On Market Games When Agents Cannot be Two Places at Once*

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Abstract

We consider markets in which agents supply their time or the services of a capital good, and the duration of the market is limited. We show that a coalitional game can be generated by such a market if and only if the characteristic function of the game is superadditive. We also characterize the vectors of balancing weights that are associated with feasible plans for forming coalitions in a time-constrained market. These are shown to be the same as the ones generated from a market with indivisible agents and lotteries.

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1 Introduction

In specifying a market, the physical process of trade is usually ignored. However, various aspects of the trading process can be important in determining which allocations are feasible in the market. In particular, suppose all exchanges must be made in person, as would be the case if the commodities supplied by individuals were their time. Or, imagine that individuals possess capital goods that provide services over time, such as a generator that provides electricity or a combine harvester. Then, the time available for trades to occur may limit the set of feasible exchanges.

To illustrate this idea, suppose three individuals are stranded on a desert island. They will be on the island for only one day, and are pondering whether to spend the time separated or in groups. Consider a plan whereby each individual agrees to spend half a day with each of the other two individuals. The time supplied by each individual equals her time endowment, but the plan is not feasible since it requires someone to be in two places at once.

The desert island story captures important features of many situations where cooperative activities take place in remote locations. Scheduling for a professional meeting, coordinating activities on a space station or assigning individuals to firms are all practical examples. The key features are that individuals supply their time and time is constrained by the duration of the event that brings the individuals together.

Timing considerations impact the feasible allocations in the market and in turn affect the class of games that can be generated from such markets.\footnote{By games, we mean coalitional games with transferable utility as defined by Shapley (1967).} Without the timing restrictions the desert island market is a direct market as defined by Shapley and Shubik (1969), and the class of games that can be generated by such markets is the class of totally balanced games. The mapping from a market to a game uses information on all the feasible allocations of the market to generate the characteristic function of the game. By restricting the set of feasible allocations of the market, the properties of games generated by the market are altered.

In this paper, we consider time-constrained direct markets. We show that a co-}

tional game can be generated by a time-constrained market if and only if its characteristic function is superadditive. We provide an example to illustrate the result, and the importance of taking account of time restrictions. We also identify the subset of vectors of balancing weights for a game that are associated with feasible coalition formations in a time-constrained environment. This subset is shown to be the same as the subset of vectors of balancing weights generated from a model with indivisi-
ble commodities and lotteries. Finally, we provide a simple characterization of these vectors when there are less than five agents.

2 The Standard Approach: Balancedness and Total Balancedness

A coalitional game with transferable utility (henceforth, a game) is a pair \((N, v)\), where \(N = \{1, 2, ..., n\}\) is the player set and \(v\) is the characteristic function defined on the family \(2^N\) of all subsets of \(N\). For \(S \subseteq N\), \(v(S)\) is the worth of the coalition \(S\), which is the maximum total utility the members of \(S\) can obtain. The value of the empty coalition is zero. The pair \((N, v)\) involves transferable utility because the characteristic function \(v\) is assumed to determine a set of feasible utility allocations \(V(S) = \{u \in \mathbb{R}^N \mid \sum_{i \in S} u_i \leq v(S)\}\) for each coalition \(S\).

For any coalition \(S \subseteq N\), let \(e^S\) denote a vector in \(\mathbb{R}_+^N\) such that \(e_i^S = 1\) if \(i \in S\) and \(e_i^S = 0\) if \(i \notin S\). For \(T \subseteq N\), let \(\Gamma(e^T)\) be the set of non negative vectors \(\gamma = (\gamma_S)_{S \subseteq N}\) such that

\[
\sum_{S \subseteq N} \gamma_S e^S = e^T. \tag{1}
\]

A vector \(\gamma \in \Gamma(e^T)\) is known as a vector of balancing weights of the subgame \((T, v)\). A subgame \((T, v)\) is balanced if

\[
\sum_{S \subseteq N} \gamma_S v(S) \leq v(T) \tag{2}
\]

holds for every \(\gamma \in \Gamma(e^T)\).

In the late sixties, Lloyd Shapley and Martin Shubik introduced the notion of a market game. A market game is a coalitional game that is generated from a market by setting the worth of any coalition in the game equal to the total utility members of the coalition can achieve in the market. The markets Shapley and Shubik consider are exchange economies in which commodities are perfectly divisible and agents’ utility functions are continuous and concave. Utility is measured in units of money, and agents can exchange commodities and transfer money in any amount. Shapley and Shubik (1969) establish that a game is a market game if and only if it is totally balanced: every subgame is balanced.

The interpretation of the vectors of balancing weights used in the test for balancedness depends on the type of interaction being modeled; particularly, on the manner in which agents participate in coalitions. The interpretation we adopt here is that agents participate in coalitions by supplying their time.\(^2\) Then, vectors of

\(^2\)This interpretation is offered as a “natural” one by Aumann (1989, p. 42). We would like to
balancing weights specify durations for all the possible coalitions that are feasible given the time endowments of the agents.

