\( \Sigma_i \); otherwise it would have been eliminated. Now consider the vectors

\[
\tilde{u}_i(s_i) = \{u_i(\sigma, s_i)\}_{s_i \in \Sigma_i}
\]

for each \( s_i \in \Sigma_i \). The set of such vectors is convex, and, by the definition of iterated dominance, \( \Sigma_i^{n+1} \) contains exactly the \( s_i \) such that \( \tilde{u}_i(s_i) \) is undominated in this set. Fix \( s_i \in \Sigma_i^{n+1} \). By the separating hyperplane theorem, there exists

\[
\sigma_i = \{\sigma_i(s_i)\}_{s_i \in \Sigma_i}
\]

such that, for all \( s_i \in \Sigma_i \),

\[
\sigma_i \cdot (\tilde{u}_i(s_i) - \tilde{u}_i(\sigma_i)) \geq 0
\]

(where a dot denotes the inner product), or

\[
u_i(s_i, \sigma_i) \geq u_i(\sigma_i, \sigma_i) \quad \forall \sigma_i \in \Sigma_i \Rightarrow s_i \in \Sigma_i^{n+1},
\]

and we conclude that \( \Sigma_i^{n+1} = \Sigma_i^{n+1} \).

**Remark** Pearce gives a different proof based on the existence of the minmax value in finite two-player zero-sum games.³ The minmax theorem, in turn, is usually proved with the separating hyperplane theorem.

The equivalence between being strictly dominated and not being a best response breaks down in games with three or more players (see exercise 2.7). The point is that, since mixed strategies assume independent mixing, the set of mixed strategies is not convex. In figure 2.2, the problem becomes that the mixed strategies no longer correspond to the set of all tangents to the efficient surface, so a strategy might be on the efficient surface without being a best response to a mixed strategy. However, allowing for correlation in the definition of rationalizability restores equivalence: A strategy is strictly dominated if and only if it is not a best response to a correlated mixed strategy of the opponents. (A correlated mixed strategy for player \( i \)'s opponents is a general probability distribution on \( S_{-i} \), i.e., an element of \( \Delta(S_{-i}) \), while a mixed-strategy profile for player \( i \)'s opponents is an element of \( \times_{s \in S} \Delta(s_1) \).) This gives rise to the notion of correlated rationalizability, which is equivalent to iterated strict dominance.

To see this, modify the proof above, replacing the subscript \( j \) with the subscript \( i \). The separating hyperplane theorem shows that if \( s_i \in \Sigma_i^{n+1} \), there is a vector

\[
\sigma_{-i} = \{\sigma_{-i}(s_i)\}_{s_i \in \Sigma_i}
\]

such that \( u_i(s_i, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}) \) for all \( s_i \in \Sigma_i \). However, \( \sigma_{-i} \) is an arbitrary probability distribution over \( S_{-i} \), and in general it cannot be interpreted as a mixed strategy, as it may involve player \( i \)'s rivals' correlating their randomizations.

### 2.1.5 Discussion

Rationalizability, by design, makes very weak predictions; it does not distinguish between any outcomes that cannot be excluded on the basis of common knowledge of rationality. For example, in the battle of the sexes (figure 1.10a), rationalizability allows the prediction that the players are certain to end up at \((F, B)\), where both get 0. \((F, B)\) is not a Nash equilibrium; in fact, both players can gain by deviating. We can see how it might nevertheless occur: Player 1 plays \( F \), expecting player 2 to play \( F \), and player 2 plays \( F \), expecting 1 to play \( B \). Thus, we might be unwilling to say \((F, B)\) wouldn't happen, especially if these players haven't played each other before. In some special cases, such as if we know that player 2's past play with other opponents has led him to expect \((B, B)\) while player 1's has led him to expect \((F, F)\), \((F, F)\) might even be the most likely outcome. However, such situations seem rare; most often we might hesitate to predict that \((F, B)\) has high probability. Rabin (1989) formalizes this idea by asking how likely each player can consider a given outcome. If player 1 is choosing a best response to his subjective beliefs \( \tilde{\sigma}_2 \) about player 2's strategy, then for any value of \( \tilde{\sigma}_2 \), player 1 must assign \((F, B)\) a probability no greater than \( \bar{\tilde{\sigma}} \). If he assigns a probability greater than \( \bar{\tilde{\sigma}} \) to player 2's playing \( B \), player 1 will play \( B \). Similarly, player 2 cannot assign \((F, B)\) a probability greater than \( \bar{\tilde{\sigma}} \). Thus, Rabin argues that we ought to be hesitant to assign \((F, B)\) a probability greater than the maximum of the two probabilities (that is, \( \bar{\tilde{\sigma}} \)).

