1. Consider the linear regression model with a lagged dependent variable

\[ Y_t = \beta_0 + \beta_1 X_t + \beta_2 Y_{t-1} + U_t, \]

for \( t = 1, \ldots, T \). Now assume that \( U_0 \) and \( Y_0 \) are known and let

\[ U_t = \rho U_{t-1} + V_t, \]

where \( \mathbb{E}[V_t] = 0 \) and \( \mathbb{E}[V_t^2] = \sigma^2 \). We know that the OLS estimator of \( \beta_2 \) is inconsistent if the errors exhibit serial dependence. Show that this is the case by determining what \( T^{-1} \sum Y_{t-1} U_t \) converges to in probability.

2. For the model in question 1, additionally assume that \( V_t \) is normally distributed. Derive the information matrix for the parameter vector \( (\beta_0, \beta_1, \beta_2, \rho, \sigma^2) \).

3. Consider the process \( Y_t = \phi Y_{t-1} + U_t \) where

\[ |\phi| < 1, \quad U_t \sim \text{IN}(0, \sigma^2), \quad t = 1, 2, \ldots, T. \]

a. Write both the full and approximate log-likelihood functions for such a process.

b. Derive the asymptotic covariance matrix for \( \hat{\theta} = (\hat{\phi}, \hat{\sigma}^2) \).

**Hint:** Since the difference between the exact and conditional log-likelihood functions is negligible in large samples, use the conditional log-likelihood function.

c. Based on (a) above, derive the equation that you would solve to obtain an estimate of \( \phi \) in each case. How would you estimate \( \phi \) based on Yule-Walker equations? What is the relationship between the three methods?

**Hint:** You may wish to concentrate out \( \sigma^2 \) to form the concentrated log-likelihood function.