Consider the game of Marienbad which, like the game of Nim, is zero-sum (there is a winner and a loser). As in Nim, there are two piles of match sticks and two players, player 1 and player 2. Let \( m \) be the number of sticks in the first pile and let \( n \) be the number of sticks in the second pile. Player 1 moves first and thereafter the players take turns. At each turn, a player can pick up any number of available matches from one of the two piles. The player who removes the last match in Marienbad loses the game (this is just the opposite of the rule for winning Nim). Prove/explain the following claims:

a) If \( m = n = 1 \), then player 1 has a winning strategy. (2 points)

\[ \begin{align*}
\text{Player 1} & \quad \text{if player one takes one stick of either piles, player two will have no option, but to take the last stick, so he will lose for sure.} \\
\text{m=0, n=1} & \quad \text{if } m = n = 1 \text{ the one who starts has a winning strategy} \\
\text{Player 2} & \\
\text{m=1, n=0} & \\
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\text{m=1, n=0} & \\
\text{m=0, n=0} & \
\end{align*} \]

b) If \( m > n \) (e.g. \( m=5, n=5 \)), then player 2 has a winning strategy. (2 points)

The player who wins is going to be the one that can leave just one stick in one of the piles. If player 2 takes exactly the same amount of sticks than player one, from the opposite pile, and when a pile gets to 0 stick, he just takes all the stick from the non-one pile he can surely win.

c) If \( m \neq n \), then player 1 has a winning strategy. (1 point)

If the two piles have different amount of sticks, then player 1 can take \( m-n \) (suppose \( m > n \)) from the bigger pile leaving them equal. This way we get to the problem in part b) but with player 2 starting, so player 1 will have a winning strategy.
b) The player who wins is going to be the one that can leave just one stick in one of the piles. If player 2 takes exactly the same amount of sticks as player one, from the opposite pile, and when a pile gets to 1 stick he just takes all the sticks from the non-one pile he can surely win.

c) If the two piles have different amount of sticks then player 1 can take m-n (suppose \( M>n \)) from the bigger pile leaving them equal. This way we get to the problem in part b) but with player 2 starting, so player 1 will have a winning strategy.
2. Use the minimax method to find the Nash equilibrium for the following zero-sum game. (3 points)

\[
\begin{array}{c|cc}
\text{Row} & \text{Left} & \text{Right} \\
\hline
\text{Up} & 2 & 5 \\
\text{Down} & 3 & 4 \\
\hline
\text{Max} & 3 & \leq
\end{array}
\]

3. Use successive elimination of dominated strategies to solve the following game. (2 points)

\[
\begin{array}{c|ccc}
\text{Row} & \text{Left} & \text{Middle} & \text{Right} \\
\hline
\text{Up} & 40 & 70 & 0 \\
\text{Down} & 50 & -10 & -40 \\
\hline
\text{Min} & 2 & 3 & 1
\end{array}
\]
4. Find the pure strategy Nash equilibria of each of the following games.

**Stag Hunt (2 points)**

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>60, 60</td>
<td>5, 30</td>
</tr>
<tr>
<td>Defect</td>
<td>30, 5</td>
<td>30, 30</td>
</tr>
</tbody>
</table>

(Defect, Cooperate) and (Defect, Defect)

**Chicken (2 points)**

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>60, 60</td>
<td>40, 70</td>
</tr>
<tr>
<td>Defect</td>
<td>70, 40</td>
<td>30, 30</td>
</tr>
</tbody>
</table>

(Defect, Cooperate) and (Cooperate, Defect)

**Prisoner’s dilemma (2 points)**

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>60, 60</td>
<td>10, 90</td>
</tr>
<tr>
<td>Defect</td>
<td>90, 10</td>
<td>30, 30</td>
</tr>
</tbody>
</table>

(Defect, Defect)
5. Consider the two-round home bargaining game discussed in class. The minimum the seller will sell his home for is $180,000 and the maximum the buyer is willing to pay is $200,000. Both players know these two amounts and are bargaining over the difference, \( M = $20,000 \). Assume the disagreement values are 0 for both players. Suppose the buyer moves first by making a proposal and the seller can accept or reject it. If the seller rejects the buyer’s proposal, the seller gets to make a counterproposal, which the buyer can then accept or reject. The game is then over (Note: if the buyer rejects in round 2 both the buyer and seller get 0). Suppose that both players discount future income using a period discount factor of \( \delta = .5 \).

   a) Use rollback to find the equilibrium for this 2-round game. What is the sale price of the home? Which player buyer or seller gets the larger share of \( M \)? (3 points)

   If the game reaches 2nd round, seller will propose $1 to buyer and keep $19999 to himself.