Following through on this interpretation it becomes apparent that for the test of balancedness described in (2) to be appropriate, the time available for forming coalitions must be unlimited. Otherwise, we could ‘accidentally’ rule out some games as candidates for market games on the basis of coalition formations that are not possible. The idea is illustrated in the following example.

**The Checkers Game**: Consider a group of \( n \) agents who arrive at a community hall to play checkers. The hall is open for one hour and each agent stays for the whole hour. Agents receive utility equal to one-half the amount of time they spend matched with another individual playing checkers. The underlying coalitional game has a characteristic function

\[
v(S) = \left\lfloor \frac{|S|}{2} \right\rfloor, \quad S \subseteq N,
\]

where \(|S|\) denotes the cardinality of \( S \), and \([a]\) denotes the greatest integer less than or equal to \( a \).

Suppose \( n \) is greater than three and even. Then, this game is balanced, but any subgame with an odd number (greater than one) of agents is not balanced. For instance, consider the subgame \((T, v)\), with \( T = \{1, 2, 3\} \). Then \( \gamma = (\gamma_S)_{S \subseteq N} \) with \( \gamma_{\{1,2\}} = \gamma_{\{1,3\}} = \gamma_{\{2,3\}} = 1/2 \), and \( \gamma_S = 0 \) otherwise, is a vector of balancing weights for the subgame, but \( \frac{1}{2}v(\{1, 2\}) + \frac{1}{2}v(\{1, 3\}) + \frac{1}{2}v(\{2, 3\}) = 1\frac{1}{2} > v(T) = 1 \). Hence, the subgame is not balanced and so the game is not totally balanced. Therefore, it cannot be represented as a market game of the type considered by Shapley and Shubik (1969).

A careful reading of the circumstances of the checkers game reveals that its disqualification as a market game may be premature. The vector of balancing weights used to violate the balancedness condition for the subgame \((T, v)\) does not correspond to a feasible plan for forming coalitions, since the proposed pairings could not take place in one hour. To properly evaluate the checkers game, and others like it, we need to consider a framework that incorporates time restrictions. In what follows, we evaluate market games in a framework where agents supply a unit of time that is equal to the time available for forming coalitions, and agents cannot be two places at once.

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3This game is a simplified version of Shubik’s “Bridge Game.” See Shubik (1971).
once. Games that can be generated from markets with these restrictions are called time-constrained market games.

3 Time-Constrained Market Games

The procedure we follow to characterize games that can be generated by time-constrained markets (time-constrained market games) is the one used by Shapley and Shubik (1969). Namely, we start with an arbitrary game and use it to generate a time-constrained direct market. Then we use the time-constrained direct market to generate a time-constrained market game. The objective is to find a necessary and sufficient condition on the initial game so that it and the generated game are the same.

Step one is to use an arbitrary game $(N, v)$ to generate a time-constrained direct market. The set of agents in the market is $N$, and the commodity space is $\mathbb{R}_+^N$. Endowments are the vectors $e^{(i)} \in \mathbb{R}_+^N$, $i = 1, \ldots, n$. Thus, each agent is endowed with one unit of her own time.

For $T \subseteq N$, let $\mathcal{P}_T$ denote the set of partitions of $T$, and $\mathcal{P} = \cup_{T \subseteq N} \mathcal{P}_T$. Then, every $P \in \mathcal{P}$ represents a set of coalitions that may be scheduled (active) simultaneously. Agents are assumed to have identical utility functions that are defined as follows: Given $x \in \mathbb{R}_+^N$,

$$u(x) = \max_{\alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+} \sum_{P \in \mathcal{P}} \alpha_P \left( \sum_{R \in P} v(R) \right)$$

subject to

(i) $\sum_{P \in \mathcal{P}} \alpha_P \left( \sum_{R \in P} e^R \right) = x$;

(ii) $\sum_{P \in \mathcal{P}} \alpha_P \leq 1$. 

(3)

The way to interpret (3) is to think of a vector $x$ as being unscheduled time, and the solution as being an optimum schedule. The weight $\alpha_P$ specifies the amount of time during which coalitions in $P$ are simultaneously active. The restriction of $\alpha$ to partitions reflects the requirement that agents cannot be two places at once. Note that $\sum_{R \in P} e^R = e^{N_P}$ where $N_P = \cup_{R \in P} R$. Thus, condition (i) is the requirement that each agent is scheduled for exactly $x_j$ units of time. Condition (ii) ensures that the schedule can be completed within one unit of time.

