Lotteries and the Law of Demand

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Summary. In economies with nonconvexities consumers can increase their expected utility by consuming lotteries. Lotteries are probability distributions over bundles in the consumption set. Standard revealed preference logic can be applied to choices in lottery space, however the implications are not readily interpretable. In this paper, we formulate the law of demand for lottery economies in terms of commodity price changes and changes in demand for commodities. The finding is that the standard expression of the compensated law of demand necessarily holds in expectation only.

1 Introduction

In economies with nonconvexities consumers sometimes prefer to have their final consumption determined by lottery. By randomizing over consumption bundles, even a risk averse consumer can achieve higher expected utility than if she chooses any single consumption bundle without randomizing. This point was first made in the context of indivisible goods in [11]. Discussions of how lotteries improve consumer welfare in situations where consumers make indivisible expenditures on education and improvements to property are found in [7] and [8], respectively.

Lottery demands are probability distributions over the consumption bundles in the consumption set. When lottery demands satisfy the weak axiom of revealed preference the implication is that for compensated price changes

$$(\Pi' - \Pi) \cdot (L' - L) \leq 0, \tag{1}$$

where Π and Π' are two price vectors that have dimension equal to the number of available commodity bundles, and L and L' are the corresponding lottery demands. The law of demand in this setting, which can be expressed as a special case of (1) where Π and Π' differ by a single component, says that the probability demanded on a commodity bundle must vary inversely with a compensated change in the price of the bundle.

This statement of the law of demand is somewhat removed from what people (economists and actual consumers) normally focus on, namely, changes in prices of commodities and changes of demands for commodities. In this paper, the law of demand for lottery economies is expressed in terms of commodity price changes and changes in demand for commodities. This is done by mapping lotteries into state-contingent consumptions, so that the resulting state-contingent demand vectors, $x(s)$, have dimension equal to the number of commodities, not the number of commodity bundles.

Suppose p and p' are two vectors of prices of individual commodities. The main result is that satisfaction of the weak axiom in lottery economies implies that

$$\int_S (p' - p) \cdot (x'(s) - x(s))ds \leq \mathbf{0}. \tag{2}$$

Thus, in suitably defined lottery economies, the standard expression of the law of demand, $(p' - p) \cdot (x' - x) \leq 0$, necessarily holds in expectation only.

In section 2, we establish the main result for the case where the commodity set has a finite number of bundles. The literature on lottery equilibrium often assumes that the consumption set has only a finite number of points because this simplifies notation, proofs, and computations (see [12], [13] and [3]). However, the main result extends quite easily to more general commodity spaces with an uncountable number of commodities (see Remark 2). Section 3 provides an example where the consumer selects consumption bundles comprised of 0 or 1 units of a single indivisible good and any quantity of a divisible one. The case of $\{0, 1\}$ goods has received much attention in the literature on labor-market lotteries ([9], [14], [15]). The example describes a scenario in which the probability demanded on the indivisible good increases when its price rises. Nevertheless, it is easily verified that the lottery law of demand, and its alternative expression provided in (5) below, holds when income is compensated for the price change. A brief discussion of the main result is provided in Section 4.

2 Model and Results

A consumer with endowed wealth w is deciding on her consumption of n goods. Her consumption set C is nonconvex, and hence she considers lotteries. In this section, we assume that she only has access to a finite set of possible consumption bundles, so her consumption set C is finite, $|C| = m$. Thus, we consider lottery demands $L = (q_1, ..., q_m)$, where q_i denotes the probability demanded on commodity bundle i, $\sum_{i=1}^{m} q_i = 1$ and $q_i \geq 0$.

Suppose the cost of lottery L is given by $\Pi \cdot L$ where $\Pi \in \Re^m_+$. Suppose that for given prices Π and wealth w the consumer has lottery demand $L(\Pi, w)$ that is homogeneous of degree zero and satisfies Walras' Law.

