Bidding Behavior

We assume people attempt to maximize their payoff from participating in an auction.

Hence, we are in a sense trying to determine their optimal bids.

However, an auction is a game in which the payoff an individual earns from any given bid depends on the bids placed by others.

Hence, the notion of an optimal or “best” bid cannot usually be defined in isolation.

We need some conjecture about how others will bid.
Bayes Nash Equilibrium

Our prediction for bidding behavior is based on the idea that people bid optimally based on their predictions about how others will bid and that these predictions are correct.

How everyone bids depends, of course, on their values.

It is assumed that each bidder knows her own value, but only knows the probability distribution function that determines other bidders’ values.

Hence, bidders must best respond to their beliefs regarding the bid functions used by other bidders and the likelihood of the values that will enter into these bid functions.

In other words, our predictions for bidding behavior are the Bayes Nash equilibrium bid functions.
A Bayes Nash equilibrium for an auction is a bid-function profile $b = (b_1(\cdot), \ldots, b_n(\cdot))$ such that for each bidder $i$ and each possible value $v_i$ for bidder $i$, the bid $b_i(v_i)$ maximizes bidder $i$’s expected payoff given the vector $b_{-i} = (b_1(\cdot), \ldots, b_{i-1}(\cdot), b_{i+1}(\cdot), \ldots, b_n(\cdot))$ of bid functions for the other $n-1$ bidders.

This definition is not quite operational yet, since we do not explain how to compute the expected payoff to a bid.

We will do this explicitly in the following sections, using the particulars of the different auction formats.
In the second-price auction the bidder who places the highest bid wins and she pays the amount of the second-highest bid.

In these auctions, your bid determines whether or not you are the auction winner, but it does not determine how much you pay.

You might think that this suggests you should bid very high to ensure that you win the auction.

- happens in laboratory experiments

Faulty logic!
Hypothetical scenario I

Suppose your value is $52, you bid $90, and the only other player bids $70. In this case, because you have the highest bid you win the auction. However, your earnings from the auction would be $52 - $70 = $-18. That is, you lose $18. In fact, whenever you bid above your value you run the risk of losing money in this way. Moreover, it is important to understand that you cannot improve your payoff by bidding more than your value! If you need to bid more than your value to win the auction, it means the price you will have to pay is more than your value, in which case you will lose money by winning the auction. If the second-highest bid is below your value, then a bid equal to your value will win the auction anyway and earn the same payoff as a higher bid.
Hypothetical scenario II

Suppose your value is $52, you bid $35, and the only other player bids $40. In this case, because you do not have the highest bid you do not win the auction. Your earnings from the auction would be 0. Any bid above $40 would have caused you to win the auction and earned you a payoff of $52 - $40 = $12. It does not cost you anything to bid as high as your value and in fact you might gain by doing so if the other player’s bid is higher than $40 but below $52. If you bid above your value, however, you run the risk of losing money if the other player bids more than your value.
**Theorem** The unique equilibrium in weakly-dominant strategies of the second-price auction consists of each bidder $i$ following the strategy

$$b^\text{II}(v_i) = v_i.$$ 

The reasoning behind this result was provided in the preceding text for the case of two bidders.

A formal argument that applies to any number of bidders is in the lecture notes.
Asymmetric equilibria

Other asymmetric equilibria in weakly dominated strategies exist.

Suppose the highest use value is $100.

Then, $b_1 = $100, $b_2, ..., b_n = 0$ is also an equilibrium.

However, $b_1 = $100 is weakly dominated by $b_1 = v_1$ and $b_i = 0$ is weakly dominated by $b_i = v_i$, for $i = 2, ..., n$.

Resale makes equilibria of this sort more likely (no longer weakly dominated).
In-class discussion of Experiments 1 and 2: SPA with no entry fee and no reserve for 2- and 5-bidder auctions.
Reserve Prices and Entry Fees

Assume that the second-price auction has an entry fee, $c$, and a reserve, $r$, where $0 \leq c \leq 100 - r \leq 100$.

Since the entry fee is paid regardless of whether or not a bidder wins the auction, it is no longer a weakly dominant strategy for bidders to bid their value.

Bidders with low values will choose not to participate in the auction because it is unlikely they will earn enough profit in the auction to cover the entry fee (or even impossible if their value is less than $c$).

