Risk Aversion

We examine the impact of risk aversion on bidding behavior in first-price auctions.

Assume there is no entry fee or reserve.

Note: Risk aversion does not affect bidding in SPA because there, a person’s bid only affects the probability that they win the auction; it does not affect their payoff.

Note: In SPA with a positive entry fee, the minimum value \( v_0 \) at which a bidder would be willing to enter the auction would be higher.

Next we will describe the equilibrium bid functions for the case of \( n \) bidders under the assumption that the bidders have constant relative risk aversion (CRRA) utility functions.
CRRA utility

Interpretation from portfolio theory: one risky asset and one risk-free asset.

If the person experiences an increase in wealth, he/she will choose to keep unchanged the fraction of the portfolio held in the risky asset.

We wish to introduce the possibility that bidders exhibit differing degrees of risk aversion.

Assume each bidder’s utility over money payoffs depends on a risk parameter $\alpha_i$.

In particular, suppose the utility over money payoffs for each bidder $i$ is given by the function $u(z) = z^{\alpha_i}$, where $0 < \alpha_i < 1$.

Note: This implies constant relative risk aversion equal to $1 - \alpha_i$. 
Risk averse means person prefers expected value of a gamble to the gamble itself.

$$\alpha_i = \frac{1}{2}$$

Option A: a gamble that gives her a wealth of $0 with probability .5 and a wealth of $100 with probability .5

Option B: $50

Utility of option A is $\frac{1}{2} \times 100 = 5$.

Utility of option B is $50^{\frac{1}{2}} \approx 7.07$
Higher degrees of risk aversion are consistent with lower values of $\alpha_i$.

Contemplate increasing the probability of the $100 prize in order to get her to accept the gamble rather than a certainty cash payment of $50.

The minimum probability $p$ that would work is determined by solving

$$p100^{\alpha_i} = 50^{\alpha_i}.$$ 

If $\alpha_i = \frac{1}{2}$, the solution is $p = .707$.

If $\alpha_i = \frac{1}{4}$, the solution is $p = .84$.

In general $p = \left(\frac{1}{2}\right)^{\alpha_i}$.

The lower $\alpha_i$ is, the higher $p$ must be. I.e., the more risk averse the person is, the more we must compensate her (by giving her favorable terms) to accept a gamble.
Suppose each bidder has a value that is drawn from the uniform distribution on \([0, 1]\).

I.e., \(F(v) = v\) for all bidders.

We will now verify that the symmetric equilibrium bid function is

\[
b^L(v_i) = \frac{n - 1}{n - 1 + \alpha_i} v_i,
\]

Once again we apply the Nash equilibrium logic directly.
Suppose bidder $i = 2, \ldots, n$ follows the strategy $b^i(v_i) = \frac{1}{\alpha_i + 1} v_i$ and consider an arbitrary bid $b$ for bidder 1.

Bidder 1 wins with bid $b$ with probability $G(b)$, which, in the case of uniform $F$, is equal to

$$
\prod_{i=2}^{n} Pr\left[\frac{n-1}{n-1 + \alpha_i} v_i < b\right] = \prod_{i=2}^{n} Pr\left[v_i < \frac{n-1 + \alpha_i}{n-1} b\right]
$$

$$
= \prod_{i=2}^{n} \frac{n-1 + \alpha_i}{n-1} b
$$

$$
= \left(\prod_{i=2}^{n} \frac{n-1 + \alpha_i}{n-1}\right) b^{n-1}.
$$

Thus, the expected payoff of bidder 1 if she bids an amount $b$ is

$$
G(b)(v_1 - b)^{\alpha_1} = \left(\prod_{i=2}^{n} \frac{n-1 + \alpha_i}{n-1}\right) b^{n-1}(v_1 - b)^{\alpha_1}.
$$
Now let’s maximize this with respect to the choice of $b$. The first-order condition for a maximum is

$$
\left(\prod_{i=2}^{n} \frac{n-1+\alpha_i}{n-1}\right) \left((n-1)b^{n-2}(v_1-b)^{\alpha_1} - b^{n-1}\alpha_1(v_1-b)^{\alpha_1-1}\right) = 0.
$$