**Remark 1a.** Without loss of generality assume that $x_1 \leq x_2 \leq \cdots \leq x_n$. Set $P^1 = \{N\}$, $P^2 = \{N \setminus \{1\}\}$, $\ldots$, $P^n = \{\{n\}\}$. Consider $\alpha: \mathcal{P} \rightarrow \mathbb{R}_+$ where $\alpha_{P^1} = x_1$, $\alpha_{P^j} = x_j - \sum_{h=1}^{j-1} x_h$ for $j = 2, \ldots, n$, and $\alpha_P = 0$ otherwise. Then, $\alpha$ satisfies the con-
straints. Therefore, the constraints of (3) are consistent.

Remark 1b. The set of points satisfying the constraints is clearly closed. Therefore, the maximum exists.

Let \( A(x) \) denote the set of solutions for (3). Then, \( \alpha \in A(x) \) is an optimum way to schedule coalitions given the constraints that the time available for scheduling coalitions is one unit and agents cannot be two places at once. In the case where \( x = e^S \), the support of the solutions to (3) is confined to partitions of \( S \). This convenient property is presented in the following lemma.

**Lemma 1** Given \( x = e^S \), if \( \alpha \) satisfies (i) and (ii) of (3) then \( \alpha_P > 0 \) implies \( P \in \mathcal{P}_S \) and \( \sum_{P \in \mathcal{P}_S} \alpha_P = 1 \).

**Proof.** When \( x = e^S \) for some \( S \subseteq N \), (i) becomes (using \( N_P = \cup_{R \in P} R \)),

\[
\sum_{P \in \mathcal{P}} \alpha_P e^{N_P} = e^S.
\]  

Thus \( \sum_{P \in \mathcal{P}} \alpha_P \geq 1 \). By (ii), \( \sum_{P \in \mathcal{P}} \alpha_P = 1 \), which implies that (4) is equivalent to

\[
\sum_{P \in \mathcal{P}} \alpha_P (e^{N_P} - e^S) = 0,
\]

or

\[
\sum_{P \in \mathcal{P}} \alpha_P (e^{N_P \setminus S} - e^{S \setminus N_P}) = 0.
\]

This last equation is in turn equivalent to

\[
\sum_{P \in \mathcal{P}} \alpha_P e^{N_P \setminus S} = 0 \text{ and } \sum_{P \in \mathcal{P}} \alpha_P e^{S \setminus N_P} = 0.
\]  

Since \( \alpha_P \geq 0 \), for all \( P \in \mathcal{P} \), (5) implies \( \alpha_P = 0 \) whenever \( N_P \neq S \). Q.E.D.

That completes the specification of the time-constrained direct market.

Step two is to use the time-constrained direct market to generate a game \((N, w)\). For this we require the following definition, that states the condition for compatibility of a collection of individual solutions to (3).

**Definition 1** For \( S \subseteq N \), \((\alpha^i)_{i \in S}\) with \( \alpha^i : \mathcal{P} \rightarrow \mathbb{R}_+ \), \( i \in S \), is compatible for \( S \) if \( \sum_{i \in S} \alpha^i \) satisfies (i) and (ii) with \( x = e^S \).
The game \((N, w)\) is now generated as follows: For each \(S \subseteq N\),

\[
w(S) = \max_{(x^i)_{i \in S} \in X^S} \sum_{i \in S} u(x^i),
\]  

(6)

where

\[
X^S = \left\{ (x^i)_{i \in S} \left| \begin{array}{l}
x^i \in \mathbb{R}^N_+, \forall i \in S; \\
\sum_{i \in S} x^i = e^S; \\
\forall i \in S, \exists \alpha^i \in A(x^i) : (\alpha^i)_{i \in S} \text{ is compatible for } S.
\end{array} \right. \right\}.
\]

**Remark 2.** Consider \((x^i)_{i \in S}\) with \(x^j = e^S\) for some \(j \in S\), and \(x^i = 0\), for \(i \in S\) with \(i \neq j\). Then, by Lemma 1, \(\alpha^j \in A(e^S)\) satisfies \(\sum_{p \in P} \alpha^j_p = 1\) and \(\sum_{p \in P} \alpha^j_p e^S = e^S\). Since \(A(0) = \{0\}\), this implies that \(\sum_{i \in S} \alpha^i\) is compatible for \(S\), for all \((\alpha^i)_{i \in S}\) such that \(\alpha^i \in A(x^i), i \in S\). Hence the set \(X^S\) is not empty.

To understand the construction of the characteristic function \(w\), it helps to think of agents for which \(x^i \neq 0\) in (6) as being managers who allocate agents’ time into optimum schedules according to (3). Then, \(w(S)\) is understood as representing the maximum total utility for the coalition \(S\) over all distributions of the total time endowment for which the managers’ solutions to (3) are compatible.