The implication is that only bidders with high enough values will enter the auction.
Let’s think about an equilibrium in which only bidders with a value above a certain cutoff, $v_0$, choose to participate.

A bidder with value $v_0$ will win the auction only if she is the only one who participates. This happens if everyone else has a value less than $v_0$.

Probability one other bidder has a value less than $v_0$ is $\frac{v_0}{100}$.

The probability all $n-1$ other bidders have a value less than $v_0$ is $\left(\frac{v_0}{100}\right)^{n-1}$

Hence, the payoff to a bidder of type $v_0$ to participating and bidding $v_0$ is $(v_0 - r)\left(\frac{v_0}{100}\right)^{n-1}$.
A bidder with value $v_0$ is indifferent between participating and not participating if

$$(v_0 - r) \left( \frac{v_0}{100} \right)^{n-1} = c.$$ 

This equation defines our critical value $v_0$.

A bidder with a value greater than $v_0$ has a payoff at least as large this, because she still wins in the scenarios where $y$ is less than $v_0$, and now she also win in scenarios where $y$ is greater than $v_0$ but less than her value.

Hence any bidder with a value greater than $v_0$ will participate in the auction.
Given that all bidders with values greater than \( v_0 \) participate we can use the same reasoning as we did in the case of no entry fee to show that value-bidding is an equilibrium.

Thus, we have established that an equilibrium of second-price auction with reserve \( r \) and entry fee \( c \) consists of each bidder following the strategy

\[
b^{II}(v_i) = \begin{cases} 
  v_i & \text{if } v_i \geq v_0 \\
  \text{No} & \text{if } v_i < v_0
\end{cases}
\]

where \( v_0 \) is the solution to \((v_0 - r) \left( \frac{v_0}{100} \right)^{n-1} = c\).

This symmetric equilibrium is in weakly dominant strategies only if \( c = 0 \).
The theory predicts that once you have paid the entry fee you should bid the same way as you would if you had not entered. This is not surprising if you think about it. The argument we gave to show that value bidding is a weakly dominant strategy does not depend on the support of values for other bidders. Yet, it has been demonstrated in laboratory experiments that subjects often bid different after paying an entry fee. In particular, experiment evidence reported in Charness, Garratt and Hartman (2008) shows that people often deduct a fraction of their entry fee from their bid. Once a subject has paid to enter the auction, any entry fee that they might have paid should have no bearing on their bidding decision. This is a nice instance of the sunk cost fallacy.
In-class discussion of Experiments 3 and 4: 2-bidder SPA with no entry fee and reserve $r = \$50$, 2-bidder SPA with entry fee $c = \$25$ and no reserve.
First-Price Auctions

In the first-price auction the bidder who places the highest bid wins and she pays the amount of her bid.

In these auctions, your bid determines whether or not you are the auction winner and the price you pay.

Since you pay your own bid it is not optimal to bid your value, as was the case for the second-price auction. This would ensure you a payoff of 0.

By lowering your bid below your value you reduce the probability that you win the auction, but you make a positive payoff when you do win.

The optimum bid must therefore balance the gain in payoff from bidding lower, and hence earning a higher profit when you win the auction, with the loss in probability of winning.
**Theorem** Suppose $F$ is uniform on $[0,100]$. The unique, symmetric equilibrium of the first-price auction consists of each bidder following the strategy

$$b^I(v_i) = \frac{(n-1)}{n} v_i$$
Verifying Equilibrium Bid Function for FPA

It is relatively easy to show that this is a symmetric equilibrium bidding strategy.

We can apply the Nash equilibrium logic directly.

Suppose bidders $i = 2, \ldots, n$ are following the strategy $b^i(v_i) = \frac{n-1}{n}v_i$ and consider an arbitrary bid $b$ for bidder 1.

Bidder 1 wins with bid $b$ with probability $\left(\frac{nb}{100(n-1)}\right)^{n-1}$

Why? Because all the other bidders are assumed to bid $\frac{n-1}{n}v_i$.

Hence the probability that any one of their bids is less than $b$ is

$Pr\left[\frac{n-1}{n}v_i < b\right] = Pr\left[v_i < \frac{n}{n-1}b\right] = \frac{n-1}{n}b = \frac{nb}{100(n-1)}$. 

