Vickery-Clark-Groves Mechanism

Source:
http://auctiontheorycourse.wordpress.com/slides-and-notes/
Previously...

• We studied **single-item auctions**

• Bidders have values $v_i$ for an item

• A winning bidder gets a utility of $u_i = v_i - p_i$
  – A losing bidder pays nothing and get $u_i = 0$
Previously...

- **Seller possible goals:**
  - Maximize *social welfare* (efficiency)
    - 2nd-price (*Vickrey*) auction
  - Maximize *revenue*
    - 2\textsuperscript{nd}-price auction with a reserve price (*Myerson*)
      - For example, reserve-price=50 for the uniform distribution on [0,100]
      - Reserve price is independent of the number of players.
      - Optimality assumes a technical assumption on the distributions.
    - Revenue equivalence
Previously …

• We saw that in single-item auctions we can achieve **efficiency** with **dominant strategies**.

• Can this be achieved in other models?
Today

• This class:
  Moving from a specific example (single-item auctions) to a more general mechanism design setting.

• **Main goal:** in the presence of incomplete information, design the right incentives such that the efficient outcome will be chosen.
Example 1: Roommates buy TV

• Consider two roommates who would like to buy a TV for their apartment.

• TV costs $100

• They should decide:
  – Do they want to buy a TV together?
  – If so, how should they share the costs?

I only watch sports

I hate sports
Example 2: Selling multiple items

• Each bidder has a value of $v_i$ for an item.

• But now we have 5 items!
  – Each bidder want only one item.

• An efficient outcome: sell the items to the 5 bidders with the highest values.
Vickrey-Clarke-Groves (VCG) mechanisms

• **Goal:** implement the *efficient outcome* in dominant strategies.

• A **general method** to do this: VCG
  – 2\textsuperscript{nd}-price auction is a special case

• **Solution (intuitively):** players should pay the “damage” they impose on society.
VCG basic idea (cont.)

In more details:

• You can maximize efficiency by:
  – Choosing the efficient outcome (given the bids)
  – Each player pays his “social cost” (how much his existence hurts the others).

\[ p_i = \text{Optimal welfare (for the other players) if player i was not participating.} - \text{Welfare of the other players from the chosen outcome} \]
VCG idea in single item auctions

- \( P_i = \)

  Optimal welfare (for the other players) if player \( i \) was not participating.

  = 2\(^{nd}\)-highest value.

  When \( i \) is not playing, the welfare will be the second highest.

Welfare of the other players from the chosen outcome

= 0.

When \( i \) wins, the total value of the other is 0.

\( \rightarrow \) By VCG payments, winners pay the 2\(^{nd}\)-highest bid, and loser pays nothing!
VCG in 5-item auctions

- $p_i =$

Optimal welfare (for the other players) if player $i$ was not participating.

=30+27+25+12+5

The five winners when $i$ is not playing.

Welfare of the other players from the chosen outcome

=30+27+25+12.

The other four winners.

What is my VCG payment?

$70 \quad $30 \quad $27 \quad $25 \quad $12 \quad $5 \quad $2
VCG in $k$-item auctions

• VCG rules for $k$-item auctions:
  – Highest $k$ bids win.
  – Everyone pay the $(k+1)^{st}$ bid.

And truthfulness is a dominant strategy here too. (we will prove now!)
Formal model

- $n$ players
- possible outcome $w_1, w_2, \ldots, w_m$
- Each player has private info $t_i$
- Each player has a value per each outcome (depends on $t_i$)
  - $v_i(t_i, w)$ where $w$ is from $\{w_1, \ldots, w_m\}$
- Goal of social planner: choose $w$ that maximizes
  $$\sum_{i=1}^{n} v_i(t_i, w)$$

Single-item auction example:

- 2 players
- $w_1 = “1 wins”, w_2 = “2 wins”$
- $t_i = v_i$ (willingness to pay)
- $v_1(v_1, w_1) = v_1$
- $v_1(v_1, w_2) = 0$
- Goal: choose a winner with the highest $v_i$. 
## Formal model

### Table

<table>
<thead>
<tr>
<th></th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$w_4$</th>
<th>$w_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>$V_1(t_1, w_1)$</td>
<td>$V_1(t_1, w_2)$</td>
<td>$V_1(t_1, w_3)$</td>
<td>$V_1(t_1, w_4)$</td>
<td>$V(t_1, w_5)$</td>
</tr>
<tr>
<td>Player 2</td>
<td>$V_2(t_2, w_1)$</td>
<td>$V_2(t_2, w_2)$</td>
<td>$V_2(t_2, w_3)$</td>
<td>$V_2(t_2, w_4)$</td>
<td>$V_2(t_2, w_5)$</td>
</tr>
<tr>
<td>Player 3</td>
<td>$V_3(t_3, w_1)$</td>
<td>$V_3(t_3, w_2)$</td>
<td>$V_3(t_3, w_3)$</td>
<td>$V_3(t_3, w_4)$</td>
<td>$V_3(t_3, w_5)$</td>
</tr>
<tr>
<td>Player 4</td>
<td>$V_4(t_4, w_1)$</td>
<td>$V_4(t_4, w_2)$</td>
<td>$V_4(t_4, w_3)$</td>
<td>$V_4(t_4, w_4)$</td>
<td>$V_4(t_4, w_5)$</td>
</tr>
</tbody>
</table>

Assume: $w_5$ maximizes efficiency

$$w^* = w_5$$

$$\sum_{i=1}^{n} v_i(t_i, w_1)$$  $$\sum_{i=1}^{n} v_i(t_i, w_3)$$  $$\sum_{i=1}^{n} v_i(t_i, w_5)$$  $$\sum_{i=1}^{n} v_i(t_i, w_2)$$  $$\sum_{i=1}^{n} v_i(t_i, w_4)$$
VCG – formal definition

• Bidders are asked to report their private values $t_i$
• Terminology: (given the reported $t_i$’s)
  – $w^*$ outcome that maximizes the efficiency.
  – Let $w^*_{-i}$ be the efficient outcome when $i$ is not playing.