**Theorem 1 (Single Manager Theorem)** \(w(S) = u(e^S)\).

**Proof.** For all \((x^i)_{i \in S} \in X^S\) there exists \(\alpha^i : P \rightarrow \mathbb{R}_+, i \in S\) such that \((\alpha^i)_{i \in S}\) is compatible. Set \(\alpha = \sum_{i \in S} \alpha^i\). Then, the compatibility of \((\alpha^i)_{i \in S}\) implies that \(\alpha\) satisfies (i) and (ii) with \(x = e^S\). Thus,

\[
u(e^S) \geq \sum_{p \in P} \alpha_p \left( \sum_{R \in P} v(R) \right).
\]

On the other hand,

\[
u(x^i) = \sum_{p \in P} \alpha^i_p \left( \sum_{R \in P} v(R) \right)
\]

for \(i \in S\). So,

\[
\sum_{i \in S} \nu(x^i) = \sum_{p \in P} \left( \sum_{i \in S} \alpha^i_p \right) \left( \sum_{R \in P} v(R) \right) = \sum_{p \in P} \alpha_p \left( \sum_{R \in P} v(R) \right).
\]

\(^4\)Recall that \(x^i\) is a vector with components \(x^i_j, j \in S\).
Thus, \( w(S) \leq u(e^S) \).

Choose \( j \in S \) and consider \((x^i)_{i \in S}\) where \( x^j = e^S \) for some \( j \in S \), and \( x^i = 0 \) for \( i \in S \) with \( i \neq j \). Then, by Remark 2, \((x^i)_{i \in S} \in X^S\). Since \( \sum_{i \in S} u(x^i) = u(e^S) \), \( w(S) \geq u(e^S) \). Q.E.D.

Theorem 1 says that in a time-constrained market the value of a coalition can be defined by giving control of the entire endowment of the coalition to a single agent. That completes the process of going from a game to a market and back to a game.

**Definition 2** A game \((N, v)\) is a time-constrained market game if \( w(S) = v(S) \), for all \( S \subseteq N \).

**Definition 3** A game \((N, v)\) is superadditive if \( v(S) + v(S') \leq v(S \cup S') \) for all \( S, S' \subseteq N \) such that \( S \cap S' = \emptyset \).

**Theorem 2** A game \((N, v)\) is a time-constrained market game if and only if it is superadditive.

**Proof.** Suppose \((N, v)\) is a time-constrained market game. Then, by Theorem 1, for any \( S \subseteq N \), \( v(S) = u(e^S) \). By Lemma 1, \( u(e^S) = \max_{P \in \mathcal{P}_S} \alpha_P \sum_{R \in P} v(R) \) subject to \( \alpha : \mathcal{P}_S \rightarrow \mathbb{R}_+ \) with \( \sum_{P \in \mathcal{P}_S} \alpha_P = 1 \). Hence, \( u(e^S) \geq \sum_{R \in P} v(R) \), for all \( P \in \mathcal{P}_S \). Therefore, \( v(S) \geq \sum_{R \in P} v(R) \), for all \( P \in \mathcal{P}_S \). This shows the game \((N, v)\) is superadditive and hence the condition is necessary.

Suppose now that the game \((N, v)\) is superadditive. For any \((x^i)_{i \in S} \in X^S\) there exist \( \alpha^i : \mathcal{P} \rightarrow \mathbb{R}_+ \) for \( i \in S \) such that

\[
\sum_{i \in S} u(x^i) = \sum_{P \in \mathcal{P}_S} \alpha_P \left( \sum_{R \in P} v(R) \right)
\]

where \( \alpha = \sum_{i \in S} \alpha^i \). Since \((\alpha^i)_{i \in S}\) is compatible for coalition \( S \), by Lemma 1, \( \sum_{P \in \mathcal{P}_S} \alpha_P = 1 \). Thus,

\[
\sum_{i \in S} u(x^i) = \sum_{P \in \mathcal{P}_S} \alpha_P \left( \sum_{R \in P} v(R) \right).
\]

Since \((N, v)\) is superadditive, \( \sum_{R \in P} v(R) \leq v(S) \) for all \( P \in \mathcal{P}_S \). Thus \( \sum_{i \in S} u(x^i) \leq v(S) \) which implies that \( w(S) \leq v(S) \). On the other hand, \( w(S) = u(e^S) \), by Theorem 1. By construction, \( u(e^S) \geq v(S) \), which implies that \( w(S) \geq v(S) \). Therefore, \( v(S) = w(S) \) showing that the condition is sufficient. Q.E.D.
4 The Checkers Game Revisited

Recall the Checkers game from Section 2. We have already shown that the game is not totally balanced for \( n \geq 3 \). However, the game is superadditive. That is,

\[
v(S) + v(S') = \left\lfloor \frac{|S|}{2} \right\rfloor + \left\lfloor \frac{|S'|}{2} \right\rfloor \leq \left\lfloor \frac{|S \cup S'|}{2} \right\rfloor = v(S \cup S'),
\]

for all \( S, S' \subseteq N \) such that \( S \cap S' = \emptyset \). Hence, by Theorem 2, there exists a time-constrained direct market that can be used to generate the game, and so it is a time-constrained market game.