Definition 1. *(WARP): The lottery demand function $L(\Pi, w)$ satisfies the weak axiom of revealed preference if for any two price-wealth situations (Π, w) and (Π', w') : If $\Pi \cdot L(\Pi', w') \leq w$ and $L(\Pi', w') \neq L(\Pi, w)$ then $\Pi' \cdot L(\Pi, w) > w'$.*

WARP implies that for compensated price changes

$$(\Pi' - \Pi) \cdot [L(\Pi', w') - L(\Pi, w)] \leq 0 \tag{3}$$

with strict inequality whenever $L(\Pi', w') \neq L(\Pi, w)$ (See, for example, p. 31, part (i) of [10]). Note that for reasons outlined in [3] no arbitrage implies there exists $p \in \Re^n_+$ such that $\Pi_i = p \cdot c_i$. Thus, (3) becomes (omitting obvious transposes)

$$\begin{pmatrix} p' \cdot c_1 - p \cdot c_1 \\ \vdots \\ p' \cdot c_m - p \cdot c_m \end{pmatrix} \cdot [L(\Pi', w') - L(\Pi, w)] \leq 0$$

or

$$\begin{pmatrix} (p' - p) \cdot c_1 \\ \vdots \\ (p' - p) \cdot c_m \end{pmatrix} \cdot [L(\Pi', w') - L(\Pi, w)] \leq 0. \tag{4}$$

Fix a continuous probability space $(S, \sum, \mu)$ where $S = [0, 1]$, $\sum$ is the Borel sets on S and μ is the Lebesgue measure. By Lemma 4 of [5] there exists $x : S \to \Re^n_+$ such that

$$L(\Pi, w) = \mu \circ x^{-1}.$$

This establishes that, given a lottery L, one can find a state contingent consumption x defined over S with the property that for each bundle c_i, $i = 1, ..., m$, the measure of the set of states $\{s : x(s) = c_i\}$ is q_i.

Given this result, (4) becomes

$$\begin{pmatrix} (p' - p) \cdot c_1 \\ \vdots \\ (p' - p) \cdot c_m \end{pmatrix} \cdot [\mu \circ x'^{-1} - \mu \circ x^{-1}] \leq 0$$

or

$$\begin{pmatrix} (p' - p) \cdot c_1 \\ \vdots \\ (p' - p) \cdot c_m \end{pmatrix} \cdot \begin{pmatrix} \int_{S_1'} ds - \int_{S_1} ds \\ \vdots \\ \int_{S_m'} ds - \int_{S_m} ds \end{pmatrix} \leq 0$$

where $\int_{S_i} ds = \int_{S:x(s)=c_i} ds$. Rewrite the left-hand-side to get

$$(p' - p) \cdot \left[c_1 \left(\int_{S_1'} ds - \int_{S_1} ds \right) + \cdots + c_m \left(\int_{S_m'} ds - \int_{S_m} ds \right) \right] =$$

$$(p'-p)\cdot\left[\left(\int_{S_1'} c_1 ds+\cdots+\int_{S_m'} c_m ds\right)-\left(\int_{S_1} c_1 ds+\cdots+\int_{S_m} c_m ds\right)\right]=$$

$$(p'-p)\cdot\left[\left(\int_{S_1'} x_1'(s)ds+\cdots+\int_{S_m'} x'(s)ds\right)-\left(\int_{S_1} x(s)ds+\cdots+\int_{S_m} x(s)ds\right)\right]=$$

$$(p'-p)\cdot\left[\int_0^1 x'(s)ds-\int_0^1 x(s)ds\right].$$

Thus we have

$$\int_0^1 (p'-p)\cdot(x'(s)-x(s))ds\leq 0.$$

For a change in the price of a single commodity i this becomes

$$\int_0^1 (p_i'-p_i)(x_i'(s)-x_i(s))ds\leq 0, \tag{5}$$

which is the desired result.