### 2.2 Correlated Equilibrium

The concept of Nash equilibrium is intended to be a minimal necessary condition for "reasonable" predictions in situations where the players must choose their strategies independently. Now consider players who may engage in preplay discussion, but then go off to isolated rooms to choose their strategies. In some situations, both players might gain if they could build a "signaling device" that sent signals to the separate rooms. Aumann's (1974) notion of a correlated equilibrium captures what could be achieved with any such signals. (See Myerson 1986 for a fuller introduction to this concept, and for a discussion of its relationship to the theory of mechanism design.)

---

³ A two-person, zero-sum game with strategy spaces \( S_i \) and \( S_j \) has a (minmax) value if

\[
\sup_{s_i} \inf_{s_j} u_i(s_i, s_j) = \inf_{s_j} \sup_{s_i} u_i(s_i, s_j).
\]

If a game has a value \( \bar{\sigma}_{ij} \) and if there exists \((s_i, s_j)\) such that \( u_i(s_i, s_j) = \bar{\sigma}_{ij} \), then \((s_i, s_j)\) is called a saddle point. Von Neumann (1928) and Fan (1952, 1953) have given sufficient conditions for the existence of a saddle point.
To motivate this concept, consider Aumann's example, presented in figure 2.4. This game has three equilibria: $(U, L_1, D, R)$, and a mixed-strategy equilibrium in which each player puts equal weight on each of his pure strategies and that gives each player 2.5. If they can jointly observe a "coin flip" (or sunspots, or any other publicly observable random variable) before play, they can achieve payoffs $(3, 3)$ by a joint randomization between the two pure-strategy equilibria. (For example, flip a fair coin, and use the strategies “player 1 plays $U$ if heads and $D$ if tails; player 2 plays $L$ if heads and $R$ if tails”). More generally, by using a publicly observable random variable, the players can obtain any payoff vector in the convex hull of the set of Nash-equilibrium payoffs. Conversely, the players cannot obtain any payoff vector outside the convex hull of Nash payoffs by using publicly observable random variables.

However, the players can do even better (still without binding contracts) if they can build a device that sends different but correlated signals to each of them. This device will have three equally likely states: $A$, $B$, and $C$. Suppose that if $A$ occurs player 1 is perfectly informed, but if the state is $B$ or $C$ player 1 does not know which of the two prevails. Player 2, conversely, is perfectly informed if the state is $C$, but he cannot distinguish between $A$ and $B$. In this transformed game, the following is a Nash equilibrium: Player 1 plays $U$ when told $A$, and $D$ when told $B$, $C$: player 2 plays $L$ when told $A$, and $R$ when told $B$, $C$. Let's check that player 1 does not want to deviate. When he observes $A$, he knows that player 2 observes $(A, B)$, and thus that player 2 will play $L$; in this case $U$ is player 1's best response. If player 1 observes $(B, C)$, then conditional on his observation he expects player 2 to play $L$ and $R$ with equal probability. In this case player 1 will average 2.5 from either of his choices, so he is willing to choose $D$. So player 1 is choosing a best response; the same is easily seen to be true for player 2. Thus, we have constructed an equilibrium in which the players' choices are correlated: The outcomes $(U, L), (D, L)$, and $(D, R)$ are chosen with probability $\frac{1}{3}$ each, and the “bad” outcome $(U, R)$ never occurs. In this new equilibrium the expected payoffs are $3\frac{1}{2}$ each, which is outside the convex hull of the equilibrium payoffs of the original game without the signaling device. (Note that adding the signaling device does not remove the "old" equilibria: Since the signals do not influence payoffs, if player 1 ignores his signal, player 2 may as well ignore hers.)

![Figure 2.4](image)

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>5.1</td>
<td>0.0</td>
</tr>
<tr>
<td>D</td>
<td>4.4</td>
<td>1.5</td>
</tr>
</tbody>
</table>

The next example of a correlated equilibrium illustrates the familiar game-theoretic point that a player may gain from limiting his own information if the opponents know he has done so, because this may induce the opponents to play in a desirable fashion.

In the game illustrated in figure 2.5, player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. In this game the unique Nash equilibrium is $(D, L, A)$, with payoffs $(1, 1, 1)$.