   Buyer then, must offer at least $1x$19999 to the seller or $0.5 \times 19999 \approx $19000

   The sale price is $19000

   Both gets the same proportion of M.

   b) Suppose the buyer’s discount factor was \( \delta_b = .8 \) while the seller’s discount factor remained at \( \delta_s = .5 \). How does your answer to part a change in this case? (1 points)

   No change is made as the seller is the one who proposes in the last round and his \( \delta \) remains 0.5.
Return to the case where both have the same discount factor of \( \delta = .5 \). Suppose now that there is no limit to the number of alternating bargaining rounds and the buyer continues to move first. What is the equilibrium price in this case? (2 points)

\[
\frac{1 - \delta}{1 - \delta^2} = \frac{.5}{1 - .25} = \frac{.5}{.75} = \frac{2}{3}
\]

\[
\frac{2}{3} \left( 20,000 \right) = \frac{40,000}{3} = 13,333
\]

**Price** = 200,000 - 13,333

6. Suppose you have a pair of dice, a red die and a white die and you roll them just once. Let A be the event that the sum of numbers from the two die add up to 7 and let B be the event that the red die comes up 1.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 4 & 5 & 6 & 1 \\
3 & 4 & 5 & 6 & 1 & 2 \\
4 & 5 & 6 & 1 & 2 & 3 \\
5 & 6 & 1 & 2 & 3 & 4 \\
6 & 1 & 2 & 3 & 4 & 5
\end{array}
\]

a) What is \( P(A) \)? \( P(B) \)? (2 points)

\[
P(A) = \frac{6}{36} = \frac{1}{6}
\]

\[
P(B) = \frac{1}{6}
\]

b) What is \( P(A \cap B) \)? (2 points)

\[
P(A \cap B) = P(A) \cdot P(B) = P(AB)
\]

\[
= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & X & X & & & \\
2 & & X & & & X \\
3 & & & X & & \\
4 & & & & X & X \\
5 & & & & & X \\
6 & X & & & & \\
\end{array}
\]
7. A public good is one for which those who do not pay for the good enjoy the same benefits as those who do pay for it. Think of publicly supported television or radio; paid members cannot exclude non-member “free-riders” from watching or listening. With this in mind, consider the following three-person game involving contributions to a public good.

- Call the three players: Larry, Curly and Moe.
- Each player has two action choices: contribute or not contribute one unit to the public good. Each player who chooses to contribute pays a cost of 1.5 units.
- If a player contributes, his payoff is the total number of units contributed by all players, himself included (so a maximum of 3 units), less the cost of making his own contribution, 1.5 units.
- If a player does not contribute, his payoff is the total number of units contributed by all other players.

(a) Write this three-player game down in normal (strategic) form. (5 points)

<table>
<thead>
<tr>
<th></th>
<th>Contribute</th>
<th>Not Contribute</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Larry</td>
<td></td>
<td></td>
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<tr>
<td>Moe</td>
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</tbody>
</table>

(b) Find the Nash equilibrium of this game. Is this equilibrium the most efficient outcome? (3 points)

The Nash equilibrium is that none of these 3 players contribute; however a more efficient outcome would be if say all 3 players contributed or if even just 2 players contributed 6/10 then they would receive payoffs greater than 0.
8. For the game shown below, let \( p \) be the probability that Player 1 plays Cooperate and let \( q \) be the probability that Player 2 plays Cooperate.

a) Find the mixed strategy Nash equilibrium of the following game. (4 points)

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 (1-p)</td>
<td>( 60, 60 )</td>
<td>( 40, 5 )</td>
</tr>
<tr>
<td>0 (p)</td>
<td>( 5, 30 )</td>
<td>( 30, 30 )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
60p + 5(1-p) &= 30p + 30(1-p) \\
60p + 5 - 5p &= 30p + 30 - 30p \\
55p &= 25 \\
p &= \frac{25}{55} = \frac{5}{11} \text{ or } 0.45
\end{align*}
\]

\[
\begin{align*}
60q + 5(1-q) &= 40q + 30(1-q) \\
60q + 5 - 5q &= 40q + 30 - 30q \\
55q + 5 &= 10q + 30 \\
45q &= 25 \\
q &= \frac{25}{45} = \frac{5}{9} \text{ or } 0.56
\end{align*}
\]

b) What is the expected payoff of player 1 under the mixed strategy Nash equilibrium? (2 points)

\[
\begin{align*}
\text{Exp Payoff} &= 60q + 5(1-q) \\
&= 60\left(\frac{5}{9}\right) + 5\left(\frac{4}{9}\right) \\
&= \frac{300 + 20}{9} = \frac{320}{9}
\end{align*}
\]