Vickery-Clark-Groves Mechanism

Source:
http://auctiontheorycourse.wordpress.com/slides-and-notes/

Previously…

- We studied single-item auctions

- Bidders have values v_i for an item

- A winning bidder gets a utility of $u_i = v_i - p_i$
  - A losing bidder pays nothing and get $u_i=0$

Previously…

- Seller possible goals:
  - Maximize social welfare (efficiency)
    - 2nd-price (Vickrey) auction
  - Maximize revenue
    - 2^{nd}-price auction with a reserve price (Myerson)
      - For example, reserve-price=50 for the uniform distribution on [0,100]
      - Reserve price is independent of the number of players.
      - Optimality assumes a technical assumption on the distributions.
    - Revenue equivalence

Previously …

- We saw that in single-item auctions we can achieve efficiency with dominant strategies.
- Can this be achieved in other models?

Today

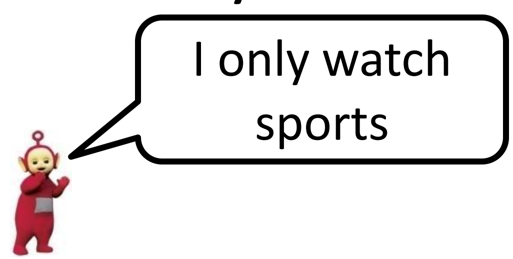
- This class:
  Moving from a specific example (single-item auctions) to a more general mechanism design setting.

- Main goal: in the presence of incomplete information, design the right incentives such that the efficient outcome will be chosen.

Example 1: Roommates buy TV

- Consider two roommates who would like to buy a TV for their apartment.

- TV costs $100
- They should decide:
  - Do they want to buy a TV together?
  - If so, how should they share the costs?



Example 2: Selling multiple items

- Each bidder has a value of v_i for an item.

- But now we have 5 items!
  - Each bidder want only one item.

- An efficient outcome: sell the items to the 5 bidders with the highest values.

Vickrey-Clarke-Groves (VCG) mechanisms

- Goal:
  implement the efficient outcome in dominant strategies.

- A general method to do this: VCG
  - 2nd-price auction is a special case

- Solution (intuitively):
  players should pay the "damage" they impose on society.

VCG basic idea (cont.)

In more details:

- You can maximize efficiency by:
  - Choosing the efficient outcome (given the bids)
  - Each player pays his "social cost" (how much his existence hurts the others).

p_i = [Optimal welfare (for the other players) if player i was not participating.] — [Welfare of the other players from the chosen outcome]

VCG idea in single item auctions

- P_i= [Optimal welfare (for the other players) if player *i* was not participating.] — [Welfare of the other players from the chosen outcome]

= 2^{nd}-highest value.

When i is not playing, the welfare will be the second highest.

= 0.

When i wins, the total value of the other is 0.

→ By VCG payments, winners pay the 2^{nd}-highest bid, and loser pays nothing!

VCG in 5-item auctions

- p_i= Optimal welfare (for the other players) if player *i* was not participating. — Welfare of the other players from the chosen outcome

=30+27+25+12+5

The five winners when *i* is not playing.

=30+27+25+12.

The other four winners.

What is my VCG payment?

$70 $30 $27 $25 $12 $5 $2

VCG in *k*-item auctions

- VCG rules for k-item auctions:
  - Highest k bids win.
  - Everyone pay the (k+1)st bid.

And truthfulness is a dominant strategy here too. (we will prove now!)

Formal model

- n players
- possible outcome $w_1, w_2, \ldots, w_m$
- Each player has private info t_i
- Each player has a value per each outcome (depends on t_i)
  - $v_i(t_i, w)$ — w is from $\{w_1, \ldots, w_m\}$
- Goal of social planner:
  choose w that maximizes

$$\sum_{i=1}^{n} v_i(t_i, w)$$

- Single-item auction example:
- 2 players
- w_1 = "1 wins",
  w_2 = "2 wins"
- $t_i = v_i$ (willingness to pay)
- $v_1(v_1, w_1) = v_1$
  $v_1(v_1, w_2) = 0$
- Goal:
  choose a winner with the highest v_i.

Formal model

$w^{*}=w_5$

	w_1	w_2	w_3	w_4	w_5
Player 1	$V_1(t_1,w_1)$	$V_1(t_1,w_2)$	$V_1(t_1,w_3)$	$V_1(t_1,w_4)$	$V(t_1,w_5)$
Player 2	$V_2(t_2,w_1)$	$V_2(t_2,w_2)$	$V_2(t_2,w_3)$	$V_2(t_2,w_4)$	$V_2(t_2,w_5)$
Player 3	$V_3(t_3,w_1)$	$V_3(t_3,w_2)$	$V_3(t_3,w_3)$	$V_3(t_3,w_4)$	$V_3(t_3,w_5)$
Player 4	$V_4(t_4,w_1)$	$V_4(t_4,w_2)$	$V_4(t_4,w_3)$	$V_4(t_4,w_4)$	$V_4(t_4,w_5)$