Now imagine that the players build a correlated device with two equally likely outcomes, $H$ ("heads") and $T$ ("tails"), and that they arrange for the outcome to be perfectly revealed to players 1 and 2, while player 3 receives no information at all. In this game, a Nash equilibrium is for player 1 to play $U$ if $H$ and $D$ if $T$, player 2 to play $L$ if $H$ and $R$ if $T$, and player 3 to play $B$. Player 3 now faces a distribution of $\frac{1}{2}(U, L)$ and $\frac{1}{2}(D, R)$, which makes $B$ a best response. Note the importance of players 1 and 2 knowing that player 3 does not know whether heads or tails prevailed when choosing the matrix. If the random variable were publicly observable and players 1 and 2 played the above strategies, then player 3 would choose matrix $A$ if $H$ and matrix $C$ if $T$, and thus players 1 and 2 would deviate as well. As we observed, the equilibrium would then give player 3 a payoff of 1.

With these examples as an introduction, we turn to a formal definition of correlated equilibrium. There are two equivalent ways to formulate the definition.

The first definition explicitly defines strategies for the "expanded game" with a correlating device and then applies the definition of Nash equilibrium to the expanded game. Formally, we identify a correlating device with a triple $(\Omega, |H|, p)$. Here $\Omega$ is a (finite) state space corresponding to the outcomes of the device (e.g., $H$ or $T$ in our discussion of figure 2.5), and $p$ is a probability measure on the state space $\Omega$.

Player 1's information about which $\omega \in \Omega$ occurred is represented by the information partition $H_i$; if the true state is $\omega$, player 1 is told that the state lies in $h_i(\omega)$. In our discussion of figure 2.4, player 1's information partition is $(\{A\}, \{B, C\})$ and player 2's partition is $(\{A, B\}, \{C\})$. In the discussion of figure 2.5, players 1 and 2 have the partition $(H, T)$; player 3's partition is the one-element set $(H, T)$.

More generally, a partition of a finite set $\Omega$ is a collection of disjoint subsets of $\Omega$ whose union is $\Omega$. An information partition $H_i$ assigns an $h_i(\omega)$
to each \( \omega \) in such a way that \( \omega \in h_i(\omega) \) for all \( \omega \). The set \( h_i(\omega) \) consists of those states that player \( i \) regards as possible when the truth is \( \omega \); the requirement that \( \omega \in h_i(\omega) \) means that player \( i \) never "wrong" in the weak sense that he never regards the true state as impossible. However, player \( i \) may be poorly informed. If his partition is the one-element set \( h_i(\omega) = \Omega \) for all \( \omega \), he has no information at all beyond his prior. (This is called the "trivial partition.").

For all \( h_i \) with positive prior probability, player \( i \)'s posterior beliefs about \( \Omega \) are given by Bayes' law: \( p(\omega|h_i) = p(\omega|h_i)p(h_i) \) for \( \omega \in h_i \), and \( p(\omega|h_i) = 0 \) for \( \omega \) not in \( h_i \).

Given a correlating device \( \{\Omega, \{H_i\}, p\} \), the next step is to define strategies for the expanded game where players can condition their play on the signal the correlating device sends them. A pure strategy for the expanded game can be viewed as a function \( a_i \) that maps elements \( h_i \) of \( H_i \)—the possible signals that player \( i \) receives—to pure strategies \( s_i \in S_i \) of the game without the correlating device. Note that if \( \omega' \in h_i(\omega) \), then necessarily \( a_i \) prescribes the same actions in states \( \omega \) and \( \omega' \). Instead of defining strategies in this way as maps from information sets to elements of \( S_i \), it will be more convenient for our analysis to use an equivalent formulation: We will define pure strategies \( a_i \) as maps from \( \Omega \) to \( S_i \) with the additional property that \( a_i(\omega) = a_i(\omega') \) if \( \omega' \in h_i(\omega) \). The formal term for this is that the strategies are \textit{adapted} to the information structure. (Mixed strategies can be defined in the obvious way, but they will be irrelevant if we take the state space \( \Omega \) to be sufficiently large. For example, instead of player 1 playing \( \{1, \overline{1}\} \) when given signal \( h_i \), we could construct an expanded state space \( \Omega \) where each \( \omega \in h_i \) is replaced by two equally likely states, \( \omega' \) and \( \omega'' \), and player 1 is told both "h" and whether the state is the single-prime or the double-prime kind. Then player 1 can use the pure strategy "play \( I \) if told \( h_i \) and single-prime, play \( \overline{I} \) if told \( h_i \) and double-prime." This will be equivalent to the original mixed strategy.)

