + \int_0^{\theta_2} x_2(\theta_1, \sigma)\, d\sigma + h_2(\theta_1).$$

Consequently,

$$\begin{aligned}
&\int_0^1\int_0^1 \left(\theta_1 x_1(\theta_1,\theta_2) + t_1(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2 \\
&= \int_0^1\int_0^1 \left(\int_0^{\theta_1} x_1(\tau, \theta_2)\, d\tau\right) d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2) d\theta_2 \\
&= \int_0^1\int_0^1 \left(\int_\tau^1 x_1(\tau, \theta_2)\, d\theta_1\right) d\tau d\theta_2 + \int_0^1 h_1(\theta_2) d\theta_2 \\
&= \int_0^1\int_0^1 (1-\tau) x_1(\tau, \theta_2) d\tau d\theta_2 + \int_0^1 h_1(\theta_2) d\theta_2 \\
&= \int_0^1\int_0^1 (1-\theta_1) x_1(\theta_1, \theta_2) d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2) d\theta_2
\end{aligned}$$

Similarly, we also have

$$\int_0^1\int_0^1 \left(\theta_2 x_2(\theta_1,\theta_2) + t_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2 = \int_0^1\int_0^1 (1-\theta_2) x_2(\theta_1, \theta_2) d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2) d\theta_2 .$$

Therefore,

$$\int_0^1\int_0^1 \left(\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2$$

$$= \int_0^1\int_0^1 \left((1-\theta_1)x_1(\theta_1,\theta_2)+(1-\theta_2)x_2(\theta_1,\theta_2)\right)d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1$$

$$= \int_0^1\int_0^1 \left(x_1(\theta_1,\theta_2)+x_2(\theta_1,\theta_2)\right)d\theta_1 d\theta_2 - \int_0^1\int_0^1 \left(\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)\right)d\theta_1 d\theta_2$$

$$+ \int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1$$

$$= 1 - \int_0^1\int_0^1 \left(\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)\right)d\theta_1 d\theta_2 + \int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1$$

$$\le 1 - \int_0^1\int_0^1 \left(\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\right)d\theta_1 d\theta_2$$

$$+ \int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1.$$

The last inequality holds because $t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\le 0$ by feasibility. Hence,

$$EC(M) = \int_0^1\int_0^1 \left(\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2$$
$$\le \frac{1}{2}+\frac{1}{2}\left(\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1\right) \quad .$$

Let us now find the maximal value of $\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1$. Since the value of this expression will remain the same when any unconditional transfer is made between agent one and agent two, we may assume that $h_1(1)=h_2(1)=\alpha$. Given that $h_1(\theta_2)$ and $h_2(\theta_1)$ are non-decreasing, we have both $h_1(\theta_2)\le\alpha$ and $h_2(\theta_1)\le\alpha$.

Without loss of generality we will also assume that agent one gets the good when $\theta_1 = 1$ and $\theta_2 < 1$, and agent twp gets the good when $\theta_2 = 1$ and $\theta_1 < 1$. (As we can see in later parts, any $M \in \mathcal{F}$ can be approximated by a sequence of mechanisms of such.) Now consider the feasibility condition at $\theta_1 = 1$:

$$h_1(\theta_2)+h_2(1)-\varphi_1(\theta_2)\le 0, \text{ or } h_1(\theta_2)\le\varphi_1(\theta_2)-\alpha .$$

Thus,

$$h_1(\theta_2)\le \min\{\,\varphi_1(\theta_2)-\alpha,\alpha\,\} = \min\{\,\varphi_1(\theta_2), 2\alpha\,\}-\alpha = g_1(\theta_2)-\alpha ,$$

where $g_1(\theta_2) = \min\{\varphi_1(\theta_2), 2\alpha\}$. Similarly, we also have

$$h_2(\theta_1) \le g_2(\theta_1) - \alpha ,$$

where $h_2(\theta_1) = \min\{\varphi_2(\theta_1), 2\alpha\}$. Hence, we have

$$\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1 \le \int_0^1 g_1(\theta_2)d\theta_2 + \int_0^1 g_2(\theta_1)d\theta_1 - 2\alpha \ .$$

From Figure 1 below, $\int_0^1 g_1(\theta_2)d\theta_2$ = area (*A*), and $\int_0^1 g_2(\theta_1)d\theta_1$ = area (*B*), hence

$$\int_0^1 g_1(\theta_2)d\theta_2 + \int_0^1 g_2(\theta_1)d\theta_1 \le 1 - (1-2\alpha)^2 .$$

