Model II

Common values can also be modelled as a special case of *interdependent values*.

In the interdependent values model

\[
\begin{align*}
v_1 &= \alpha s_1 + \gamma s_2 \\
v_2 &= \alpha s_2 + \gamma s_1
\end{align*}
\]

where \( s_1 \) and \( s_2 \) are private signals of bidders 1 and 2, \( \alpha \geq 0 \) is the weight a bidder puts on her own signal and \( \gamma \geq 0 \) is the weight she puts on her opponent’s signal.
We consider the case where $\alpha = \gamma = 1$.

In this case, $v_1 = v_2$, which is a case of common values.

The oil lease example fits this model when each signal $s_i$ is determined independently and $s_i = 0$ or 3 with equal probability.

In what follows we will consider a more general treatment of the private signals.

We will assume throughout that the signals $s_i$ are drawn independently from the uniform distribution on $[0, 100]$. 
Claim: The first-price auction has a symmetric Nash equilibrium in which each bidder bids \( s_i \).

Proof.

Suppose bidder 2 bids \( s_2 \) and consider an arbitrary bid \( b_1 \) for bidder 1.

We need to write down bidder 1’s expected payoff as a function of her bid \( b_1 \) and show that this is maximized at \( b_1 = s_1 \).

We derive bidder 1’s expected payoff as a function of her bid in three steps.
Step 1. Compute the probability that bidder 1 wins with bid $b_1$.

Bidder 1 wins only if her bid is higher than bidder 2’s bid, i.e., $b_1 > s_2$.

Since we assume that signals are uniform on $[0, 100]$, this happens with probability $\frac{b_1}{100}$.

Thus, bidder 1 wins the auction with bid $b_1$ with probability $\frac{b_1}{100}$.
**Step 2.** Compute bidder 1’s expected value of the item if she wins at bid $b_1$.

Remember that each bidder’s common value for the good it equal to the sum of the private signals.

Bidder 1 knows $s_1$, but she does not know $s_2$.

Before the auction begins, the expected value of $s_2$ is simply 50.

But this assumes the random variable $s_2$ can take on any value between 0 and 100.

Bidder 1 is interested in the expected value of $s_2$ only in the event that she wins the auction with a bid $b_1$.

Given the proposed bidding strategy of bidder 2, this only happens if $s_2 < b_1$. 
Hence, we want the expected value of the random variable $\tilde{s}_2$ conditional on $s_2 < b_1$.

We know that bidder 2’s signal is between 0 and $b_1$.

Since all these possibilities are equally likely, the expected value of bidder 2’s signal is simply half its maximum value, or $\frac{b_1}{2}$.

Hence, bidder 1’s expected value of the item for bidder 1 if she wins with bid $b_1$ is

$$s_1 + \frac{b_1}{2}.$$
Example

If bidder 1 were to win with a bid of 90 dollars, then bidder 2’s signal could be as high as 90 dollars.

However, the expected value of bidder 2’s signal would be half that, or 45 dollars.

The expected value of the item for bidder 1 is $s_1 + 45$. 
Step 3. Compute the expected price bidder 1 pays if she wins with bid $b_1$.

Since this is a first-price auction, the price she pays if she wins is her own bid, $b_1$.

That’s it! We are ready to write down the expected payoff. It is

\[
\frac{b_1}{100} \left( s_1 + \frac{b_1}{2} - b_1 \right) = \frac{b_1}{100} \left( s_1 - \frac{b_1}{2} \right).
\]
To maximize this with respect to the choice of $b_1$ we take the derivative of

$$\frac{b_1}{100}s_1 - \frac{b_1^2}{200}.$$ 

w.r.t $b_1$ and set it equal to 0:

$$s_1 - b_1 = 0.$$ 

The solution is $b_1 = s_1$, as required.

The same argument can be used to show that $b_2 = s_2$ is a best response to $b_1 = s_1$.

That completes the proof.
Claim: The second-price auction has a symmetric Nash equilibrium in which each bidder bids $2s_i$.

Proof.

Suppose bidder 2 bids $2s_2$ and consider an arbitrary bid $b_1$ for bidder 1.

We need to write down bidder 1’s expected payoff as a function of her bid $b_1$ and show that this is maximized at $b_1 = 2s_1$.

We derive bidder 1’s expected payoff as a function of her bid in three steps.
Step 1. Compute the probability that bidder 1 wins with bid $b_1$.