Remark 1. Suppose L is such that q_i is a rational number for $i=1,...,m$. Then we can duplicate the above process using a finite set of k equally-probable extrinsic states to get

$$\sum_{j=1}^{k}(p'-p)\cdot(x'(s_j)-x(s_j))\leq 0.$$

Remark 2. Redefine the consumption set C to be any (possibly nonconvex) subset of $\Re^n_+$. Let $\Delta(C)$ denote the set of probability measures over the set C. An individual lottery demand is an element of $\Delta(C)$, $\delta: B(C)\rightarrow\Re_+$, where $B(C)$ denotes the Borel subsets of a given set C. We have assumed that lottery prices are such that a unit of probability on commodity bundle $c\in C$ costs $p\cdot c$. Thus we can define an optimally chosen lottery, given commodity prices p and income w, by $\delta(p,w)$.[1]

A wide class of examples will involve solutions to the lottery choice problem that have countable support. For instance, this will be true if $C=Z_+^{\ell_1}\times\Re_+^{\ell_2}$ where ℓ_1 is the number of indivisible goods and ℓ_2 is the number of divisible goods, and her utility function is strictly concave on $\Re^{\ell_2}_{++}$. Risk averse consumers will not engage in unnecessary randomization, and hence the support of the chosen distribution will typically be quite small.

[1] For a discussion of the consumer's lottery choice problem in this environment see [5].

Suppose that the consumer always demands strictly positive probability on a finite number of commodity bundles. Then, given two price-income combinations p, w and p', w', let $\{c_1, ..., c_m\}$ denote the joint list of commodity bundles that receive strictly positive probability by the consumer in either of her selected lotteries. Let $L = \{q_1, ..., q_m\}$ and $L' = \{q'_1, ..., q'_m\}$ denote m-dimensional vectors where $q_i = \delta(c_i; p, w)$ and $q'_i = \delta(c_i; p', w')$, for $i = 1, ..., m$. Note that for each vector L and L' the components are non-negative and sum to 1. The statement of WARP and the implications for compensated price changes can be evaluated using these finite-vector expressions of lottery demands, as shown in Section 3.1.

3 Example

Consider a consumer with endowed wealth w choosing consumption of an indivisible good $x_1 \in \{0, 1\}$ and perfectly divisible (composite) good x_2. Suppose utility for the consumer is represented by a von Neumann-Morgenstern utility function $u = U(x)$. U is twice differentiable with respect to x_2, $U(1, x_2) > U(0, x_2)$ for all x_2, and U is strictly increasing and strictly concave in x_2. Suppose the price of the indivisible good is p_1 and the price of the divisible good is 1. Any money that is not spent on the indivisible good is spent on the divisible good. Thus, without lotteries, and provided $p_1 < w$, the consumer's problem is simply a choice between two alternatives, $(0, w)$ and $(1, w - p_1)$. The former solution occurs whenever $U(0, w) > U(1, w - p_1)$ while the latter occurs whenever $U(1, w - p_1) > U(0, w)$. If $U(1, w - p_1) = U(0, w)$ then both $(0, w)$ and $(1, w - p_1)$ are solutions.

Using z as a running variable for wealth, indirect utility is denoted by

$$F(z; p) = \max[U(0, z), \, U(1, z - p_1)],$$

and is shown as the upper envelope of the two utility curves in Figure 1. The key to understanding the figure is to recognize that consuming the indivisible good requires an expenditure of p_1. Thus the upper curve in Figure 1, denoted $U(1, z - p_1)$, is just the curve $U(1, z)$ shifted to the right by the cost of the indivisible good. Given our assumptions on utility it is easy to establish the existence of a reservation price for the indivisible good, below which the indivisible good is always consumed and above which it is not. The existence of a reservation price implies that according to the standard analysis the indivisible good cannot be a Giffen good. Because of the indivisibility, however, consumers may increase their utility exante by randomizing over final consumption bundles. This is because indivisibility leads to risk loving by consumers.

The indirect utility shown in Figure 1 is of the form posed by [2]. Thus the consumer can gain from a gamble. This is shown in Figure 2. To obtain Figure 2, take Figure 1 and form the least concave hull by drawing a straight

