Dynamic Identification of DSGE Models

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1 Why is the Problem Non-standard?

2 Setup
   ■ Model
   ■ Observables

3 Identification Analysis
   ■ Observational Equivalence
   ■ Rank and Order Conditions

4 Illustration
   ■ Example 1: An and Schorfheide (2007)
   ■ Example 2: Stochastic Growth Model

5 Conclusion
Motivation

- In estimation of DSGE models, some parameters are always fixed
- How many (if any) should we fix?
- Does fixing the parameters guarantee identification of the remaining parameters?

Why is the Problem Non-standard?

- Models are dynamic with non-iid shocks.
- Likelihood may not exist because of stochastic singularity.
- Reduced form parameters are generally not identified.
Why is the Problem Non-standard?

**Literature**

1. multivariate ARMA models
   - rule out ‘stochastically singular’ models

2. classical rank conditions
   - require the reduced form parameters to be identified

3. objective function approaches: likelihood or GMM
   - depend on the population moments of the data

4. approaches based on observational equivalence
   - nonlinear models: this paper
Summary of the Results

1. show that the reduced form parameters of DSGE models are generally not identifiable
2. provide rank and order conditions for identifying the structural parameters from the observable spectrum
   - exploits all information in the autocovariances (spectrum) of the observed variables
   - exploits restrictions on observationally equivalent transformations of the model
   - depends only on the system matrices in the solution equations
   - should be evaluated prior to collecting data
3. analyze several examples
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5. Conclusion
Setup

Reduced form Solution:

\[
K_{t+1} = P(\theta) K_t + Q(\theta) Z_t
\]
\[
(n_K \times 1) (n_K \times n_K) (n_K \times n_Z)
\]
\[
W_t = R(\theta) K_t + S(\theta) Z_t
\]
\[
(n_W \times 1) (n_W \times n_K) (n_W \times n_Z)
\]
\[
Z_{t+1} = \Psi(\theta) Z_t + \epsilon_{t+1}
\]
\[
(n_Z \times 1) (n_Z \times n_Z) (n_Z \times 1)
\]

- \(K_t\): endogenous (state) variables
- \(W_t\): other endogenous (‘jump’) variables
- \(Z_t\): latent exogenous shocks
- \(\theta\): parameter vector of interest (dim \(n_\theta\))
Assumptions

1. For every $\theta \in \Theta$, $\{\epsilon_t\} \sim WN(0, \Sigma(\theta))$ and $\Sigma(\theta)$ nonsingular.

2. For every $\theta \in \Theta$ and any $z \in \mathbb{C}$, 
   \[ \det(\text{Id} - \Psi(\theta)z) \det(\text{Id} - P(\theta)z) = 0 \implies |z| > 1. \]
   Moreover, for every $(\theta, \tilde{\theta}) \in \Theta^2$, $P(\theta)$ and $\Psi(\tilde{\theta})$ have no eigenvalues in common.

3. $n_K \leq n_Z \leq n_K + n_W$.

4. For every $\theta \in \Theta$, we have: (i) $Q(\theta)$ full rank; (ii) $(Q(\theta)', S(\theta)')'$ full rank.

5. $P(\cdot), Q(\cdot), R(\cdot), S(\cdot), \Psi(\cdot), \Sigma(\cdot)$ continuously differentiable on $\Theta$. 
Observables

1. Lemma 1 shows that $Y_t$ is weakly stationary

$$
Y_t \equiv \begin{pmatrix} K_t \\ W_t \end{pmatrix} = \sum_{j=0}^{\infty} h(j; \theta) \epsilon_{t-j} = H(L; \theta) \epsilon_t
$$

- in general, $n_Z \leq n_Y$ and $h(0; \theta) \neq \text{Id}_{n_Y}$

2. Lemma 2 shows that $Y_t$ is invertible:

$$
\epsilon_t = \sum_{j=0}^{\infty} g(j; \theta) Y_{t-j} = G(L; \theta) Y_t
$$

- $\epsilon_t$ is fundamental for $Y_t$
- rank of $H(z; \theta)$ is $n_Z$ and a left inverse $G(z; \theta)$ exists
Transfer (impulse response) function:

\[ H(z; \theta) = \begin{pmatrix} z[\text{Id} - P(\theta)z]^{-1}Q(\theta)[\text{Id} - \Psi(\theta)z]^{-1} \\
\{ R(\theta)z[\text{Id} - P(\theta)z]^{-1}Q(\theta) + S(\theta) \} [\text{Id} - \Psi(\theta)z]^{-1} \end{pmatrix} \]