This is easily demonstrated for the case of \( N = \{1, 2, 3, 4\} \). For \( S \) such that \( |S| \leq 2 \), it is obvious that \( v(S) \) and \( w(S) \) are the same. Consider \( S = \{1, 2, 3\} \). Then, a solution of (3) with \( x = e^S \) is \( \alpha \in A(e^S) \) where

\[
\alpha_P = \begin{cases} 
1 & \text{for } P = \{\{1, 2\}, \{3\}\} \\
0 & \text{otherwise}
\end{cases}
\]

Hence, \( u(e^S) = v(\{1, 2\}) + v(\{3\}) = 1 \). Consider \( S = N \). Then, a solution of (3) with \( x = e^N \) is \( \alpha' \in A(e^N) \) where

\[
\alpha'_P = \begin{cases} 
1 & \text{for } P = \{\{1, 2\}, \{3, 4\}\} \\
0 & \text{otherwise}
\end{cases}
\]

Hence, \( u(e^N) = v(\{1, 2\}) + v(\{3, 4\}) = 2 \). By Theorem 1, \( w(S) = u(e^S) \), for all \( S \subseteq N \). Hence, \( w(S) = v(S) \), for all \( S \subseteq N \).

5 A Comparison with the Lottery Model

In Garratt and Qin (1997), we consider a scenario in which each agent can participate in only one coalition (i.e., agents are indivisible), but participation and ownership of coalitions are determined randomly. In this context, we show that the set of joint lotteries over ‘partitions’ of the agents of any subgame induces a strict subset of the set of vectors of balancing weights for the subgame.

To see how this works, fix \( T \subseteq N \), and let \( S \) be the set of \( T \)-tuples \((S^i)_{i \in T}\) of coalitions (possibly empty) that form a pseudo-partition of \( T \) (i.e., \( S^i \cap S^j = \emptyset \) for \( i \neq j \), and \( \cup_{i \in T} S^i = T \)), and let \( s \) denote a typical element of \( S \). For example, if \( T = \{1, 2\} \), then \( S = \{(T, \emptyset, (\emptyset, T), (\{1\}, \{2\}), (\{2\}, \{1\})\} \). Elements \( s \in S \) are called allocations, since they specify a coalition for each agent. Let \( S^i(S) \) denote the set of allocations that give \( S \) to player \( i \), and let \( S(S) = \cup_{i \in T} S^i(S) \) denote the set of allocations that give \( S \) to some player \( i \in T \).
Let \( \mathcal{L} \) denote the set of probability distributions over \( \mathcal{S} \), and let \( \Gamma(e^T) \) denote the set of vectors of balancing weights for the game with agent-set \( T \) (See the appendix for the definition of \( \Gamma(e^T) \)). Each \( \ell \in \mathcal{L} \) induces a \( \gamma \in \Gamma(e^T) \) as follows (Garratt and Qin (1997), Proposition 1): For \( S \notin T \), \( \gamma_S = 0 \), and for \( S \subseteq T \)

\[
\gamma_S = \ell(S) = \sum_{i \in T} \sum_{s \in S_i} \ell(s). \tag{7}
\]

For example, if \( T = \{1, 2\} \) then \( \gamma_{\{1\}} = \ell(\{1\}, \{2\}) + \ell(\{2\}, \{1\}) \) and \( \gamma_T = \ell(T, \emptyset) + \ell(\emptyset, T) \). Thus, \( \gamma_{\{1\}} + \gamma_T = 1 \). Similarly, \( \gamma_{\{2\}} + \gamma_T = 1 \). Hence, \( \sum_{S \subseteq T} \gamma_S e^S = e^T \). The set of vectors of balancing weights induced in this fashion are denoted by \( \Gamma^L(e^T) \).

In the current scenario, with time restrictions, a vector \( \alpha \) satisfying (i) and (ii) of (3) with \( x = e^T \), induces a vector \( \gamma \in \Gamma(e^T) \) as follows: For \( S \notin T \), \( \gamma_S = 0 \), and for \( S \subseteq T \)

\[
\gamma_S = \sum_{P \in \mathcal{P} : P \supseteq S} \alpha_P. \tag{8}
\]

The inclusion, \( \gamma \in \Gamma(e^T) \), follows from Lemma 1. Denote the set of such \( \gamma \)'s by \( \Gamma^t-c(e^T) \).