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The probability that all of the other n-1 bids is less than b is, assuming independent value draws, \( \left( \frac{nb}{100(n-1)} \right)^{n-1} \).

Thus, the expected payoff of bidder 1 if she bids an amount b is

\[
\left( \frac{nb}{100(n-1)} \right)^{n-1} (v_1 - b)
\]

\[
= \left( \frac{n}{100(n-1)} \right)^{n-1} b^{n-1} v_1 - \left( \frac{n}{100(n-1)} \right)^{n-1} b^n.
\]
Now maximize \( \left( \frac{n}{100(n-1)} \right)^{n-1} b^{n-1} v_1 - \left( \frac{n}{100(n-1)} \right)^{n-1} b^n \) with respect to the choice of \( b \).

The first-order condition for a maximum is

\[
\left( \frac{n}{100(n-1)} \right)^{n-1} \left[ (n-1)b^{n-2}v_1 - nb^{n-1} \right] = 0
\]

or

\[
(n-1)v_1 - nb = 0
\]

The solution is

\[
b^* = \frac{(n-1)}{n} v_1.
\]

Hence, if bidder 1 believes that everyone else will follow the symmetric bidding strategy she best responds by following it as well. Since this is true for any bidder, we have verified that \( b^I(v_i) = \frac{(n-1)}{n} v_i \), for \( i = 1, \ldots, n \), is a symmetric Nash equilibrium in the case of uniform \( F \).
In-class discussion of Experiments 5 and 6: FPA with no entry fee and no reserve for 2- and 5-bidder auctions.
Reserve Prices and Entry Fees

Assume that the first-price auction has an entry fee, \( c \), and a reserve, \( r \), where \( 0 \leq c \leq 100 - r \leq 100 \).

In the case of \( F \) uniform and \( n = 2 \), an equilibrium of \( \text{FPA}(r,c) \) consists of each bidder following the strategy

\[
b^I(v_i) = \begin{cases} 
\frac{1}{2} \frac{v_i^2 - v_0^2}{v_i} + r \frac{v_0}{v_i} & \text{if } v_i \geq v_0 \\
\text{No} & \text{if } v_i < v_0 
\end{cases},
\]

where \( v_0 \) is defined implicitly by the equation \( \frac{v_0}{100}(v_0 - r) = c \).

The value \( v_0 \) is same as in the second-price auction.

This is the unique symmetric equilibrium.
In-class discussion of Experiments 7 and 8: 2-bidder FPA with no entry fee and reserve \( r = 50 \), 2-bidder FPA with entry fee \( c = 25 \) and no reserve.

**Useful Information:** For \( r = 0 \) and \( c = 25 \), we have \( v_0 = 50 \) and hence

\[
b^I(v_i) = \begin{cases} 
\frac{1}{2} v_i - \frac{1250}{v_i} & \text{if } v_i \geq 50 \\
\text{No} & \text{if } v_i < 50
\end{cases}.
\]

For \( r = 50 \) and \( c = 0 \), we have \( v_0 = 50 \) and hence

\[
b^I(v_i) = \begin{cases} 
\frac{1}{2} v_i + \frac{1250}{v_i} & \text{if } v_i \geq 50 \\
\text{No} & \text{if } v_i < 50
\end{cases}.
\]
Risk Aversion

We examine the impact of risk aversion on bidding behavior in first-price auctions.

Assume there is no entry fee or reserve.

Note: Risk aversion does not affect bidding in SPA because there, a person’s bid only affects the probability that they win the auction; it does not affect their payoff.

Note: In SPA with a positive entry fee, the minimum value $v_0$ at which a bidder would be willing to enter the auction would be higher.

Next we will describe the equilibrium bid functions for the case of $n$ bidders under the assumption that the bidders have constant relative risk aversion (CRRA) utility functions.
CRRA utility

Interpretation from portfolio theory: one risky asset and one risk-free asset.

If the person experiences an increase in wealth, he/she will choose to keep unchanged the fraction of the portfolio held in the risky asset.

We wish to introduce the possibility that bidders exhibit differing degrees of risk aversion.

Assume each bidder’s utility over money payoffs depends on a risk parameter $\alpha_i$.