Spectrum:

\[ \Omega(z; \theta) \equiv \sum_{j=-\infty}^{+\infty} E(Y_t Y_{t+j}')z^{-j} = H(z; \theta) \Sigma(\theta) H(z^{-1}; \theta)' \]

\[ \Omega(z; \theta) \text{ only depends on the reduced form parameter} \]

\[ \Lambda(\theta) \equiv (P(\theta), Q(\theta), R(\theta), S(\theta), \Psi(\theta), \Sigma(\theta)) \]
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Identification Problem

- given $\theta_0$, we know $\Lambda(\theta_0)$, from which we can compute $\Omega(z; \theta_0)$
- the identification problem: given $\Omega(z; \theta_0)$, can we recover the $\theta_0$ that generated it?
- mathematically, $\theta_0$ is **identifiable from the spectrum** if for every $z \in \mathbb{C}$:

\[
H(z; \theta_0)\Sigma(\theta_0)H(z^{-1}; \theta_0)' = H(z; \theta_1)\Sigma(\theta_1)H(z^{-1}; \theta_1)'
\]

implies

\[
\theta_1 = \theta_0
\]
Identification Analysis
Observational Equivalence

Two sources of observational equivalence:

1. Equivalence of transfer functions:
   - each $H(z; \theta)$ can be obtained by many combinations of $(P(\theta), Q(\theta), R(\theta), S(\theta))$ in the transfer function
   - all such combinations are characterized in Lemma 3 (‘minimum state’) 

2. Equivalence of spectral densities:
   - there are many couples $(H(z; \theta), \Sigma(\theta))$ that can jointly produce the same spectrum
   - all such couples are characterized in Lemma 4 (‘minimum phase’)
Proposition 1

Let Assumptions 1 through 4 hold. Then $\theta_0$ and $\theta_1$ generate the same spectrum for $Y_t$ if and only if:

$$P(\theta_1) = P(\theta_0) \quad \text{and} \quad R(\theta_1) = R(\theta_0)$$

and there exists a full rank $n_Z \times n_Z$ matrix $U$ such that:

$$Q(\theta_1) = Q(\theta_0)U$$
$$S(\theta_1) = S(\theta_0)U$$
$$\Psi(\theta_1) = U^{-1}\Psi(\theta_0)U$$
$$\Sigma(\theta_1) = U^{-1}\Sigma(\theta_0)U^{-1}'.$$
Implications:

1. the reduced form parameter $\Lambda(\theta)$ is generally not identifiable (only its components $P(\theta)$ and $R(\theta)$ are)

2. Proposition 1 defines a system of

   $$(n_K + n_W)(n_K + n_Z) + 2n_Z^2$$

   equations

   (components of $\Lambda(\theta)$) in

   $$n_{\theta} + n_Z^2$$

   unknowns

   (components of $\theta$ and $U$)

3. identification at $\theta_0$ requires that the unique solution to this system be $\theta_1 = \theta_0$ and $U = \text{Id}$

4. this system is finite so we can write necessary and sufficient conditions for local uniqueness (local identification)
Rank and Order Conditions

Let

\[
\Delta(\theta) \equiv \begin{pmatrix}
\frac{\partial \text{vec}P(\theta)}{\partial \theta} & 0_{n_K^2 \times n_Z^2} \\
\frac{\partial \text{vec}Q(\theta)}{\partial \theta} & \text{Id}_{n_Z} \otimes Q(\theta) \\
\frac{\partial \text{vec}R(\theta)}{\partial \theta} & 0_{n_K n_W \times n_Z^2} \\
\frac{\partial \text{vec}S(\theta)}{\partial \theta} & \text{Id}_{n_Z} \otimes S(\theta) \\
\frac{\partial \text{vec}\Psi(\theta)}{\partial \theta} & \text{Id}_{n_Z} \otimes \Psi(\theta) - \Psi(\theta)' \otimes \text{Id}_{n_Z} \\
\frac{\partial \text{vec}\Sigma(\theta)}{\partial \theta} & -(\text{Id}_{n_Z^2} + T_{n_Z,n_Z})(\Sigma(\theta) \otimes \text{Id}_{n_Z})
\end{pmatrix}
\]
Proposition 2