Assume: w_5 maximizes efficiency

$$\sum_{i=1}^{n} v_i(t_i, w_1)$$

$$\sum_{i=1}^{n} v_i(t_i, w_3)$$

$$\sum_{i=1}^{n} v_i(t_i, w_5)$$

$$\sum_{i=1}^{n} v_i(t_i, w_2)$$

$$\sum_{i=1}^{n} v_i(t_i, w_4)$$

VCG – formal definition

- Bidders are asked to report their private values t_i
- Terminology: (given the reported t_i's)
  - w^* outcome that maximizes the efficiency.
  - Let w^*_{-i} be the efficient outcome when i is not playing.

- The VCG mechanism:
  - Outcome w* is chosen.
  - Each bidder pays: $\sum_{j \neq i} v_j(t_j, w^*_{-i}) - \sum_{j \neq i} v_j(t_j, w^*)$

The total value for the other when player *i* is not participating

The total value for the others when *i* participates

Truthfulness of VCG - Proof

- The VCG mechanism:
  - Outcome w* is chosen.
  - Each bidder pays: $\sum_{j \neq i} v_j(t_j, w_{-i}^*) - \sum_{j \neq i} v_j(t_j, w^*)$

- <u>Method of proof</u>: we will assume that there is a profitable lie for some player i, and this will result in a contradiction.

Truthfulness of VCG - Proof

- Buyer's utility (when **w*** is chosen):

$$
\begin{aligned}
v_i(t_i, w^*) - p_i &= v_i(t_i, w^*) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) - \sum_{j \neq i} v_j(t_j, w^*) \right) \\
&= v_i(t_i, w^*) + \sum_{j \neq i} v_j(t_j, w^*) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) \right) \\
&= \sum_{i=1}^{n} v_i(t_i, w^*) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) \right) \\
&= SocialWelfare(w^*) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) \right)
\end{aligned}
$$

- Assume: bidder i reports a lie t' → *outcome x is chosen.*

- Buyer's utility (when ***x*** is chosen):

$$SocialWelfare(x) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) \right)$$

Truthfulness of VCG - Proof

- Buyer's utility **from truth** (**w*** is chosen):

$$SocialWelfare(w^*) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) \right)$$

- Buyer's utility **from lying** (***x*** is chosen):

$$SocialWelfare(x) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) \right)$$

- Lying is good when:

$$SocialWelfare(x) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) \right) > SocialWelfare(w^*) - \left(\sum_{j \neq i} v_j(t_j, w_{-i}^*) \right)$$

- Impossible since w* maximizes social welfare!

Truthfulness of VCG - intuition

- The trick is actually quite simple:
  - By lying, players may be able to change the outcome.
  - But their utility depends on the total efficiency.
  - → Therefore, players want the efficient outcome to be chosen. Lying my ruin this.

Example 1: Roommates buy TV

- TV cost \$100
- Bidders are willing to pay v_1 and v_2
  - Private information.

- VCG ensures:
  - Efficient outcome (buy if $v1+v2>100$)
  - Truthful revelation.

<u>In our model:</u>

Welfare when buying: v_1+v_2

Welfare when not buying: 100 (saved the construction cost)

Example 1: Roommates buy TV

- Let's compute VCG payments.
- Consider values $v_1=70$, $v_2=80$.
  - With player 1: value for the others is 80.
  - Without player 1: welfare is 100.
  - → $p_1= 100-80=20$
  - Similarly: $p_2 = 100-70 = 30$
  - Total payment received: $20+30 < 100$
- Cost is not covered!

In general, $p_1=100-v_2$, $p_2=100-v_1$

$$p_1+p_2 = 100-v_1+100-v_2 = 100-(v_1+v_2-100) < 100$$

- Whenever we build, cost is not covered.

Example 1: Roommates buy TV

Conclusion: in some cases, the VCG mechanism is not budget-balanced.

(spends more than it collects from the players.)

This is a real problem!

There isn't much we can do:

It can be shown that there is no mechanism that is both efficient and budget balanced.

- Even in simple settings: one seller and one buyer with private values.
- "Myerson-Satterthwaite theorem"

Context: Public goods

- The roommate problem is knows as the "public good" problem.

- Consider a government that wants to build a bridge.
  - When to build? If the total welfare is greater than the cost.
  - How the cost is shared?
  - Efficiency vs. Budget Balance (cannot achieve both).
- Another example: cable infrastructure.

Summary: VCG

- Efficiency is desired in various settings.
- We saw: one can always achieve this with (dominant-strategy) equilibrium.
  - "implementation"

- This is the only general goal that is known to be "implementable".

- Pros: No distributional assumptions, strong equilibrium concept, individually rational.
- Cons: not budget balanced, prone to other manipulations.