**Definition 2.4A.** A \textit{correlated equilibrium} \( s \) relative to information structure \( \{\Omega, \{H_i\}, p\} \) is a Nash equilibrium in strategies that are adapted to this information structure. That is, \((s_1, \ldots, s_n)\) is a correlated equilibrium if, for every \( i \) and every adapted strategy \( a_i \),

\[
\sum_{\omega \in \Omega} p(\omega)u_i(s_i(\omega), s_{-i}(\omega)) \geq \sum_{\omega \in \Omega} p(\omega)u_i(s_i(\omega), s_{-i}(\omega)).
\]

This definition, where the distribution \( p \) over \( \Omega \) is the same for all players, is sometimes called an "objective correlated equilibrium" to distinguish it from "subjective correlated equilibria" where players may disagree on prior beliefs and each player \( i \) is allowed to have different beliefs \( p_i \). We say more about subjective correlated equilibrium in section 2.3.

Definition 2.4A, which requires that \( a_i \) maximize player \( i \)'s "ex ante" payoff—her expected payoff before knowing which \( h_i \) contains the true state—implies that \( a_i \) maximizes player \( i \)'s payoff \textit{conditional on} \( h_i \) for each \( h_i \) that player \( i \) assigns positive prior probability (this conditional payoff is often called an "interim" payoff). That is, (2.1) is equivalent to the condition that, for all players \( i \), information sets \( h_i \) with \( p(h_i) > 0 \), and all \( s_i \),

\[
\sum_{(\omega, h_i) \in \Omega \times H_i} \frac{p(\omega|h_i)u_i(s_i(\omega), s_{-i}(\omega))}{p(h_i|s_i)} \geq \sum_{(\omega, h_i) \in \Omega \times H_i} \frac{p(\omega|h_i)u_i(s_i(\omega), s_{-i}(\omega))}{p(h_i|s_i)}.
\]

When all players have the same prior, any \( h_i \) with \( p(h_i) = 0 \) is irrelevant, and all states \( \omega \in h_i \) can be omitted from the specification of \( \Omega \). New issues arise when the priors are different, as we will see when we discuss Brandenburger and Dekel 1987.

An awkward feature of this definition is that it depends on the particular information structure specified, yet there are an infinite number of possible state spaces \( \Omega \) and many information structures possible for each. Fortunately there is a more concise way to define correlated equilibrium. This alternative definition is based on the realization that any joint distribution over actions that forms a correlated equilibrium for some correlating device can be attained as an equilibrium with the "universal device" whose signals to each player constitute a recommendation of how that player should play. In the example of figure 2.4, player 1 would be told "play \( D \)" instead of "the state is \( (B, C) \)," and player 1 would be willing to follow this recommendation so long as, when he is told to play \( D \), the conditional probability of player 2 being instructed to play \( R \) is \( \frac{1}{2} \). (Those familiar with the literature on mechanism design will recognize this observation as a version of the "revelation principle"; see chapter 7.)

**Definition 2.4B.** A \textit{correlated equilibrium} is any probability distribution \( p(\cdot) \) over the pure strategies \( S_1 \times \cdots \times S_n \) such that, for every player \( i \) and every function \( d_i(\cdot) \) that maps \( S_i \) to \( S_i \),

\[
\sum_{s_i \in S_i} p(s(\cdot)|s_i)u_i(s_i, s_{-i}) \geq \sum_{s_i \in S_i} p(s(\cdot)|s_i)u_i(d_i(s_i), s_{-i}).
\]

Just as with definition 2.4A, there is an equivalent version of the definition stated in terms of maximization conditional on each recommendation: \( p(\cdot) \) is a correlated equilibrium if, for every player \( i \) and every strategy \( s_i \) with \( p(s_i) > 0 \),

\[
\sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i)u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i)u_i(s'_i, s_{-i}) \quad \forall s'_i \in S_i.
\]

That is, player \( i \) should not be able to gain by disobeying the recommendation to play \( s_i \) if every other player obeys his recommendation.

Let us explain why the two definitions of correlated equilibrium are equivalent. Clearly an equilibrium in the sense of definition 2.4B is an equilibrium according to definition 2.4A—just take \( \Omega = S \), and \( h_i(s) = \{s' \mid s'_i = s_i\} \).
Conversely, if $\sigma$ is an equilibrium relative to some $(\Omega, \{H_i\}, p)$ as in definition 2.4A, set $p(s)$ to be the sum of $\hat{p}(o)$ over all $o \in \Omega$ such that $\sigma(o) = s_i$ for all players $i$. Let us check that no player $i$ can gain by disobeying any recommendation $s_i \in S_i$. (The only reason this isn't completely obvious is that there may have been several information sets $h_i$ where player $i$ played $s_i$, in which case his information has been reduced to $s_i$ alone.) Set