In particular, suppose the utility over money payoffs for each bidder $i$ is given by the function $u(z) = z^{\alpha_i}$, where $0 < \alpha_i < 1$.

Note: This implies constant relative risk aversion equal to $1 - \alpha_i$. 
Risk averse means person prefers expected value of a gamble to the gamble itself.

\[ \alpha_i = \frac{1}{2} \]

Option A: a gamble that gives her a wealth of $0 with probability .5 and a wealth of $100 with probability .5

Option B: $50

Utility of option A is \[ .5 \times 100^{\frac{1}{2}} = .5 \times 10 = 5. \]

Utility of option B is \[ 50^{\frac{1}{2}} \approx 7.07 \]
Higher degrees of risk aversion are consistent with lower values of $\alpha_i$.

Contemplate increasing the probability of the $100 prize in order to get her to accept the gamble rather than a certainty cash payment of $50.

The minimum probability $p$ that would work is determined by solving

$$p100^{\alpha_i} = 50^{\alpha_i}.$$ 

If $\alpha_i = \frac{1}{2}$, the solution is $p = .707$.

If $\alpha_i = \frac{1}{4}$, the solution is $p = .84$.

In general $p = \left(\frac{1}{2}\right)^{\alpha_i}$.

The lower $\alpha_i$ is, the higher $p$ must be. I.e., the more risk averse the person is, the more we must compensate her (by giving her favorable terms) to accept a gamble.
Suppose each bidder has a value that is drawn from the uniform distribution on $[0, 100]$.

I.e., $F(v) = \frac{v}{100}$ for all bidders.

We will now verify that the symmetric equilibrium bid function is

$$b^*(v_i) = \frac{n - 1}{n - 1 + \alpha_i} v_i,$$

Once again we apply the Nash equilibrium logic directly.
Suppose there are just two bidders, assume bidder 2 follows the strategy \( b^I(v_i) = \frac{1}{1+\alpha_2} v_2 \) and consider an arbitrary bid \( b \) for bidder 1.

Bidder 1 wins with bid \( b \) with probability

\[
Pr\left[ \frac{1}{1+\alpha_2} v_2 < b \right] = Pr\left[ v_2 < (1 + \alpha_2) b \right] = \frac{(1 + \alpha_2) b}{100}.
\]

Thus, the expected utility of bidder 1 if she bids an amount \( b \) is

\[
\frac{(1 + \alpha_2) b}{100} (v_1 - b)^{\alpha_1}.
\]
Now let’s maximize this with respect to the choice of $b$. The first-order condition for a maximum is

$$
\left(\frac{1 + \alpha_2}{100}\right) \left[(v_1 - b)^{\alpha_1} - b\alpha_1(v_1 - b)^{\alpha_1 - 1}\right] = 0.
$$

Dividing both sides by $\left(\frac{1 + \alpha_2}{100}\right) (v_1 - b)^{\alpha_1 - 1}$ yields

$$(v_1 - b) - b\alpha_1 = 0$$

or

$$b^* = \frac{1}{1 + \alpha_1} v_1.$$

Hence, if bidder 1 believes that bidder 2 will follow the symmetric bidding strategy she best responds by following it as well.
Note: The equilibrium bid functions are still linear and have an intercept of 0.

However, now the slope is somewhere between the risk-neutral prediction of $\frac{n-1}{n}$ that corresponds to $\alpha_1 = 1$ and a slope of 1, the slope that corresponds to $\alpha_1 = 0$.

Note: Each bidder's strategy depends on her own risk-aversion parameter, $\alpha_i$, but not on those of the other bidders.

Hence, we do not require that individuals know or make any conjectures about the risk aversion of their opponents!
The result that equilibrium bids do not depend on opponent’s degree of risk aversion may seem surprising, but it is actually apparent from the risk-neutral case.

Specifically, the best response of a bidder in a first price auction involving two bidders is to bid one-half of her value regardless of the fraction between 1/2 and 1 that her opponent bids.

See problems 17.6 and 17.7 in *Workouts in Intermediate Economics, 7th Edition* by Bergstrom and Varian.

In the case of a risk-averse opponent, if we set $\alpha_1 = 1$ and vary $\alpha_2$ from 0 to 1 we trace out the same scenarios.