• The VCG mechanism:
  – Outcome $w^*$ is chosen.
  – Each bidder pays: $\sum_{j \neq i} v_j(t_j, w^*_{-i}) - \sum_{j \neq i} v_j(t_j, w^*)$

The total value for the other when player $i$ is not participating
The total value for the others when $i$ participates
Truthfulness of VCG - Proof

• The VCG mechanism:
  – Outcome $w^*$ is chosen.
  – Each bidder pays: $\sum_{j \neq i} v_j(t_j, w^*) - \sum_{j \neq i} v_j(t_j, w^*)$

• **Method of proof:** we will assume that there is a profitable lie for some player $i$, and this will result in a contradiction.
Truthfulness of VCG - Proof

• **Buyer’s utility (when \( w^* \) is chosen):**

\[
v_i(t_i, w^*) - p_i = v_i(t_i, w^*) - \left( \sum_{j \neq i} v_j(t_j, w^*_j) - \sum_{j \neq i} v_j(t_j, w^*_j) \right)
\]

\[
= v_i(t_i, w^*) + \sum_{j \neq i} v_j(t_j, w^*_j) - \left( \sum_{j \neq i} v_j(t_j, w^*_j) \right)
\]

\[
= \sum_{i=1}^{n} v_i(t_i, w^*) - \left( \sum_{j \neq i} v_j(t_j, w^*_j) \right)
\]

\[
= SocialWelfare(w^*) - \left( \sum_{j \neq i} v_j(t_j, w^*_j) \right)
\]

• **Assume:** bidder \( i \) reports a lie \( t' \) \( \rightarrow \) outcome \( x \) is chosen.

• **Buyer’s utility (when \( x \) is chosen):**

\[
SocialWelfare(x) - \left( \sum_{j \neq i} v_j(t_j, w^*_j) \right)
\]
Truthfulness of VCG - Proof

• **Buyer’s utility from truth** (*w*\(^*\) is chosen):

\[
\text{SocialWelfare}(w^*) - \left( \sum_{j \neq i} v_j(t_j, w^*_{-i}) \right)
\]

• **Buyer’s utility from lying** (*x* is chosen):

\[
\text{SocialWelfare}(x) - \left( \sum_{j \neq i} v_j(t_j, w^*_{-i}) \right)
\]

• Lying is good when:

\[
\text{SocialWelfare}(x) - \left( \sum_{j \neq i} v_j(t_j, w^*_{-i}) \right) > \text{SocialWelfare}(w^*) - \left( \sum_{j \neq i} v_j(t_j, w^*_{-i}) \right)
\]

• Impossible since *w*\(^*\) maximizes social welfare!
Truthfulness of VCG - intuition

• The trick is actually quite simple:
  — By lying, players may be able to change the outcome.
  — But their utility depends on the total efficiency.

→ Therefore, players want the efficient outcome to be chosen. Lying my ruin this.
Example 1: Roommates buy TV

- TV cost $100
- Bidders are willing to pay $v_1$ and $v_2$
  - Private information.
- VCG ensures:
  - Efficient outcome (buy if $v_1 + v_2 > 100$)
  - Truthful revelation.

In our model:
- Welfare when buying: $v_1 + v_2$
- Welfare when not buying: 100 (saved the construction cost)
Example 1: Roommates buy TV

• Let’s compute VCG payments.

• Consider values $v_1=70$, $v_2=80$.
  - With player 1: value for the others is 80.
  - Without player 1: welfare is 100.
  $\Rightarrow p_1 = 100-80 = 20$
  - Similarly: $p_2 = 100-70 = 30$
  - Total payment received: 20+30 < 100

• Cost is not covered!

In general, $p_1 = 100-v_2$, $p_2 = 100-v_1$

\[ p_1 + p_2 = 100-v_1 + 100-v_2 = 100-(v_1+v_2-100) < 100 \]

• Whenever we build, cost is not covered.
Example 1: Roommates buy TV

Conclusion: in some cases, the VCG mechanism is not budget-balanced. (spends more than it collects from the players.)

This is a real problem!

There isn’t much we can do: It can be shown that there is no mechanism that is both efficient and budget balanced.

– Even in simple settings: one seller and one buyer with private values.

– “Myerson-Satterthwaite theorem”
Context: Public goods

• The roommate problem is known as the "public good" problem.

• Consider a government that wants to build a bridge.
  – When to build? If the total welfare is greater than the cost.
  – How the cost is shared?
  – Efficiency vs. Budget Balance (cannot achieve both).

• Another example: cable infrastructure.
Summary: VCG

• Efficiency is desired in various settings.

• We saw: one can always achieve this with (dominant-strategy) equilibrium.
  – “implementation”

• This is the only general goal that is known to be “implementable”.

• **Pros:** No distributional assumptions, strong equilibrium concept, individually rational.

• **Cons:** not budget balanced, prone to other manipulations.