Midterm
May 2, 2012

This exam is worth a total of 40 Points. You have 1 hour and 15 minutes to complete this exam. Good Luck!

1. A slave has just been thrown to the lions in the Roman Colosseum. Three lions are chained down in a line, with Lion 1 closest to the slave. Each lion’s chain is short enough that he can only reach the two players immediately adjacent to him. The game proceeds as follows. First Lion 1 decides whether or not to eat the slave. If Lion 1 has eaten the slave, then Lion 2 decides whether or not to eat Lion 1 (who is the too heavy to defend himself). If Lion 1 has not eaten the slave, then Lion 2 has no choice: he cannot try to eat Lion 1, because a fight would kill both lions. Similarly, if Lion 2 has eaten Lion 1, then Lion 3 decides whether or not to eat Lion 2. Each lion’s preferences are fairly natural: best (payoff=4) is to eat and stay alive, next best (payoff=3) is to stay alive but go hungry, next (payoff=2) is to eat and be eaten, and worst (payoff=1) is to go hungry and be eaten.
   a. Draw the game tree, with payoffs, for this three player game. (2 points)

   b. What is the rollback equilibrium to this game? Make sure to describe the strategies, not just the payoffs. (2 points)

   If the game gets to Lion 3, he will always choose to eat Lion 2 because no one is left to eat him (payoff 4). Then, Lion 2, knowing if he eats Lion 1, he will be eaten by Lion 3 (payoff 2), will choose to not eat Lion 1 (payoff 3). Lion 1 then knows Lion 2 will not eat him, so he will eat the slave. Ultimately, the equilibrium is for Lion 1 to eat and Lion 2 to not eat, Lion 3 to eat if given the option.

   c. Is there a first-mover advantage to this game? Explain. (1 point)

   Yes, because each lion prefers to stay alive and go hungry than to eat and be eaten, so the second lion will never choose to eat the first and the first will always be able to eat the slave and not be eaten.
2. Consider the following table:

<table>
<thead>
<tr>
<th>Row</th>
<th>North</th>
<th>South</th>
<th>East</th>
<th>West</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>1,1</td>
<td>0,2</td>
<td>1,1</td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td>1,3</td>
<td>1,2</td>
<td>2,3</td>
<td>1,1</td>
</tr>
<tr>
<td>Wind</td>
<td>3,2</td>
<td>2,1</td>
<td>1,3</td>
<td>0,3</td>
</tr>
<tr>
<td>Fire</td>
<td>2,0</td>
<td>3,0</td>
<td>1,1</td>
<td>2,2</td>
</tr>
</tbody>
</table>

a. Does either Row or Column have a dominant strategy? (1 point)

No, no one strategy for either player is always dominant.

b. Use iterated elimination of strictly dominated strategies to reduce the game as much as possible. Give the order in which the eliminations occur and show the resulting reduced form of the game. (3 points)

1. Column → East > South, South eliminated
2. Row → Fire > Earth, Earth eliminated
3. Column → East > North, North eliminated
4. Row → Water > Wind, Wind eliminated

| Column |
|--------|-------|-------|
| East   | 2,3   | 1,1   |
| West   | 1,1   | 2,2   |

Row Water

| Fire  |
|-------|-------|-------|
|       | 1,1   |       |
|       | 2,2   |       |

c. Is the original game dominance solvable? Explain. (1 point)

No, once reduced to East/West and Water/Fire, there are no more dominated strategies.

d. State the pure strategy Nash equilibrium (or equilibria) of this game. (2 points)

By best response, both Water, East and Fire, West are Nash equilibria.
3. Two individuals are involved in a synergistic relationship. If both individuals devote more effort to the relationship they are both better off. For any given effort level of individual $j$, the return to individual $i$’s effort first increases then decreases. Specifically, an effort level is a non-negative number, and individual $i$’s preferences (for $i = 1, 2$) are represented by the payoff function $\pi_i(1 + a_i - a_j)$, where $a_i$ is $i$’s effort level and $a_j$ is the other individual’s effort level. Effort levels are chosen simultaneously.

a. Sketch the reaction function for each individual. **(3 points)**

\[ \pi_1 = a_1(1 + a_2 - a_1) = a_1 + a_1a_2 - a_1^2 \]
\[ \pi_2 = a_2(1 + a_1 - a_2) = a_2 + a_2a_1 - a_2^2 \]

\[ \frac{\partial \pi_1}{\partial a_1} = 1 + a_2 - 2a_1 = 0 \Rightarrow a_1 = \frac{1 + a_2}{2} \]

\[ \frac{\partial \pi_2}{\partial a_2} = 1 + a_1 - 2a_2 = 0 \Rightarrow a_2 = \frac{1 + a_1}{2} \]

b. State the Nash equilibrium effort levels. **(2 points)**

Nash equilibrium:

\[ a_1 = 1 + \left( \frac{1 + a_1}{2} \right) = \frac{1}{2} + \frac{1 + a_1}{2} \]
\[ a_1 = \frac{3}{4} + \frac{a_1}{4} = 0 \quad 3 \frac{a_1}{4} = 0 \quad a_1 = 1 \]

\[ a_2 = \frac{1 + 1}{2} = 1 \]