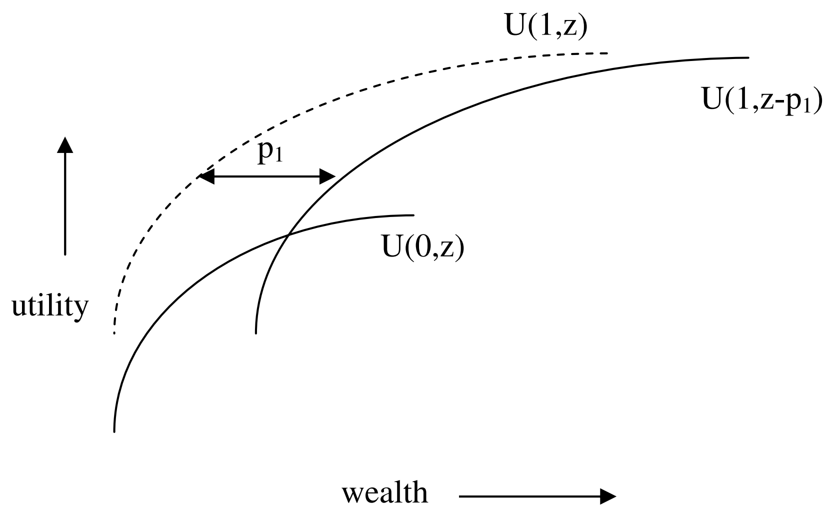

Fig. 1. Indirect utility

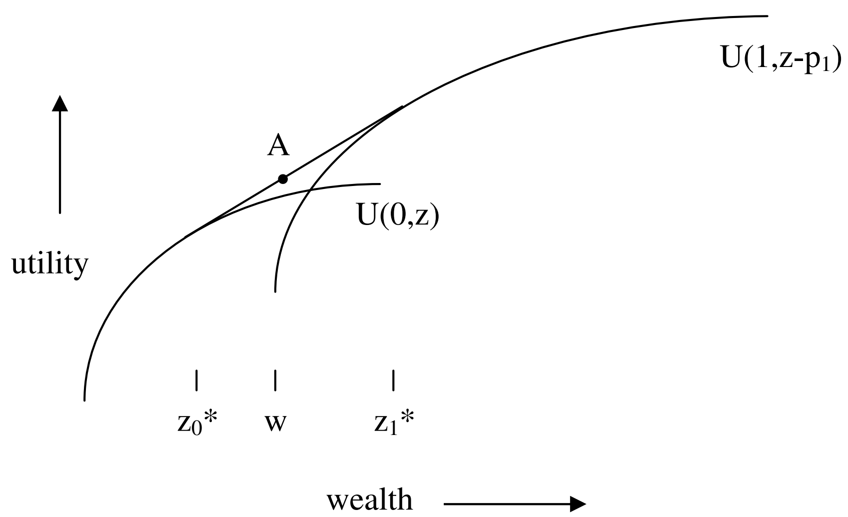

Fig. 2. Optimum gamble

line that is just tangent to the two utility curves. Point z_0^* is the left tangency on the curve $U(0, z)$. Point z_1^* is the right tangency on the curve $U(1, z-p_1)$. Suppose endowed wealth w lies somewhere between z_0^* and z_1^*. Point A is on the linear segment used to form the convex hull of the indirect utility frontier, directly above w. It is apparent from Figure 2 that the consumer obtains her highest possible expected utility at point A by trading all of her wealth for a gamble in money that pays either z_1^* or z_0^*. If the gamble pays z_1^* then she purchases the indivisible good. If it pays z_0^*, she does not. The payoff z_1^* is defined to be the wealth level that corresponds to the decision to purchase the indivisible good. Thus, the probability of receiving the payoff z_1^*, to be denoted by λ^*, is the expected demand for the indivisible good.

Actuarial fairness implies that

$$\lambda^* = \frac{w - z_0^*}{z_1^* - z_0^*}. \tag{6}$$

Assuming the consumer is an expected utility maximizer, her optimization problem is written formally as

$$\max_{\lambda, z_0, z_1} (1-\lambda)U(0, z_0) + \lambda U(1, z_1 - p_1)$$

subject to

$$\lambda = \frac{w - z_0}{z_1 - z_0}$$

$$0 \leq \lambda \leq 1$$

$$z_0 \geq 0,\ z_1 \geq 0.$$

The first order conditions for an interior solution are given by (6),

$$U_2(0, z_0^*) = U_2(1, z_1^* - p_1), \tag{7}$$

and

$$U(1, z_1^* - p_1) = U(0, z_0^*) + U_2(0, z_0^*)[z_1^* - z_0^*]. \tag{8}$$

Despite the non-concavity of the objective function and the nonlinearity of the constraint (6), the values of z_0^*, z_1^* and λ^* that solve the first order conditions represent a global maximum (see [7]).