Let Assumptions 1 through 5 hold. If the rank of $\Delta(\theta)$ remains constant in a neighborhood of $\theta_0$, then a necessary and sufficient rank condition for $\theta_0$ to be locally identified from the spectrum of $\{Y_t\}$ is:

$$\text{rank } \Delta(\theta_0) = n_\theta + n_Z^2.$$ 

Moreover, a necessary order condition is:

$$n_\theta \leq (n_K + n_W)(n_Z + n_K) + \frac{n_Z(n_Z + 1)}{2}.$$
Identification Under A Priori Restrictions

When the rank or order conditions fail we need:

\[ n_\varphi \text{ a priori restrictions } \quad \varphi(\theta) = 0 \]

Let

\[ \Delta_\varphi(\theta) \equiv \begin{pmatrix} \frac{\partial \varphi(\theta)}{\partial \theta} & 0_{n_\varphi \times n_Z^2} \\ \frac{\partial \operatorname{vec} P(\theta)}{\partial \theta} & 0_{n_\varphi^2 \times n_Z^2} \\ \frac{\partial \operatorname{vec} Q(\theta)}{\partial \theta} & \operatorname{Id}_{n_Z^2} \otimes Q(\theta) \\ \frac{\partial \operatorname{vec} R(\theta)}{\partial \theta} & 0_{n_K n_W \times n_Z^2} \\ \frac{\partial \operatorname{vec} S(\theta)}{\partial \theta} & \operatorname{Id}_{n_Z^2} \otimes S(\theta) \\ \frac{\partial \operatorname{vec} \Psi(\theta)}{\partial \theta} & \operatorname{Id}_{n_Z^2} \otimes \Psi(\theta) - \Psi(\theta)' \otimes \operatorname{Id}_{n_Z^2} \\ \frac{\partial \operatorname{vec} \Sigma(\theta)}{\partial \theta} & -(\operatorname{Id}_{n_Z^2} + T_{n_Z, n_Z})(\Sigma(\theta) \otimes \operatorname{Id}_{n_Z^2}) \end{pmatrix} \]
Proposition 3

Let Assumptions 1 through 5 hold and assume that $\phi$ is continuously differentiable on $\Theta$. If the rank of $\Delta \phi(\theta)$ remains constant in a neighborhood of $\theta_0$, then a necessary and sufficient \textbf{rank} condition for $\theta_0$ to be locally identified from the spectrum of $\{Y_t\}$ and the a priori restrictions $\phi(\theta_0) = 0$ is:

$$\text{rank } \Delta \phi(\theta_0) = n_\theta + n_Z^2.$$ 

Moreover, a necessary \textbf{order} condition for identification is:

$$n_\phi \geq n_\theta - \left[ (n_K + n_W)(n_Z + n_K) + \frac{n_Z(n_Z + 1)}{2} \right].$$
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Illustration
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Conclusion
Example 1: An and Schorfheide (2007)

\[ y_t = E_t y_{t+1} + g_t - E_t g_{t+1} - \frac{1}{\tau} (r_t - E_t \pi_{t+1} - E_t z_{t+1}) \]

\[ \pi_t = \beta E_t \pi_{t+1} + \frac{\tau (1 - \nu)}{\nu \pi^2 \phi} (y_t - g_t) \]

\[ c_t = y_t - g_t \]

\[ r_t = \rho_r r_{t-1} + (1 - \rho_r) \psi_1 \pi_t + (1 - \rho_r) \psi_2 (y_t - g_t) + \epsilon_{rt} \]

\[ g_t = \rho_g g_{t-1} + \epsilon_{gt} \]

\[ z_t = \rho_z z_{t-1} + \epsilon_{zt} \]

with \( \epsilon_{rt} \sim WN(0, \sigma_r^2) \), \( \epsilon_{gt} \sim WN(0, \sigma_g^2) \), \( \epsilon_{zt} \sim WN(0, \sigma_z^2) \) mutually uncorrelated

