1. Consider the regression model

\[ y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t \]

where \( \{\varepsilon_t\}_{t=1,...,T} \) is a mean zero white noise process.

a. For \( |\beta_1| < 1 \), find what the OLS estimator of \( \beta_1 \) converges to in probability. Is \( \hat{\beta}_1 \) unbiased?

b. Suppose \( |\beta_1| < 1 \) and \( \varepsilon_t = \rho \varepsilon_{t-1} + u_t \), where \( |\rho| < 1 \) and \( \{u_t\}_{t=1,...,T} \) is a sequence of independent identically distributed random variables. Find what \( \hat{\beta}_1 \) converges to in probability. How does your answer change if \( \varepsilon_t = \rho \varepsilon_{t-1} + u_t \)? If your answer differs from those given in parts a. or b., explain intuitively why it differs.

c. Suppose \( |\beta_1| = 1 \), and \( \varepsilon_t \) is a white noise process. Find what \( \hat{\beta}_1 \) converges to in probability. How does your answer change if \( \varepsilon_t = \rho \varepsilon_{t-1} + u_t \)? If your answer differs from those given in parts a. or b., explain intuitively why it differs.

d. Derive the limiting distribution of the OLS estimator both when \( |\beta_1| < 1 \) and when \( |\beta_1| = 1 \).

2. Consider the model

\[ y_t = \rho_1 y_{t-1} + \varepsilon_t \tag{1} \]

where \( \rho_1 = 1.1, \varepsilon_t \sim \text{IN}(0, \sigma^2) \).

a. Is the ordinary least squares estimator of \( \rho_1 \) consistent? Is it unbiased?

b. Using equation (1), describe how you would test the null hypothesis of a unit root \( (H_0: \rho_1 = 1) \).
c. Suppose you estimate

\[ \Delta y_t = \beta_0 + \beta_1 y_{t-1} + \eta_t, \]  

where \( \Delta y_t = y_t - y_{t-1} \). How do the parameters and error term of equation (2) relate to those of equation (1)? How would you test the null hypothesis of a unit root using (2)?

3. Consider the sequences \( \{y_{1t}\}_{t=1}^{T} \) and \( \{y_{2t}\}_{t=1}^{T} \) that are both known to be driven by a unit root process. However, economic theory suggests that these sequences are linked over time, leading to the model

\[ y_{1t} = y_{2t} \beta + u_{1t} \quad t = 1, ..., T \]
\[ y_{2t} = \rho y_{2t-1} + u_{2t} \quad t = 1, ..., T \]

where \( u_t = [u_{1t}, u_{2t}]' \) is a mean zero, I(0), period-t disturbance vector, and \( \rho = 1 \).

a. Explain the intuition behind the corrections made to the OLS estimator by the fully-modified ordinary least squares (FMOLS) estimator of \( \beta \).

b. Discuss the important issues that arise in implementing the FMOLS estimator.

c. Describe an alternative linear method of estimation. What are some of the advantages and disadvantages of this method, relative to FMOLS?

d. Suppose that \( \{y_{1t}\} \) and \( \{y_{2t}\} \) are in fact I(0) and not I(1) (i.e., \( |\rho| < 1 \)). If \( u_t \sim \text{IN}(0, \Sigma) \), and \( \Sigma \) is a diagonal matrix, can you find a consistent estimator that is more efficient than FMOLS? What is FMOLS equivalent to under these assumptions? If we drop the assumption that \( \Sigma \) is diagonal, how does this change your answers to d.?