Each vector of balancing weights in \( \Gamma^t-c(e^T) \) corresponds to a feasible plan for forming coalitions in the time-constrained model, while each vector of balancing weights in \( \Gamma^L(e^T) \) corresponds to a feasible (stochastic) plan for forming coalitions in the lottery model. The next result shows that the sets \( \Gamma^L(e^T) \) and \( \Gamma^t-c(e^T) \) are the same.

**Theorem 3** \( \Gamma^L(e^T) = \Gamma^t-c(e^T) \).

**Proof.** By definition, \( \gamma \in \Gamma^L(e^T) \) is induced by a probability distribution over partitions on \( T \). Thus, \( \Gamma^L(e^T) \subseteq \Gamma^t-c(e^T) \) is trivial.

Let \( \gamma \in \Gamma^t-c(e^T) \). Then there exists \( \alpha : \mathcal{P} \to \mathbb{R}_+ \) satisfying (i) and (ii) of (3) with \( x = e^T \) such that \( \gamma \) is induced by \( \alpha \) according to (8). By Lemma 1, there exists a collection of partitions of \( T \), \( \{P^1, ..., P^m\} \), for which \( \alpha_{P^k} > 0 \) and \( \sum_{k=1}^m \alpha_{P^k} = 1 \). For \( k = 1, ..., m \), let \( \hat{P}^k \) be a \( T \)-tuple of coalitions such that \( S \in \hat{P}^k \) with \( S \neq \emptyset \) if and only if \( S \in P^k \). Consider a probability distribution \( \ell \) such that \( \ell(\hat{P}^k) = \alpha_{P^k}, k = 1, ..., m \), and \( \ell(s) = 0 \) for all other \( s \in \mathcal{S} \). Then \( \ell \) induces the same distribution of coalition durations that we started with, and hence \( \gamma \in \Gamma^L(e^T) \). Thus, \( \Gamma^t-c(e^T) \subseteq \Gamma^L(e^T) \). Q.E.D.

We conclude this section by pointing out that for \( |T| \leq 4 \), there is a simple and intuitive characterization of \( \Gamma^L(e^T) \) and \( \Gamma^t-c(e^T) \). Some additional notation and definitions are required.
**Definition 4** A family $\mathcal{B}$ of coalitions contained in $T$ is directly connected if $S \cap S' \neq \emptyset$ for any two elements $S, S'$ in $\mathcal{B}$.

Let $\mathcal{B}$ denote the collection of all such families, and define $t : \mathcal{B} \times \mathbb{R}^{|T|}_+ \to \mathbb{R}_+$ by

$$t(\mathcal{B}, \gamma) = \sum_{S \in \mathcal{B}} \gamma_S, \quad \forall (\mathcal{B}, \gamma) \in \mathcal{B} \times \mathbb{R}^{|T|}_+.$$  

Assuming that agents cannot be two places at once, the coalitions in a directly connected family must form sequentially. Thus, $t(\mathcal{B}, \gamma)$ is the amount of time required to form the coalitions in $\mathcal{B}$ given the durations of coalitions $S \in \mathcal{B}$ prescribed by $\gamma$. Let

$$\tau(\gamma) = \max_{\mathcal{B} \in \mathcal{B}} t(\mathcal{B}, \gamma).$$

**Proposition 1** Suppose $|T| \leq 4$. A vector $\gamma$ is in $\Gamma^{\leq c}(e^T)$ if and only if $\gamma \in \Gamma(e^T)$ and $\tau(\gamma) = 1$.

**Proof.** We only prove the case of $|T| = 4$, because if it holds for this case then it must hold for all $|T| \leq 4$. First we prove necessity. Let $\gamma \in \Gamma^{\leq c}(e^T)$. Fix $i$ and consider the family $\mathcal{B}^i = \{S \in T : i \in S\}$. This is a directly connected family. Furthermore, $t(\mathcal{B}^i, \gamma) = \sum_{S \in \mathcal{B}} \gamma_S = 1$, and hence $\tau(\gamma) \geq 1$.

On the other hand, for any directly connected family $\mathcal{B}$, Lemma 1 of Garratt and Qin (1997) implies that $S(S) \cap S(S') = \emptyset$ for all $S, S' \in \mathcal{B}$. Hence, by (7) and Theorem 3

$$t(\mathcal{B}, \gamma) = \sum_{S \in \mathcal{B}} \gamma_S = \sum_{S \in \mathcal{B}} \ell(S(S)) \leq 1,$$

(since $\cup_{S \in \mathcal{B}} S(S) \subseteq S$ and $\ell(S) = 1$), and so $\tau(\gamma) \leq 1$. Combined we have $\tau(\gamma) = 1$, showing that the condition is necessary.