\( \bar{\pi} \) is steady state inflation rate
$P, Q, R, S$ representation of An and Schorfheide’s (2007) model is obtained by letting:

\[
Z_t \equiv \begin{pmatrix}
\epsilon_{rt} \\
g_t \\
z_t
\end{pmatrix}, \quad \epsilon_t \equiv \begin{pmatrix}
\epsilon_{rt} \\
\epsilon_{gt} \\
\epsilon_{zt}
\end{pmatrix} \quad (n_Z = 3)
\]

\[
K_t \equiv r_{t-1} \quad (n_K = 1)
\]

\[
W_t \equiv \begin{pmatrix}
y_t \\
\pi_t \\
c_t
\end{pmatrix} \quad (n_W = 3)
\]
Version I: $n_\theta = 13$

\[ \theta = (\tau, \beta, \nu, \phi, \bar{\pi}, \psi_1, \psi_2, \rho_r, \rho_g, \rho_z, \sigma_r^2, \sigma_g^2, \sigma_z^2) \]

- order condition:

\[ n_\theta = 13 < (n_K + n_W)(n_Z + n_K) + \frac{n_Z(n_Z + 1)}{2} = 31 \]

so the order condition in Proposition 2 holds

- rank condition:

\[ \text{rank } \Delta(\theta_0) = 21 < n_\theta + n_Z^2 = 22 \]

not surprising as An and Schorfheide (2007) noted that the components $\nu$, $\phi$, and $\bar{\pi}$ are not separately identifiable
**Version II: \( n_\theta = 11 \)**

\[
\kappa \equiv \frac{\tau (1 - \nu)}{\nu \pi^2 \phi}
\]
\[
\theta = (\tau, \beta, \kappa, \psi_1, \psi_2, \rho_r, \rho_g, \rho_z, \sigma_r^2, \sigma_g^2, \sigma_z^2)
\]

**order condition:**
\[
n_\theta = 11 < (n_K + n_W)(n_Z + n_K) + \frac{n_Z(n_Z + 1)}{2} = 31
\]

**rank condition:**
\[
\text{rank } \Delta(\theta_0) = 20 = n_\theta + n_Z^2
\]
Example 2: Stochastic Growth Model

$\max \{C_t, K_t\} \quad E_t \left[ \sum_{s=0}^{\infty} \beta^{t+s} \left( \frac{C_{t+s}^{1-\nu}}{1-\nu} \right) \right]$

- production technology: $Q_t = Z_t K_t^\alpha (1 - \Phi_t)$
- feasibility: $Q_t = C_t + K_{t+1} - (1 - \delta) K_t$
- exogenous technology process: $Z_0$ given $Z_t = Z_0 \exp(z_t)$ where $z_t = \psi z_{t-1} + \epsilon_t$ and $\epsilon_t \sim WN(0, \sigma^2)$
- adjustment cost: $\Phi_t(K_{t+1}, K_t) = \frac{\phi}{2} \left[ \frac{K_{t+1}}{K_t} - 1 \right]^2$
Example 2: Stochastic Growth Model

Reduced Form Solution:

\[ k_t = \lambda_{kk} k_{t-1} + \lambda_{kZ} z_t \]
\[ c_t = \lambda_{ck} k_{t-1} + \lambda_{cZ} z_t \]
\[ z_{t+1} = \psi z_t + \epsilon_{t+1}, \quad \epsilon_t \sim WN(0, \sigma^2) \]

The parameters of interest are:

\[ \theta \equiv (\alpha, \beta, \delta, \phi, \nu, \psi, \sigma^2) \]

The reduced form parameters are:

\[ \lambda \equiv (\lambda_{kk}, \lambda_{kZ}, \lambda_{ck}, \lambda_{cZ}, \psi, \sigma^2) \]

- each \( \lambda \) is a nonlinear function of \( \theta \)
- analytic expressions are provided (Appendix D)
We consider 3 versions of the model, all with

\[ \theta_0 = (0.36, 0.95, 0.025, \phi, \nu, 0.85, 0.04). \]

**Model I: \( \nu = 1, \phi = 0 \)**

\[ n_{\theta} = 5 = (n_K + n_W)(n_Z + n_K) + \frac{n_Z(n_Z + 1)}{2} \Rightarrow \text{order holds} \]

\[
\text{rank } \Delta(\theta_0) = 6 = n_{\theta} + n_Z^2 \implies \text{rank condition holds}
\]