Therefore,

$$\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1 \le 1 - (1-2\alpha)^2 - 2\alpha = 2\alpha(1-2\alpha) \ .$$

The maximum on the right is achieved when $\alpha = 1/4$ and has a value of $1/4$. As a result, for all mechanisms in which $h_1(\theta_2)$ and $h_2(\theta_1)$ are increasing,

$$EC(M) \le \frac{1}{2} + \frac{1}{2}\left(\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1\right) \le \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} = \frac{5}{8} \ .$$

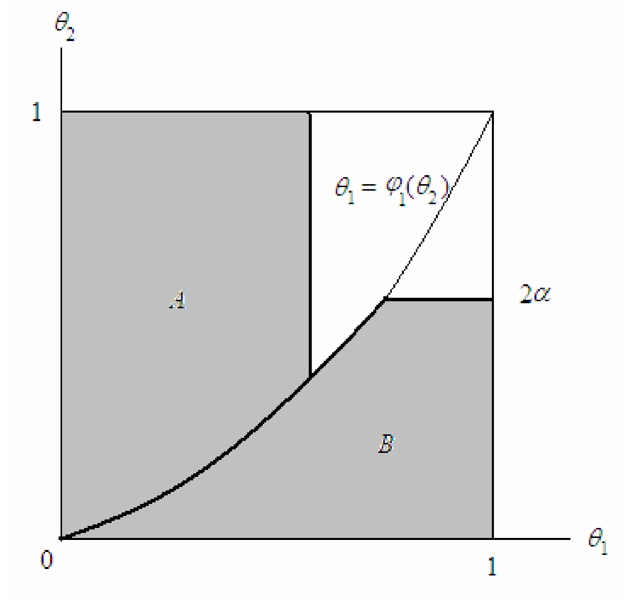


Figure 1

Part 2 Next, we consider mechanisms in which both $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ are step functions. For any integer *m*, we consider mechanisms of the following form. There are $2m$ parameters: $0 \le x_1 \le \ldots \le x_m \le 1$ and $0 \le y_1 \le \ldots \le y_m \le 1$, which define $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ (see Figure 2):

$$\varphi_1(\theta_2) = \begin{cases} 0\,, & \theta_2 \in [0, y_1] \\ x_i\,, & \theta_2 \in (y_i, y_{i+1}], \quad \forall i = 1, \ldots, m \end{cases}$$

$$\varphi_2(\theta_1) = \begin{cases} y_i\,, & \theta_1 \in [x_{i-1}, x_i)\,, \quad \forall i = 1, \ldots, m \\ 1\,, & \theta_1 \in [x_m, 1]\,. \end{cases}$$

(We adopt the convention $x_0 = y_0 = 0$ and $x_{m+1} = y_{m+1} = 1$).

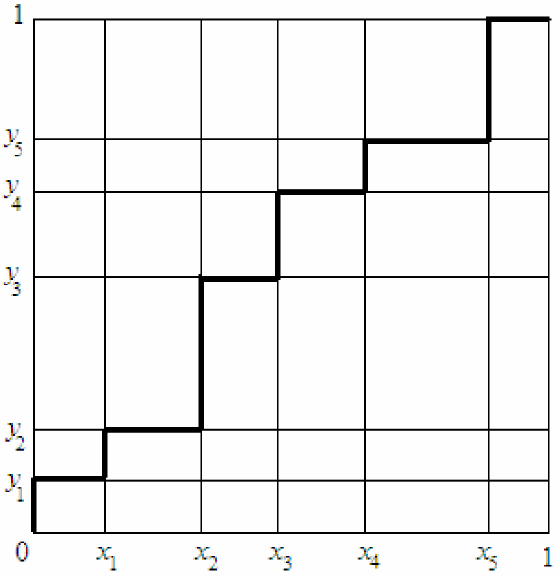


Figure 2

For any given pair of $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$, there is a continuum of functions $h_1(\theta_2)$ and $h_2(\theta_1)$ that are consistent with $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ in the sense that together they form a feasible strategy-proof mechanism. Again, $\frac{5}{8}$ remains the upper-bound of $EC(M)$ if we can show $\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1 \le \frac{1}{4}$ for all such $h_1(\theta_2)$ and $h_2(\theta_1)$.