Nash equilibrium: $$(1, 1)$$
4. Suppose you have a pair of dice, a red die and a white die and you roll them just once. Let A be the event that the sum of numbers from the two die add up to 6 and let B be the event that the red die comes up 5.

a. What is $P(A)$? $P(B)$? (2 points)

$$P(A) = \frac{5}{36}$$

$$P(B) = \frac{1}{6}$$

b. What is $P(A \cap B)$? (1 point)

$$P(A \cap B) = \frac{1}{36}$$

5. For the game shown below, let $p$ be the probability that Player 1 plays Up and let $q$ be the probability that Player 2 plays Left.

a. Find the mixed strategy Nash equilibrium of the following game. (3 points)

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>10, 10</td>
<td>5, 0</td>
</tr>
<tr>
<td>Down</td>
<td>0, 5</td>
<td>30, 30</td>
</tr>
</tbody>
</table>

$$10p + 5(1-p) = 2p + 30(1-p)$$

$$10p + 5 - 5p = 2p + 30 - 30p$$

$$35p = 25$$

$$p = \frac{5}{7}$$

b. What is the expected payoff of player 1 under the mixed strategy Nash equilibrium? (1 points)

$$2p + 5(1-2p) = 10 \left(\frac{5}{7}\right) + 5 \left(\frac{2}{7}\right) = \frac{50}{7}$$
6. Use the minimax method to find the Nash equilibrium for the following zero-sum game. (3 points)

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Middle</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>6</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Straight</td>
<td>7</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Down</td>
<td>2</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

Row \直 \左 Middle

7. Use sequential elimination of never best responses to solve the following game.

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Middle</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>0.6</td>
<td>6, 4</td>
<td>4, 0</td>
</tr>
<tr>
<td>Straight</td>
<td>4, 0</td>
<td>2, 6</td>
<td>2, 4</td>
</tr>
<tr>
<td>Down</td>
<td>8, 6</td>
<td>4, 4</td>
<td>0, 4</td>
</tr>
</tbody>
</table>

State the solution and describe the steps that lead to this solution. (3 points)

1. For Row, Straight is never best
2. For Column, Right is never best
3. For Column, Middle is never best
4. For Row, Up is never best

So Down, Left is the solution
8. Consider the two-round home bargaining game discussed in class. The minimum the seller will sell his home for is \$190,000 and the maximum the buyer is willing to pay is \$200,000. Both players know these two amounts and are bargaining over the difference, M=\$10,000. Assume the disagreement values are 0 for both players. Suppose the buyer moves first by making a proposal and the seller can accept or reject it. If the seller rejects the buyer's proposal, the seller gets to make a counterproposal, which the buyer can then accept or reject. The game is then over (Note: if the buyer rejects in round 2 both the buyer and seller get 0). Suppose that both players discount future income using a period discount factor of \( \delta = 0.8 \).

a. Use rollback to find the equilibrium for this 2-round game. What is the sale price of the home? (3 points)

If the seller is allowed to make a proposal, he will take all \$10,000 which is only worth \$8,000 to him by turn 2. Thus, the buyer must offer at least \( \frac{10,000 \cdot 0.8}{1 - 0.8^2} = 8,000 \) in turn one.

sale price = \$198,000

b. Suppose the buyer's discount factor was \( \delta_b = 0.5 \) while the seller's discount factor remained at \( \delta_s = 0.8 \)? How does your answer to part a change in this case? (1 point)

It doesn't only the seller's discount matters.

c. Return to the case where both have the same discount factor of \( \delta = 0.8 \).

Suppose now that there is no limit to the number of alternating bargaining rounds and the buyer continues to move first. What is the equilibrium price in this case? (2 points)

\[
\lambda \cdot \frac{1 - 0.5}{1 - 0.8^2} = \frac{1 - 0.8}{0.8^2} = \frac{0.20}{0.36} \cdot \frac{5}{9}(\$10,000)
\]

\[
(\text{amount buyer keeps}) = \frac{4}{9}(\$10,000)
\]

\[
\text{price} = \$190,000 + \lambda(\$10,000)
\]
9. Alice, Bob and Charlie are meeting for lunch at a park. Each of them has a choice of bringing one order of sushi or showing up with nothing. They do not communicate before showing up to the park. Each person gets a consumption utility equivalent to $20 if there are two or three orders of sushi, and a consumption utility of 0 if there is one or zero orders of sushi. Keep in mind, however, that the cost to a person of picking up an order of sushi is $10. So overall utility for each person depends on both the number of orders of sushi the group brings and their own cost if they purchased an order themselves.

a) Write this three-player game down in normal (strategic) form. (2 points)

b) Find the Nash equilibrium of this game. (2 points)

By best response, the equilibria are that none of them bring, or any of the three situations in which two people bring and one does not.

(4 equilibria)