Interior solutions occur when prices and endowed wealth are such that w lies in a nonconcave region of $F(z;p)$. If prices and endowed wealth are such that $F(z;p)$ is concave in z at w and $F(z;p) = U(0, w)$ then $\lambda^* = 0$ is chosen and $x_2 = w$ is the consumption of the divisible good. If prices and endowed wealth are such that $F(z;p)$ is concave at w and $F(w;p) = U(1, w - p_1)$ then $\lambda^* = 1$ is chosen and $x_2^* = w - p_1$ is the consumption of the divisible good.

It is now possible to analyze how the optimum λ responds to changes in p_1. Let $R(0, z_0) = -U_{22}(0, z_0)/U_2(0, z_0)$ and $R(1, z_1) = -U_{22}(1, z_1 -$

$p_1)/U_2(1, z_1 - p_1)$ denote the Arrow-Pratt measures of absolute risk aversion at points on each of the curves, $U(0, z_0)$ and $U(1, z_1 - p_1)$ respectively. Then, from the first order equations,

$$\partial\lambda^*/\partial p_1 = \frac{-[(1-\lambda^*)/R(0, z_0^*) + \lambda^*/R(1, z_1^*) + \lambda^*(z_1^* - z_0^*)]}{(z_1^* - z_0^*)^2}. \qquad (9)$$

The sign of (9) can be positive. Thus there can be a positive price effect on the expected demand for the indivisible good.

For equation (9) to be positive it is necessary that $z_1^* - z_0^* < 0$. Clearly, this is not the case for the consumer with indirect utility drawn in Figure 2. Rather, endowed wealth must fall in a nonconcave region of $F(z; p)$ where $U(0, z)$ cuts $U(1, z - p_1)$ from below. This is shown in the Figure 3. To achieve the situation shown in Figure 3, the curve $U(1, z - p_1)$ must cross the curve $U(0, z)$ more than once. In instances where $U(1, z - p_1)$ crosses the curve $U(0, z)$ from above the potential exists for a Giffen effect. The consumer with endowed wealth w in Figure 3 obtains maximum expected utility at point B. The payoffs in the optimum gamble for this consumer satisfy $z_1^* - z_0^* < 0$.