Dividing both sides by $\left(\prod_{i=2}^{n} \frac{n-1+\alpha_i}{n-1}\right) b^{n-2}(v_1-b)^{\alpha_1-1}$ yields

$$
(n-1)(v_1-b) - b\alpha_1 = 0
$$

or

$$
b^* = \frac{n-1}{n-1+\alpha_1} v_1.
$$

Hence, if bidder 1 believes that all the other bidders will follow the symmetric bidding strategy she best responds by following it as well.
Note: The equilibrium bid functions are still linear and have an intercept of 0.

However, now the slope is somewhere between the risk-neutral prediction of $\frac{n-1}{n}$ that corresponds to $\alpha_1 = 1$ and a slope of 1, the slope that corresponds to $\alpha_1 = 0$.

Note: Each bidders strategy depends on her own risk-aversion parameter, $\alpha_i$, but not on those of the other bidders.

Hence, we do not require that individuals know or make any conjectures about the risk aversion of their opponents!
The result that equilibrium bids do not depend on opponent’s degree of risk aversion may seem surprising, but it is actually apparent from the risk-neutral case.

Specifically, the best response of a bidder in a first price auction involving two bidders is to bid one-half of her value regardless of the fraction between 1/2 and 1 that her opponent bids.

See problems 17.6 and 17.7 in *Workouts in Intermediate Economics, 7th Edition* by Bergstrom and Varian.

In the case of a risk-averse opponent, if we set $\alpha_1 = 1$ and vary $\alpha_2$ from 0 to 1 we trace out the same scenarios.
So far, we have been focusing on first- and second-price auctions.

For these auction formats we already demonstrated that, in the case where each of the $n$ bidders’ values are generated independently from the uniform distribution, the expected revenue to the seller is equal to $\frac{n-1}{n+1}$.

In fact, the result that the expected revenue is the same in these auction formats is true even if values are generated from a non-uniform distribution.

Moreover, revenue equivalence extends to a much broader class of auction formats, which we will refer to as standard auctions.
Define a \textit{standard auction} to be any auction mechanism in which (i) the object is sold to person with highest value, and (ii) the expected payment of bidder with the lowest possible value is 0 in equilibrium.

Assume that all bidders are risk neutral.

Assume that the bidders’ values are independently and identically distributed according to some distribution $F$.

We do not require that $F$ be the uniform distribution.

\textbf{Revenue Equivalence Theorem} Any symmetric equilibrium of a standard auction $A$, involving a strictly increasing bid function yields the same expected revenue to the seller.
## Table: Revenue Comparison

<table>
<thead>
<tr>
<th></th>
<th>Theoretical</th>
<th>SPA</th>
<th>FPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 bidders</td>
<td>33.33</td>
<td>36.58</td>
<td>57.76</td>
</tr>
<tr>
<td>5 bidders</td>
<td>66.67</td>
<td>68.21</td>
<td>75.46</td>
</tr>
</tbody>
</table>
### Revenue With a Reserve Price or Entry Fee

<table>
<thead>
<tr>
<th></th>
<th>r=0, c=0</th>
<th>r=50, c=0</th>
<th>r=0, c=25</th>
</tr>
</thead>
<tbody>
<tr>
<td>observed</td>
<td>36.58</td>
<td>45.69</td>
<td>41.93</td>
</tr>
<tr>
<td>theory</td>
<td>33.33</td>
<td>41.67</td>
<td>41.67</td>
</tr>
</tbody>
</table>

**Table:** SPA Revenue

<table>
<thead>
<tr>
<th></th>
<th>r=0, c=0</th>
<th>r=50, c=0</th>
<th>r=0, c=25</th>
</tr>
</thead>
<tbody>
<tr>
<td>observed</td>
<td>57.76</td>
<td>50.06</td>
<td>52.01</td>
</tr>
<tr>
<td>theory</td>
<td>33.33</td>
<td>41.67</td>
<td>41.67</td>
</tr>
</tbody>
</table>

**Table:** FPA Revenue