Although $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$ are step functions, $h_1(\theta_2)$ and $h_2(\theta_1)$ are not necessarily step functions. Let us find another pair of step functions $\tilde{h}_1(\theta_2)$ and $\tilde{h}_2(\theta_1)$ that are also consistent with $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$. Moreover, $\tilde{h}_1(\theta_2)$ and $\tilde{h}_2(\theta_1)$ dominate $h_1(\theta_2)$ and $h_2(\theta_1)$. Consider the feasible condition on all small rectangles $[x_i, x_{i+1}]\times[y_j, y_{j+1}]$ ($\forall i = 0,\dots,m$ and $\forall j = 0,\dots,m$),

$$h_1(\theta_2) + h_2(\theta_1) \le x_j, \quad \forall i \ge j \text{, and}$$

$$h_1(\theta_2) + h_2(\theta_1) \le y_{i+1}, \quad \forall i < j .$$

Notice that the right hand sides of all inequalities above are constant. Hence, when we take the supremum of $h_1(\theta_2)$ over $[y_j, y_{j+1}]$ and the supremum of $h_2(\theta_1)$ over $[x_i, x_{i+1}]$, these inequalities still hold. Hence, let $h_{1j} = \sup_{\theta_2 \in [y_j, y_{j+1}]} h_1(\theta_2)$ and $h_{2i} = \sup_{\theta_1 \in [x_i, x_{i+1}]} h_2(\theta_1)$, we should have

$$h_{1j} + h_{2i} \le x_j, \quad \forall i \ge j \text{, and}$$

$$h_{1j} + h_{2i} \le y_{i+1}, \quad \forall i < j .$$

Thus, we obtain two step functions $\tilde{h}_1(\theta_2)$ and $\tilde{h}_2(\theta_1)$ with $\tilde{h}_1(\theta_2) = h_{1j}$ on $[y_j, y_{j+1}]$ ($\forall j = 0,\dots,m$) and $\tilde{h}_2(\theta_1) = h_{2i}$ on $[x_i, x_{i+1}]$ ($\forall i = 0,\dots,m$) that are also consistent with $\varphi_1(\theta_2)$ and $\varphi_2(\theta_1)$. Since $h_1(\theta_2) \le \tilde{h}_1(\theta_2), \forall \theta_2 \in [0,1]$ and $h_2(\theta_1) \le \tilde{h}_2(\theta_1), \ \forall \theta_1 \in [0,1]$, we must have

$$\int_0^1 h_1(\theta_2)d\theta_2 + \int_0^1 h_2(\theta_1)d\theta_1 \le \int_0^1 \tilde{h}_1(\theta_2)d\theta_2 + \int_0^1 \tilde{h}_2(\theta_1)d\theta_1 .$$

Let us now find an upper-bound for the sum of the integrals of $\tilde{h}_1$ and $\tilde{h}_2$. By the construction,

$$\int_0^1 \tilde{h}_1(\theta_2)d\theta_2 + \int_0^1 \tilde{h}_2(\theta_1)d\theta_1 = \sum_{j=0}^{m}\left(y_{j+1} - y_j\right)h_{1j} + \sum_{i=0}^{m}\left(x_{i+1} - x_i\right)h_{2i} \ .$$

The maximal value of the right hand side cannot be larger than the maximal value of the following maximization problem, in which $x_1,\dots,x_m$ and $y_1,\dots,y_m$, together with $h_{11},\dots,h_{1(m+1)}$ and $h_{21},\dots,h_{2(m+1)}$, are also free choice variables:

$$
\begin{aligned}
\max \quad & \sum_{j=0}^{m}\left(y_{j+1}-y_j\right)h_{1j}+\sum_{i=0}^{m}\left(x_{i+1}-x_i\right)h_{2i} \\
s.t. \quad & h_{1j}+h_{2i}-x_j \le 0, \quad \forall i \ge j \\
& h_{1j}+h_{2i}-y_{i+1} \le 0, \quad \forall i < j \\
& 0 \le x_1 \le \ldots \le x_m \le 1, \\
& 0 \le y_1 \le \ldots \le y_m \le 1.
\end{aligned}
$$