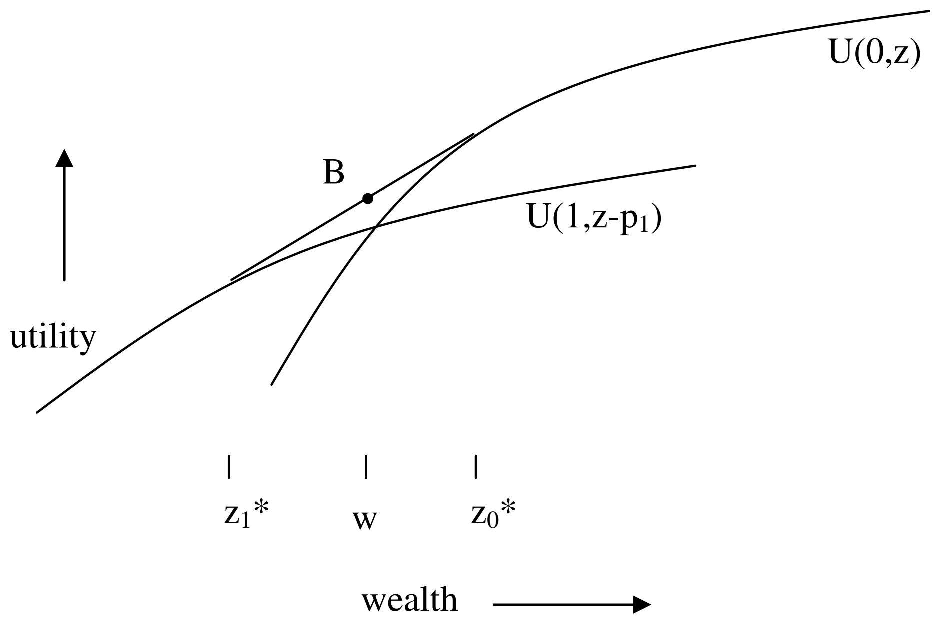

Fig. 3. Upward sloping demand

[1] defines the indivisible good to be inferior if its reservation price is a decreasing function of income. This will be true at wealth levels near a point where the curve $U(0, z)$ cuts the curve $U(1, z - p_1)$ from below. Thus, at endowed wealth w, the indivisible good is an inferior good. Not surprisingly, inferiority of the indivisible good is only a necessary condition for equation (9) to have a positive sign.

Next, an example is presented in which the probability demanded by the consumer for the indivisible good varies directly with own price. Suppose the consumer's utility function is given by

$$U(0, x_2) = \begin{cases} [(\frac{7.425}{8.9} - .75)^{.1}/(\frac{7.425}{8.9})^{.99}]x_2^{.99} & \text{if} \quad 0 \leq x_2 \leq \frac{7.425}{8.9} \\ (x_2 - .75)^{.1} & \text{if} \quad x_2 > \frac{7.425}{8.9} \end{cases}$$

and $U(1, x_2) = .4 + .5x_2^{.9}$. Using two pieces to construct the function $U(0, x_2)$ is a convenient way to make the term $z_1^* - z_0^*$ in equation (9) as large a negative number as possible, while maintaining large positive values of absolute risk aversion at each of the solution points.[2] The function $U(0, \cdot)$ is, by construction, continuous and differentiable.

Suppose endowed wealth is $w = .5$ and the price of the divisible good is 1. We are interested in how λ^* varies with p_1. For $p_1 \leq .0744$, the indirect utility function $F(z; p)$ is concave at $w = .5$ and $F(.5; p) = U(1, .5 - p_1)$. Thus, for $p_1 \leq .0744$, $\lambda^* = 1$. For $p_1 \geq .42451$, the indirect utility function $F(z; p)$ is again concave at $w = .5$ only now $F(.5; p) = U(0, .5)$. Thus, for $p_1 \geq .42451$, $\lambda^* = 0$. For values of p_1 in between .0744 and .42451, $F(z; p)$ is convex at $w = .5$, and λ^* is obtained by solving equations (7) and (8) with the given functional forms to get z_0^* and z_1^*, and then substituting these values into (6).

It is not possible to obtain closed form solutions for z_0^* and z_1^*. Therefore, the solutions are obtained numerically using Newton's method. Over a range of prices corresponding to an interior solution, λ^* varies directly with its own price. For example at $p_1 = .2$, $\lambda^* \approx .607307$ and at $p_1' = .4$, $\lambda^* \approx .7\dot{7}$. In between those prices λ^* increases monotonically with p_1.

3.1 Verification of the Lottery Law of Demand

Consider the commodity price vectors $p = (.2, 1)$ and $p' = (.4, 1)$ from the above example. At $p_1 = .2$, $z_0^*(p) = .885084$ and $z_1^*(p) = .250100$. So $z_1^*(p) - p_1 = .050100$. At $p_1' = .4$, $z_0^*(p') = .839582$ and $z_1^*(p') = .401265$. So $z_1^*(p') - p_1' = .001265$. Assuming an interior solution, the optimal lottery δ corresponding to commodity prices p and income w is

$$\delta(c; p, w) = \begin{cases} (1 - \frac{w - z_0^*(p)}{z_1^*(p) - z_0^*(p)}) & \text{if } c = (0, z_0^*(p)) \\ \frac{w - z_0^*(p)}{z_1^*(p) - z_0^*(p)} & \text{if } c = (1, z_1^*(p) - p_1) \\ 0 & \text{otherwise} \end{cases}$$

[2] An example constructed from a single functional form is provided in Garratt [4].