Bidder 1 wins only if her bid is higher than bidder 2’s bid, i.e., $b_1 > 2s_2$.

Or, equivalently, bidder 1 wins with bid $b_1$ if $s_2 < \frac{b_1}{2}$.

Since we assume that signals are uniform on $[0,100]$, this happens with probability $\frac{b_1}{200}$.

Thus, bidder 1 wins the auction with bid $b_1$ with probability $\frac{b_1}{200}$.
Step 2. Compute bidder 1’s expected value of the item if she wins at bid $b_1$.

Remember that each bidder’s common value for the good it equal to the sum of the private signals.

Bidder 1 knows $s_1$, but she does not know $s_2$.

She needs to compute the expected value of $s_2$ conditional on the event that she wins the auction with a bid $b_1$, that is, conditional on $s_2 < \frac{b_1}{2}$.
We want the expected value of the random variable $\tilde{s}_2$ conditional on $s_2 < \frac{b_1}{2}$.

We know that bidder 2’s signal is between 0 and $\frac{b_1}{2}$.

Since all these possibilities are equally likely, the expected value of bidder 2’s signal is simply half its maximum value, or $\frac{b_1}{4}$.

Hence, bidder 1’s expected value of the item for bidder 1 if she wins with bid $b_1$ is

$$s_1 + \frac{b_1}{4}.$$
Example

If bidder 1 were to win with a bid of 60 dollars, then bidder 2’s signal could be as high as 30 dollars.

However, the expected value of bidder 2’s signal would be half that, or 15 dollars.

The expected value of the item for bidder 1 is $s_1 + 15$. 
Step 3. Compute the expected price bidder 1 pays if she wins with bid $b_1$.

Since this is a second price auction, the price she pays if she wins is bidder 2’s bid, which is $2s_2$.

The expected value of bidder 2’s bid depends on the bid $b_1$ because bidder 1 does not win the auction unless $b_1 > 2s_2$.

We need to compute the expected value of $2s_2$ conditional on $b_1 > 2s_2$.

But, again, since all values between 0 and $b_1$ are equally likely this is simply

$$\frac{b_1}{2}.$$
That’s it! We are ready to write down the expected payoff.

It is

\[
\frac{b_1}{200} \left( s_1 + \frac{b_1}{4} - \frac{b_1}{2} \right) \\
\text{Step 1} \quad \text{Step 2} \quad \text{Step 3}
\]

\[
= \frac{b_1}{200} (s_1 - \frac{b_1}{400}).
\]

To maximize this with respect to the choice of \( b_1 \) we take the derivative and set it equal to 0:

\[
\frac{s_1}{2} - \frac{b_1}{4} = 0.
\]

The solution is \( b_1 = 2s_1 \), as required.

The same argument can be used to show that \( b_2 = 2s_2 \) is a best response to \( b_1 = 2s_1 \). That completes the proof.
The equilibrium we just verified is appealing because it is symmetric, but it is not unique.

While we will not prove it, there are other asymmetric equilibria to this auction.

Namely, for any $\lambda > 0$, $b_1(s_1) = (1 + \lambda)s_1$ and $b_2(s_2) = (1 + \frac{1}{\lambda})s_2$ is a Nash equilibrium.
Revenue Equivalence in Common Value Auction Model II

Given the equilibrium bid functions, the expected payment of a bidder is the same in both the first- and second-price auction formats.

In the first-price auction, bidder $i$ wins with bid $s_i$ with probability $\frac{s_i}{100}$, and pays $s_i$.

Hence, bidder $i$’s expected payment in the first-price common value auction is

\[
\frac{s_i^2}{100}.
\]
In the second-price auction, bidder $i$ wins with probability

$$\text{Prob}[2\bar{s}_j < 2s_i] = \text{Prob}[\bar{s}_j < s_i] = \frac{s_i}{100}$$

and she pays

$$E[2\bar{s}_j | 2s_j < 2s_i] = \frac{2s_i}{2} = s_i.$$  

Hence, bidder $i$’s expected payment in the second-price common value auction is

$$\frac{s_i^2}{100}.$$
So, in both the first and second-price, common-value auction expected revenue is

\[2E\left[ \frac{s_i^2}{100} \right] = 2 \int_0^{100} \frac{s_i^2}{100} \frac{1}{100} ds_i = 2 \left[ \frac{s_i^3}{30000} \right]_0^{100} = 66.67.\]

Revenue equivalence holds in this setting!