We do not know how to solve this problem directly. Instead we will try to find sufficient information about the solution so that we may solve it indirectly. We let $d_j = y_{j+1}-y_j$ and $\tilde{d}_i = x_{i+1}-x_i$, and rewrite the original problem as:

$$
\begin{aligned}
\max \quad & \sum_{j=0}^{m} d_j h_{1j}+\sum_{i=0}^{m}\tilde{d}_i h_{2i} \\
s.t. \quad & h_{1j}+h_{2i}-\sum_{k=0}^{j-1}\tilde{d}_k \le 0, \quad \forall i \ge j \\
& h_{1j}+h_{2i}-\sum_{k=0}^{i} d_k \le 0, \quad \forall i < j \\
& d_j \ge 0, \forall j \\
& \tilde{d}_i \ge 0, \ \forall i \\
& \sum_{j=0}^{m} d_j = 1, \\
& \sum_{i=0}^{m} \tilde{d}_i = 1.
\end{aligned}
$$

Suppose (h^*, d^*) is the optimal solution. Consider the first order condition for d_j:

$$h_{1j}^* + \sum_{k=j}^{m}\sum_{l=k+1}^{m}\lambda_{kl} + \mu_j - \alpha = 0,$$

in which $\lambda_{kl} \ge 0$, and $\mu_j d_j^* = 0$. When $d_j^* = 0$, $y_{j+1}^* = y_j^*$. So the value of h_{1j}^* is irrelevant for the optimization problem. We only need to consider those j for which $d_j^* > 0$. In such cases, $\mu_j = 0$. As a result,

$$h_{1j}^* = \alpha - \sum_{k=j}^{m}\sum_{l=k+1}^{m}\lambda_{kl} \ .$$

Thus, h_{1j}^* is increasing in j. Similarly, we can show that h_{2i}^* is also increasing in i. Hence, the optimal solution of the problem (h^*, d^*) defines a feasible strategy-proof mechanism in which the two h functions are increasing. Then, by what we have shown in Part 1, the upper-bound of $EC(M)$ is $\frac{5}{8}$.

Part 3 At last we show that any mechanism $M \in \mathcal{F}$ can be approximated by a sequence of "step" mechanisms that have the form as described in Part 2. We show that for any integer $n \geq 2$ there is a step mechanism M_n with

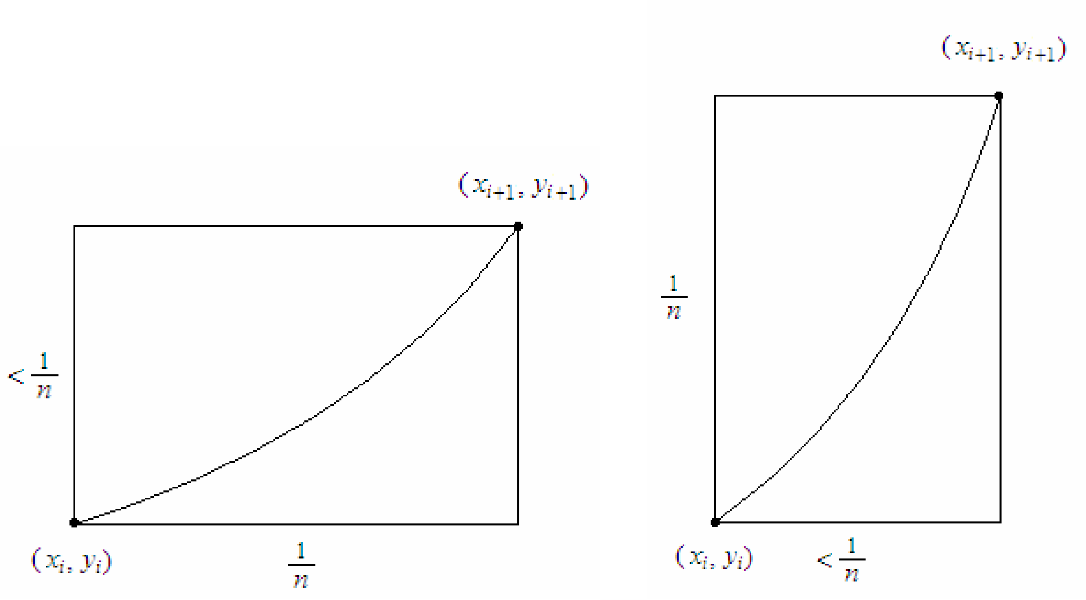
$$EC(M) \leq EC(M_n) + \frac{5}{n}.$$

Since we have already shown $EC(M_n) \leq \frac{5}{8}$, we must have $EC(M) \leq \frac{5}{8}$ when we take the limit of this inequality as $n \to \infty$.