$$J_i(s_i) = \{o | \sigma(o) = s_i\},$$

so that $p(J_i(s_i)) = p(s_i)$ is the probability that player $i$ is told to play $s_i$. If we view each pure-strategy profile $\sigma(o)$ as a degenerate mixed strategy that places probability 1 on $s_i = \sigma_i(o)$, then the probability distribution on opponents' strategies that player $i$ believes he faces, conditional on being told to play $s_i$, is

$$\sum_{o \in \Omega^{\text{-}i}} \frac{\hat{p}(o) \cdot p(s_i)}{\hat{p}(J_i(s_i))},$$

which is a convex combination of the distributions conditional on each $h_i$ such that $\sigma_i(h_i) = s_i$. Since player $i$ could not gain by deviating from $\sigma_i$ at any such $h_i$, he cannot gain by deviating when this finer information structure is replaced by the one that simply tells him his recommended strategy.

A pure-strategy Nash equilibrium is a correlated equilibrium in which the distribution $p(\cdot)$ is degenerate. Mixed-strategy Nash equilibria are also correlated equilibria. Just take $p(\cdot)$ to be the joint distribution implied by the equilibrium strategies, so that the recommendations made to each player convey no information about the play of his opponents.

Inspection of the definition shows that the set of correlated equilibria is convex, so the set of correlated equilibria is at least as large as the convex hull of the Nash equilibria. This convexification could be attained by using only public correlating devices. But, as we have seen, nonpublic (imperfect) correlation can lead to equilibria outside the convex hull of the Nash set.

Since Nash equilibria exist in finite games, correlated equilibria do too. Actually, the existence of correlated equilibria would seem to be a simpler problem than the existence of Nash equilibria, because the set of correlated equilibria is defined by a system of linear inequalities and is therefore convex; indeed, Hart and Schmeidler (1989) have provided an existence proof that uses only linear methods (as opposed to fixed-point theorems). One might also like to know when the set of correlated equilibria differs "greatly" from the convex hull of the Nash equilibria, but this question has not yet been answered.

One may take the view that the correlation in correlated equilibria should be thought of as the result of the players receiving "endogenous" correlated signals, so that the notion of correlated equilibrium is particularly appropriate in situations with preplay communication, for then the players might be able to design and implement a procedure for obtaining correlated, private signals. When players do not meet and design particular correlated devices, it is plausible that they may still observe exogenous random signals (i.e., "sunspots" or "moonspots") on which they can condition their play. If the signals are publicly observed they can only serve to convexify the set of Nash equilibrium payoffs. But if the signals are observed privately and yet are correlated, they also allow imperfectly correlated equilibria, which may have payoffs inside the convex hull of Nash equilibria, such as (3, 3) in figure 2.4. (Aumann (1987) argues that Bayesian rationality, broadly construed, implies that play must correspond to a correlated equilibrium, though not necessarily to a Nash equilibrium.)

2.3 Rationalizability and Subjective Correlated Equilibria

In matching pennies (figure 1.10a), rationalizability allows player 1 to be sure he will outguess player 2, and player 2 to be sure he'll outguess player 1; the players' strategic beliefs need not be consistent. It is interesting to note that this kind of inconsistency in beliefs can be modeled as a kind of correlated equilibrium with inconsistent beliefs. We mentioned the possibility of inconsistent beliefs when we defined subjective correlated equilibrium, which generalizes objective correlated equilibrium: by allowing each player $i$ to have different beliefs $p_i(\cdot)$ over the joint recommendation $s \in S$. That notion is weaker than rationalizability, as is shown by figure 2.6 (which is drawn from Brandenburger and Dekel 1987). One subjective correlated equilibrium for this game has player 1's beliefs assign probability 1 to (U, L) and player 2's beliefs assign probability 2 to each of (U, L) and (D, L). Given his beliefs, player 2 is correct to play L. However, that

<table>
<thead>
<tr>
<th></th>
<th>U</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2.0</td>
<td>1.1</td>
</tr>
<tr>
<td>D</td>
<td>1.1</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Figure 2.6

Barany (1988) shows that if there are at least four players ($l \geq 4$), any correlated equilibrium of a strategic-form game coincides with a Nash equilibrium of an extended game in which the players engage in endless conversations (cheap talk) before they play the strategic-form game in question. If there are only two players, then the set of Nash equilibria with cheap talk coincides with the subset of correlated equilibria induced by perfectly correlated signals (i.e., publicly observed randomizing devices).