Optimal Auctions

We wish to analyze the decision of a seller who sets a reserve price when auctioning off an item to a group of \( n \) bidders.

Consider a seller who chooses an optimal reserve for a second-price auction with one bidder.

Clearly the seller who faces a single bidder should set a positive reserve, otherwise the sale price will be zero. In fact, the optimal reserve is equal to the monopoly price. It can be obtained by solving

\[
\max_r (1 - \frac{r}{100})r \iff \max_r r - \frac{r^2}{100}
\]

The first-order-condition is

\[
1 - \frac{r}{50} = 0
\]

So, the optimal reserve (or monopoly price) is

\[
r = 50.
\]
Note that if the seller had her own use value for the item, $v_s$, then her optimization problem becomes

$$\max_{r} \left(1 - \frac{r}{100}\right)r + \frac{r}{100}v_s \iff \max_{r} r - \frac{r^2}{100} + \frac{r}{100}v_s$$

The first-order-condition is

$$1 - \frac{r}{50} + \frac{v_s}{100} = 0$$

So, the optimal reserve (or monopoly price) is

$$r = 50 + \frac{v_s}{2}.$$
Consider two scenarios. In scenario 1 the seller sells an item using a first-price auction with no reserve to 2 bidders. In scenario 2 the seller sells an item to a single bidder, but sets the optimal reserve. Assume the bidder(s)’ value is drawn from the uniform distribution on \([0, 100]\). Compute the expected seller revenue in each case.

**Answer:** In scenario 1, the expected revenue to the seller is 33.33. In scenario 2, the optimal reserve is 50, and seller’s expected revenue is (since there is only one bidder)

\[
.5 \times 0 + .5(50) = 25.
\]

So the expected revenue in the case where there are two bidders is greater.
There are two striking facts, which are verified in the lecture notes.

Both facts apply to the situation where both the seller and the bidders are risk neutral.

1. The optimal reserve does not depend on the number of bidders.

2. The optimal reserve is the same for the first- and second-price auctions.

Using the above calculations and taking these two facts for granted, one now already knows what the optimal reserve is for both first- and second price auctions, for any number of bidders!

Note: For auctions with large numbers of bidders the seller would be indifferent between having the ability to set a reserve and add one more bidder.
In-class discussion of Experiment 9: Reserve choice for 2 and 5 bidder auctions. The formula for computing the optimal reserve values for each treatment is found in Section 6.1 of the textbook. The experiment covers four cases: \( n = 2, v_s = 0 \Rightarrow r = 50; \ n = 2, v_s = 30 \Rightarrow r = 65; \ n = 5, v_s = 0 \Rightarrow r = 50; \) and \( n = 5, v_s = 30 \Rightarrow r = 65. \) Begin by comparing average observed revenue in each of these four cases to the theoretical predictions. Then examine four plots of 20 rounds of data from the reserve price experiment. Plots are \( n = 2, n = 5 \) for \( v_s = 0; \ n = 2, n = 5 \) for \( v_s = 30; \ v_s = 0, v_s = 30 \) for \( n = 2; \ v_s = 0, v_s = 30 \) for \( n = 5. \)
Saliency of the Reserve Price Decision

In our evaluation of the experimental data for the reserve price experiment we might wish to ask whether or not subjects had significant incentive to behave according to the theory.

We conducted 20 rounds of the experiment and we might ask whether subjects’ reserve choices improved over time.

If the answer is no, it could be because subjects are not responding to incentives or it could be because the incentives are very small. In what follows, we evaluate how costly it is to make mistakes in the reserve price decision. We present these calculations for each treatment considered in Experiment 9. We also provide the range of reserve price choices that result in a loss of less than 50 cents relative to the optimal reserve choice in each case.
The expected payoff for the seller with value $v_s$ and reserve price $r$ is

$$\left[\left(\frac{r}{100}\right)^n - \frac{2n}{n+1}\left(\frac{r}{100}\right)^{n+1} + \frac{n-1}{(n+1)}\right] \times 100 + \left(\frac{r}{100}\right)^n v_s$$

We now compute expected payoff of the seller as a function of the reserve choice for each of the experimental treatments we conducted in class.
$n = 2$ and $v_s = 0$

**Figure:** The seller’s expected payoff as a function of the reserve when $n = 2$ and $v_s = 0$. 
For the optimal reserve, $r = 50$, seller expected payoff is $41.67$. Reserve choices that are lower or higher than $50$ cause a significant reduction in expected payoff. The 50 cent optimal bounds are the solution to

$$41.67 - 100 \times \left(\left(\frac{r}{100}\right)^2 - \frac{4}{3} \left(\frac{r}{100}\right)^3 + \frac{1}{3}\right) = 0.50,$$

i.e., $r = 42.58$ and $r = 56.75$. 

Rod Garratt  
ECON 177: Auction Theory With Experiments
$n = 5$ and $v_s = 0$

**Figure:** The seller’s expected payoff as a function of the reserve when $n = 5$ and $v_s = 0$. 
For the optimal reserve, $r = $65, seller expected payoff is $67.19. Note, that in the case of 5 bidders it is not costly to choose a reserve that is too low. However, reserves that are too high can be quite costly. This is reflected in the 50-cent optimal bounds. The 50 cent optimal bound is the solution to

$$67.19 - 100 \times \left( \left( \frac{r}{100} \right)^5 - \frac{5}{3} \left( \frac{r}{100} \right)^6 + \frac{2}{3} \right) = .50,$$

i.e., $r = $20.40 and $r = $59.82.
$n = 2$ and $v_s = 30$

**Figure:** The seller’s expected payoff as a function of the reserve when $n = 2$ and $v_s = 30$. 
For the optimal reserve, \( r = $65 \), seller expected payoff is $51.64. Reserve choices that are lower or higher than $50 cause a significant reduction in expected payoff. The 50 cent optimal bounds are the solution to

\[
51.64 - 100 \times \left( 1.3 \left( \frac{r}{100} \right)^2 - \frac{4}{3} \left( \frac{r}{100} \right)^3 + \frac{1}{3} \right) = .50,
\]

i.e., \( r = $58.57 \) and \( r = $71.03 \).
$n = 5$ and $v_s = 30$

**Figure:** The seller’s expected payoff as a function of the reserve when $n = 5$ and $v_s = 30$. 

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For the optimal reserve, \( r = \$65 \), seller expected payoff is $69.18. Note, that in the case of 5 bidders it is not costly to choose a reserve that is too low. However, reserves that are too high can be quite costly. This is reflected in the 50-cent optimal bounds. The then the 50 cent optimal bound is the solution to

\[
69.18 - 100 \times \left(1.3 \left( \frac{r}{100} \right)^5 - \frac{5}{3} \left( \frac{r}{100} \right)^6 + \frac{2}{3} \right) = .50,
\]

i.e., \( r = \$55.93 \) and \( r = \$71.57 \).