We now construct the desired mechanism M_n. Let A_1 and A_2 denote the regions in $[0,1] \times [0,1]$ where agent one and agent two get the good respectively. We first construct a sequence $\{(x_i, y_i)\}_{i=1,\ldots,m}$ with each (x_i, y_i) on the border that separates A_1 and A_2. Pairs in the sequence are determined recursively: Suppose (x_i, y_i) is already determined, (x_{i+1}, y_{i+1}) is defined as follows:

(1) if $\varphi_2\left(x_i + \frac{1}{n}\right) \leq y_i + \frac{1}{n}$, then $x_{i+1} = x_i + \frac{1}{n}$, and $y_{i+1} = \varphi_2\left(x_i + \frac{1}{n}\right)$, or

(2) if $\varphi_2\left(x_i + \frac{1}{n}\right) > y_i + \frac{1}{n}$, then $x_{i+1} = \inf\left\{x \mid \varphi_2(x) > y_i + \frac{1}{n}\right\}$, and $y_{i+1} = y_i + \frac{1}{n}$.



Case (1) **Case (2)**

Figure 3

The construction stops when both x_i and y_i are greater than or equal $1-\frac{1}{n}$. Since $\max\{x_{i+1}-x_i,\, y_{i+1}-y_i\}=\frac{1}{n}$, the sequence will terminate in no more than $2n-1$ steps. We now use $\{(x_i, y_i)\}_{i=1,\ldots,m}$ to construct M_n:

A. The allocation of the good. On any diagonal rectangle $[x_i, x_{i+1}]\times[y_i, y_{i+1}]$, the good is given to agent one if $x_i \geq y_i$, and to agent two if $x_i < y_i$. (The idea behind this is to ensure that the new allocation is "nearly" welfare-improving.) The allocation of the good everywhere else remains the same as in *M*.

B. The transfers. The new $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are obtained by the new allocation rule as specified in A. Obviously, $\left|\varphi_1(\theta_2)-\tilde{\varphi}_1(\theta_2)\right|<\frac{1}{n}$ and $\left|\varphi_2(\theta_1)-\tilde{\varphi}_2(\theta_1)\right|<\frac{1}{n}$. The new $\tilde{h}_1$ and $\tilde{h}_2$ are simply defined by $\tilde{h}_1=h_1-\frac{1}{n}$ and $\tilde{h}_2=h_2-\frac{1}{n}$.

Let us verify that the feasibility condition still holds for M_n. When the allocation remain the same agent *i*,

$$\tilde{h}_1(\theta_2)+\tilde{h}_2(\theta_1)=h_1(\theta_2)-\frac{1}{n}+h_2(\theta_1)-\frac{1}{n}\leq\varphi_i(\theta_j)-\frac{2}{n}<\tilde{\varphi}_i(\theta_j)\ .$$

A switch of allocation of the good from one agent to the other can take place only on some diagonal rectangle $[x_i, x_{i+1}]\times[y_i, y_{i+1}]$. Nevertheless, the switching is designed to be "nearly" welfare-improving. When the good is switched from agent two to agent one,

$$\tilde{\varphi}_1(\theta_2)=x_i\geq y_i\geq y_{i+1}-\frac{1}{n}\geq\varphi_2(\theta_1)-\frac{1}{n}\ .$$

Hence,

$$\tilde{h}_1(\theta_2)+\tilde{h}_2(\theta_1)=h_1(\theta_2)-\frac{1}{n}+h_2(\theta_1)-\frac{1}{n}\leq\varphi_2(\theta_1)-\frac{2}{n}<\tilde{\varphi}_1(\theta_2)\ .$$

Similarly, when the good is switched from agent one to agent two, we also have

$$\tilde{h}_1(\theta_2)+\tilde{h}_2(\theta_1)<\tilde{\varphi}_2(\theta_1)\ .$$

Therefore, M_n is feasible.