As indicated in Remark 2, we can express these lotteries in terms of finite vectors, L and L', and demonstrate that (3) holds. Since there are only two distinct commodity bundles that receive positive probability for each set of commodity prices and income, we can evaluate the implications of WARP using vectors with four components. Let $c_1 = (0, z_0^*(p)) = (0, .885084)$, $c_2 = (1, z_1^*(p) - p_1) = (1, .050100)$, $c_3 = (0, z_0^*(p')) = (0, .839582)$, $c_4 = ((1, z_1^*(p') - p_1') = (1, .001265)$. Define

$$\Pi = c \cdot p = \begin{cases} .885084 \text{ if } c = c_1 \\ .250100 \text{ if } c = c_2 \\ .839582 \text{ if } c = c_3 \\ .201265 \text{ if } c = c_4 \end{cases}$$

and

$$\Pi' = c \cdot p' = \begin{cases} .885084 \text{ if } c = c_1 \\ .450100 \text{ if } c = c_2 \\ .839582 \text{ if } c = c_3 \\ .401265 \text{ if } c = c_4 \end{cases}$$

So $\Pi' - \Pi = (0, .2, 0, .2)$.

Let

$$L = (1 - \frac{w - z_0^*(p)}{z_1^*(p) - z_0^*(p)}, \frac{w - z_0^*(p)}{z_1^*(p) - z_0^*(p)}, 0, 0) = (.392693, .607307, 0, 0)$$

denote the finite vector corresponding to the lottery demand $\delta(p, w)$. The change in income required to compensate the consumer for the price change from Π to Π' is $\Delta w = (\Pi' - \Pi) \cdot L = .121461$. Setting $w' = .5 + \Delta w = .621461$ we get that $\frac{w' - z_0^*(p')}{z_1^*(p') - z_0^*(p')} = \frac{.621461 - .839582}{.401265 - .839582} = .497633$.

Let

$$L' = (0, 0, 1 - \frac{w' - z_0^*(p')}{z_1^*(p') - z_0^*(p')}, \frac{w' - z_0^*(p')}{z_1^*(p') - z_0^*(p')}) = (0, 0, .502367, .497633)$$

denote the finite vector corresponding to the lottery demand $\delta(p', w')$. So $L' - L = (-.392693, -.607307, .502367, .497633)$. Thus $(\Pi' - \Pi) \cdot (L' - L) = -.021935$ which is less than zero, as required.

To demonstrate the main result let

$$x(s) = \begin{cases} c_1 \text{ if } s \in [0, .392693) \\ c_2 \text{ if } s \in [.392693, 1] \end{cases}$$

and

$$x'(s) = \begin{cases} c_3 \text{ if } s \in [0, .502367) \\ c_4 \text{ if } s \in [.502367, 1] \end{cases}$$

Then we have $p_1' - p_1 = .2$ and

$$x'_1(s) - x_1(s) = \begin{cases} 0 & \text{if } s \in [0, .392693) \\ -1 & \text{if } s \in [.392693, .502367) \\ 0 & \text{if } s \in [.502367, 1] \end{cases}$$

So (5) becomes

$$\int_0^1 (p'_1 - p_1)(x'_1(s) - x_1(s))ds = \int_{.392693}^{.502367} .2(-1)ds = -.021935 \leq 0.$$

4 Discussion

The practical implication of the analysis is conveyed by the following scenario. Each year Alice undertakes a gamble to determine whether or not to take a trip to Hawaii. The odds of the gamble depend on how much money she "bets" and the price of the trip. When trips are cheap she can get better odds for the same money than when trips are expensive. Nevertheless, the example shows that she may increase her probability of going to Hawaii when the price increases. This means that empirically, we could observe her going to Hawaii more often in years when the price is high than when it is low. Of course, many years of observations would be required to establish upward sloping demand. Even if Alice chooses a lower probability of going to Hawaii when the price rises, it is possible that she will lose the more-favorable gamble the first year and win the less-favorable gamble the second year. However, the law of demand does not rule out the possibility of upward sloping demand curves in actual quantities or in consumption of probabilities. The law requires that when price rises, Alice's income compensated demand is necessarily lower. This is established by the main result and demonstrated for the case of upward sloping (uncompensated) demand by the example.

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