Model I is identifiable at \( \theta_0 \) from the spectrum of \( \{(k_{t-1}, c_t)\}' \).
Model II: $\nu > 0$ unrestricted and $\phi = 0$

Here $n_\theta = 6$, so

$$n_\theta - (n_K + n_W)(n_Z + n_K) - \frac{n_Z(n_Z + 1)}{2} = 1 \Rightarrow \text{need } n_\phi = 1$$

Let $\nu_0 = 2$. Then

- $\phi(\theta) = \beta - 0.95$, rank $\Delta_\phi(\theta_0) = 6 < n_\theta + n_Z^2 = 7$
- $\phi(\theta) = \delta - 0.025$, rank $\Delta_\phi(\theta_0) = 7$

$\theta_0$ is conditionally identifiable by the restriction $\delta = 0.025$. 
Model III: $\nu > 0$ and $\phi > 0$ unrestricted

Here $n_\theta = 7$, so

$$n_\theta - (n_K + n_W)(n_Z + n_K) - \frac{n_Z(n_Z + 1)}{2} = 2 \Rightarrow \text{need } n_\phi = 2$$

Let $\nu_0 = 2$ and $\phi_0 = 0.03$.

For the rank condition to hold we need:

$$\text{rank } \Delta_\phi(\theta_0) = n_\theta + n_Z^2 = 8$$
Fixing 2 components of $\theta$ gives the following ranks of $\Delta\phi(\theta_0)$:

<table>
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<th>$\theta_i$</th>
<th>$\theta_j$</th>
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<th>$\delta$</th>
<th>$\phi$</th>
<th>$\nu$</th>
<th>$\psi$</th>
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Conclusion

Existing Identification Analysis:
- check Hessian of the likelihood
- compare properties of the VAR implied by the DSGE model with a factor augmented VAR
- gradient of the mapping from the deep to the reduced form parameters assumed to be identified

Drawbacks:
- the methods depend on parameters that are possibly not consistently estimated
- the reduced parameters themselves are not identifiable

We provide simple to evaluate order and rank conditions that only depend on the functional form of the solution matrices
Lemma 3: Similarity Transform

Let Assumptions 1 to 4 hold. Consider the realizations 
\((P(\theta_0), Q(\theta_0), R(\theta_0), S(\theta_0), \Psi(\theta_0))\) and 
\((P(\theta_1), Q(\theta_1), R(\theta_1), S(\theta_1), \Psi(\theta_1))\) of the transfer functions 
\(H(z; \theta_0)\) and \(H(z; \theta_1)\), respectively, with \((\theta_0, \theta_1) \in \Theta^2\). Then the two realizations are equivalent, i.e. 
\(H(z; \theta_0) = H(z; \theta_1)\) for every \(z \in \mathbb{C}\), if and only if 
\[P(\theta_1) = P(\theta_0), \quad R(\theta_1) = R(\theta_0)\]

and there exists a full rank \(n_Z \times n_Z\) matrix \(T\) such that: 
\[Q(\theta_1) = Q(\theta_0)T^{-1}, \quad S(\theta_1) = S(\theta_0)T^{-1}, \quad \Psi(\theta_1) = T\Psi(\theta_0)T^{-1}.\]
Lemma 4: Spectral Decomposition

Let Assumptions 1 to 4 hold. Consider two couples \((H(z; \theta_0), \Sigma(\theta_0))\) and \((H(z; \theta_1), \Sigma(\theta_1))\) with spectral densities \(\Omega(z; \theta_0)\) and \(\Omega(z; \theta_1)\), respectively, where \((\theta_0, \theta_1) \in \Theta^2\). Then the two couples are equivalent, i.e. \(\Omega(z; \theta_0) = \Omega(z; \theta_1)\) for every \(z \in \mathbb{C}\), if and only if there exists a full rank \(n_Z \times n_Z\) matrix \(U\) such that:

\[
\text{for every } z \in \mathbb{C} \quad H(z; \theta_1) = H(z; \theta_0)U
\]

and

\[
U\Sigma(\theta_1)U' = \Sigma(\theta_0).
\]