To establish sufficiency, let $\gamma \in \Gamma(e^T)$ and $\tau(\gamma) = 1$. Set $\mathcal{B}(\gamma) = \{S \subseteq T \mid \gamma_S > 0\}$ and choose $S_1$ from $\mathcal{B}(\gamma)$ such that $|S_1| = \max\{|S| \mid S \in \mathcal{B}(\gamma)\}$. If $|S_1| = 1$, then $\mathcal{B}(\gamma)$ is a partition of $T$, and so the proof is completed. Otherwise, we will show that there exists a partition $P$ of $T$ containing $S_1$. Suppose $|S_1| > 1$. If $S_1 \neq T$, then there exists $S \in \mathcal{B}(\gamma)$ such that $S \cap S_1 = \emptyset$. This is true because otherwise $\{S_1\} \cup \{S \in \mathcal{B}(\gamma) \mid S \ni i\}$ would be directly connected for any $i \in T \setminus S_1$, and so $\tau(\gamma) \geq 1 + \gamma_{S_1} > 1$.

Choose $S_2$ from $\mathcal{B}(\gamma)$ such that $S_2 \cap S_1 = \emptyset$ and $|S_2| = \max\{|S| \mid S \in \mathcal{B}(\gamma) \}$ and $S \cap S_1 = \emptyset$. Since $|T| = 4$, and $|S_2| \leq |S_1|$, $S_1 \cup S_2 \neq T$ implies $|S_2| = 1$. Thus, in this case, the above argument can be repeated to show that there exists $S_3$ in $\mathcal{B}(\gamma)$
such that $S_3 \cap S_1 = \emptyset$ and $S_3 \cap S_2 = \emptyset$. Since $|S_1| > 1$ and $|T| = 4$, $P^1 = \{S_1, S_2, S_3\}$ is a partition of $T$. Now define $\bar{\gamma} = (\bar{\gamma}_S)$ by

$$
\bar{\gamma}_S = \begin{cases} 
\gamma_S & \text{if } S \not\in P^1; \\
\gamma_S - \min_{S \in P^1} \gamma_S & \text{if } S \in P^1.
\end{cases}
$$

(9)

Then, $\bar{\gamma} \in \Gamma(((1 - \alpha_1)e^T)$, where $\alpha_1 = \min_{S \in P^1} \gamma_S$. Let $\bar{D}$ be any directly connected family from $B(\bar{\gamma}) = \{S \subseteq T \mid \bar{\gamma}_S > 0\}$. We will show that $\sum_{S \in \bar{D}} \bar{\gamma}_S \leq (1 - \alpha_1)$.

Suppose first that $|S_1| = 3$. In this case, $S_1 \cup S_2 = T$. Since $|T| = 4$ and $|S_1| = 3$, $S_1 \not\in \bar{D}$ if and only if $S_2 \in \bar{D}$. Thus, by (9), $\sum_{S \in \bar{D}} \bar{\gamma}_S = \sum_{S \in \bar{D}} \gamma_S - \min_{S \in P^1} \gamma_S \leq 1 - \alpha_1$.

Suppose now that $|S_1| = 2$. Then, $S \in \bar{D}$ implies $|S| \leq 2$. In this case, because $\bar{D}$ is directly connected, if $\{i\} \in \bar{D}$ or $|\bar{D}| \leq 2$, then $\bar{D} \subseteq \{S \in B(\bar{\gamma}) \mid S \ni i\}$ for $i \in \bigcap_{S \in \bar{D}} S$. Hence, $\sum_{S \in \bar{D}} \bar{\gamma}_S \leq \sum_{S \ni \bigcap_{S \in \bar{D}} S} \bar{\gamma}_S = 1 - \alpha_1$. Assume that $|\bar{D}| > 2$ and $|S| = 2$ for all $S \in \bar{D}$. Since a family containing more than three 2-agent coalitions cannot be directly connected, we must have $|\bar{D}| = 3$. Thus, either $\bar{D} = \{\{i, j\}, \{i, k\}, \{i, m\}\}$ or $\bar{D} = \{\{j, k\}, \{j, m\}, \{k, m\}\}$. In the first case, $\bar{D} \subseteq B^i$ which implies that $\sum_{S \in \bar{D}} \bar{\gamma}_S \leq 1 - \alpha_1$.

Let $\bar{D} = \{\{j, k\}, \{j, m\}, \{k, m\}\}$. If $|S_2| = 2$, then $S_1 \in \bar{D}$ or $S_2 \in \bar{D}$ because $\{S_1, S_2\}$ is a partition of $T$. Thus, $\sum_{S \in \bar{D}} \bar{\gamma}_S \leq 1 - \alpha_1$. If $|S_2| = 1$, then $S_1 \in \bar{D}$ because otherwise there would exist $S \in B(\bar{\gamma})$ such that $S \cap S_1 = \emptyset$ and $|S| = 2$ contradicting $|S_2| = 1$. Again, $\sum_{S \in \bar{D}} \bar{\gamma}_S = \sum_{S \in \bar{D}} \gamma_S - \min_{S \in P^1} \gamma_S \leq 1 - \alpha_1$. We have thus shown that $\tau(\bar{\gamma}) \leq 1 - \alpha_1$. Since $\bar{\gamma} \in \Gamma(((1 - \alpha_1)e^T)$ we also have $\tau(\bar{\gamma}) \geq 1 - \alpha_1$, and hence $\tau(\bar{\gamma}) = 1 - \alpha_1$.