Finally, let us compare $EC(M)$ and $EC(M_n)$. Given the construction in A, on each diagonal rectangle $[x_i, x_{i+1}]\times[y_i, y_{i+1}]$, we have

$$\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)\le \theta_1\tilde{x}_1(\theta_1,\theta_2)+\theta_2\tilde{x}_2(\theta_1,\theta_2)+\frac{1}{n}\ .$$

When we integrate the above inequality over $[x_i, x_{i+1}]\times[y_i, y_{i+1}]$,

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}}\int_{y_i}^{y_{i+1}}\left(\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2\\
&\le \int_{x_i}^{x_{i+1}}\int_{y_i}^{y_{i+1}}\left(\theta_1\tilde{x}_1(\theta_1,\theta_2)+\theta_2\tilde{x}_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2+\frac{1}{n}\left(x_{i+1}-x_i\right)\left(y_{i+1}-y_i\right).\\
&\le \int_{x_i}^{x_{i+1}}\int_{y_i}^{y_{i+1}}\left(\theta_1\tilde{x}_1(\theta_1,\theta_2)+\theta_2\tilde{x}_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2+\frac{1}{n^3}
\end{aligned}$$

Summing them over from 1 to $m\left(\le 2n\right)$,

$$\begin{aligned}
&\int_0^1\int_0^1\left(\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2\\
&\le \int_0^1\int_0^1\left(\theta_1\tilde{x}_1(\theta_1,\theta_2)+\theta_2\tilde{x}_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2+\frac{2n}{n^3}\\
&\le \int_0^1\int_0^1\left(\theta_1\tilde{x}_1(\theta_1,\theta_2)+\theta_2\tilde{x}_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2+\frac{1}{n}
\end{aligned}$$

Thus,

$$\begin{aligned}
EC(M)&=\int_0^1\int_0^1\left(\theta_1 x_1(\theta_1,\theta_2)+\theta_2 x_2(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\right)\, d\theta_1 d\theta_2\\
&\le \int_0^1\int_0^1\left(\theta_1\tilde{x}_1(\theta_1,\theta_2)+\theta_2\tilde{x}_2(\theta_1,\theta_2)\right)d\theta_1 d\theta_2+\frac{1}{n}+\int_0^1\int_0^1\left(\tilde{t}_1(\theta_1,\theta_2)+\tilde{t}_2(\theta_1,\theta_2)\right)d\theta_1 d\theta_2+\frac{4}{n}\\
&=EC(M_n)+\frac{5}{n}
\end{aligned}$$

Q.E.D.

3. Concluding Remarks

When an indivisible good is to be allocated between two agents, we find a mechanism that, among all feasible strategy-proof mechanisms, maximizes the sum of agents' utilities on average. It gives the good to agent one and gives an option to agent two, which allows agent two to buy the good from agent one at a price of one half. In addition to being the most efficient one, this mechanism also satisfies individual rationality and budget-balanced-ness. It can also be used to construct a (random) mechanism that also treats both agents equally.

We have assumed that both agents' types are uniformly distributed on [0, 1], our analysis remains valid as long as agents' types are independently and identically distributed with any other distribution on [0, 1]. This is not surprising once we realize that the set of feasible strategy-proof mechanisms are the same regardless of the distribution of types. The optimal mechanism is the same except that the exercise price of the option is now the mean of the agent's types.

The next natural extension of our work would be the case of *n* agents with one single indivisible good. However, the highly geometric nature of our current analysis does not allow an immediate adaptation. At this stage we are not even sure what might be the form of the optimal mechanism. Our conjecture is that there is still an optimal mechanism that satisfies the budget-balanced-ness. A possible candidate is the mechanism in which agent one plays the role of a seller and runs a second-price auction of the good for the other $n-1$ agents with a reservation price α. Such a mechanism never runs a deficit, and is also reasonably efficient in allocating the good for some value of α. The best value of α can be found by maximizing the sum of the expected utilities of all agents (the transfers are ignored since the budget is always balanced):

$$\alpha^* = \arg\max\left(\frac{1}{2}\alpha^{n-1} + \int_{\alpha}^{1} y d(y^{n-1})\right) .$$

Hence, $\alpha^* = \frac{1}{2}$. Notice that this reservation value is the same as that in Myerson's mechanism in which the expected revenue of the seller is maximized. It would be remarkable that the maximum efficiency in allocating an indivisible good among *n* agents can be achieved when we give the good to one of the agents and direct him to conduct a revenue maximizing auction with the other $n-1$ agents as buyers. (Would that provide another justification for privatization?) It is purely speculative at this point, and further research must be carried out to settle the issue.

Finally, we want to repeat a point we raised earlier. As most current work on strategy-proof mechanisms has been predominantly axiomatic, it is desirable that we can broaden the scope of the research using alternative approaches. In this paper we adopt the optimal mechanism design approach to study strategy-proof mechanisms, which leads us to discover some important mechanisms that have not received economists' attentions yet. We hope that the same approach will also lead to more interesting findings in other models.

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