Consider $\gamma^1 = (\gamma^1)$ defined by $\gamma^1_S = \bar{\gamma}_S/(1 - \alpha_1)$, for all $S \subseteq T$. Since $\bar{\gamma} \in \Gamma(((1 - \alpha_1)e^T)$ and $\tau(\bar{\gamma}) = 1 - \alpha_1$, we have $\gamma^1 \in \Gamma(e^T)$ and $\tau(\gamma^1) = 1$. Moreover, $|B(\gamma^1)| \leq |B(\gamma)| - 1$. Since $|B(\gamma)|$ is finite, by repeating the above procedure a finite number of times, we obtain a finite number $m$ of partitions, $P^1, P^2, \ldots, P^m$, and positive numbers $\alpha_{P^1}, \ldots, \alpha_{P^m}$ such that

$$
\gamma_S = \sum_{k : S \in P^k} \alpha_{P^k},
$$

for $S \subseteq T$ with $\gamma_S > 0$. Q.E.D.

**Remark 3.** Given $s \in S$, $S \subseteq N$, and $i \in N$, define $b_{is}(s)$ by $b_{is}(s) = 1$ if $i \in S$ and $S \subseteq s$; $b_{is}(s) = 0$ otherwise. Let $B(s)$ be the matrix whose rows are indexed by $i \in N$, columns are indexed by $S \subseteq N$, and whose $(i, S)$ entry is $b_{is}(s)$. Then, for any $\ell \in L$, the entries of $B = \sum_{s \subseteq S} \ell(s)B(s)$ form a balanced vector in $\Gamma^L(e^N)$. Let $\gamma \in \Gamma(e^N)$ be given. Consider $B(\gamma) = (b_{is}(\gamma))$ where $b_{is}(\gamma) = \gamma_S$ if $i \in S$ and $b_{is}(\gamma) = 0$ otherwise. Finding conditions under which $\gamma \in \Gamma^L(e^N)$ is equivalent to
finding conditions on $B(\gamma)$ under which $B(\gamma) = \sum_{s \in S} \ell(s)B(s)$ for some $\ell \in \mathcal{L}$. This brings to mind a well known result by Birkhoff and von Neumann (1963, Lemma 2, p. 46) which says that any doubly-stochastic matrix can be written as a randomization of permutation matrices.\(^5\) However, we cannot apply the Birkhoff/von Neumann result since $B(\gamma)$ is not a square matrix. Furthermore, our conditions do not guarantee the existence of an $\ell \in \mathcal{L}$ for which $B(\gamma) = \sum_{s \in S} \ell(s)B(s)$ if $|T| > 4$. Let $|T| = 5$, and consider the vector $\gamma$ with $\gamma_{(1,2)} = \gamma_{(1,3)} = \gamma_{(2,4)} = \gamma_{(3,5)} = \gamma_{(4,5)} = 1/2$, and $\gamma_S = 0$ otherwise. Clearly $\gamma \in \Gamma(e^T)$ and $\tau(\gamma) = 1$, but $\gamma \notin \Gamma_{\tau}(e^T)$.

6 Concluding Remarks

This paper began with the observation that the interpretation of vectors of balancing weights as feasible plans for forming coalitions depends upon the notion that commodities supplied by agents are divisible in a very strong sense. Namely, they can be divided and separated. Time is divisible in a weaker sense. Fractions of an agent’s time can be supplied to different coalitions, but deliveries must take place sequentially, not simultaneously. We have argued that if the time available for participating in coalitions is limited then some possibilities for forming coalitions must be ruled out. This is good news for those interested in finding market interpretations of games where agents supply their time. It means that in situations where the total amount of time available for forming coalitions is limited, the appropriate test for whether or not a game is a market game is much weaker than total balancedness. All that is required, is that the characteristic function of the game be superadditive.

A model in which agents supply their time is compared to one with indivisible commodities and lotteries. While conceptually the two models are quite different, we identify a sense in which they are identical: The feasible allocations for each of the models generate the same set of vectors of balancing weights. Furthermore, we provide a simple characterization of these vectors when there are four or fewer agents. This is an initial step towards finding a useful characterization of the feasible allocations of the time-constrained (or lottery) model for any number of agents.

References


\(^5\) This result is used to simplify the characterization of feasible lottery allocations in Hylland and Zeckhauser (1979) and Garratt and Qin (1996).


