Two-Stage Least Squares Estimation in Simultaneous Equation Models

Given that the parameters of an equation in a simultaneous equation model are identified, one then turns to estimation of the parameters. Consider the identified simultaneous equation model

\[
Y_{1t} = \alpha_1 + \alpha_2 Y_{2t} + \alpha_3 X_{1t} + U_{1t},
\]

\[
Y_{2t} = \beta_1 + \beta_2 Y_{1t} + \beta_3 X_{2t} + U_{2t},
\]

for which \(Y_{1t}\) and \(Y_{2t}\) are the jointly endogenous variables. The structural equations contain regressors that are correlated with the error. To understand the correlation: If \(U_{1t}\) increases, then \(Y_{1t}\) increases ⇒ \(Y_{2t}\) increases (assuming \(\beta_2 > 0\)) so \(U_{1t}\) and \(Y_{2t}\) are (positively) correlated. In essence, when \(U_{1t}\) is positive, \(Y_{2t}\) tends to be above its mean and both increase \(Y_{1t}\) (assuming \(\alpha_2 > 0\)). Because \(U_{1t}\) is unobserved, we attribute all of the increase in \(Y_{1t}\) to \(Y_{2t}\), thereby overestimating \(\alpha_1\). Because the source of the bias is the simultaneous determination of \(Y_{1t}\) and \(Y_{2t}\), the bias is referred to as simultaneity bias. (The OLS estimators are not only biased, they are inconsistent.)

The problem of endogenous regressors is also revealed by attempting to interpret the coefficients. The coefficient \(\alpha_2\) is designed to capture the effect of a small change in \(Y_{2t}\) holding \(X_{1t}\) constant. Yet a change in \(Y_{2t}\) (caused, say, by a change in \(U_{2t}\)) leads to a change in \(Y_{1t}\), which then feeds back on \(Y_{2t}\) through the second equation, which again affects \(Y_{1t}\) and so on. We see that \(\alpha_2\) captures the effects of all the feedbacks and so represents some mix of the effect of \(Y_{2t}\) on \(Y_{1t}\) and the effect of \(Y_{1t}\) on \(Y_{2t}\). Further, consider \(\alpha_3\), which is designed to capture the effect of a small change in \(X_{1t}\) on \(Y_{1t}\) holding \(Y_{2t}\) constant. Yet \(Y_{2t}\) cannot be held constant as \(Y_{1t}\) changes, implying that all the coefficients estimators are biased by simultaneity.

As noted earlier, a natural way to mitigate the bias would be to replace the endogenous regressors with instruments. Because a good instrument is hard to find, the idea is to construct the instruments from the predetermined regressors and then form IV estimators. The method is termed two-stage least squares (2SLS) estimation, in which the first stage constructs the instruments and the second stage constructs IV estimators of the parameters of interest.

To begin, we must create instruments. A natural set of variables from which to construct the instruments is the set of predetermined regressors in the model.
A natural way to form the instruments is to select the linear combination of the predetermined regressors that is most highly correlated with the endogenous regressor. To do so, we estimate the first-stage equations

\[
Y_{1t} = \eta_1 + \eta_2 X_{1t} + \eta_3 X_{2t} + V_{1t}, \\
Y_{2t} = \theta_1 + \theta_2 X_{1t} + \theta_3 X_{2t} + V_{2t}.
\]

The two equations from the first stage are termed \textit{reduced-form} equations, as they express the endogenous variables wholly in terms of predetermined regressors. The parameters of the reduced form can be estimated by OLS, as the regressors are not correlated with the error. In special cases, researchers are interested in the parameters of the reduced form rather than the structure. For example, if one wondered about the effect of a change in \(X_{1t}\) on \(Y_{1t}\) after all the dynamic interactions subside, then the answer is obtained by estimating \(\eta_2\).

Because the reduced form is obtained by algebraically solving each of the structural equations for one endogenous variable, the reduced form coefficients and errors are functions of the structural parameters and errors. To see this, we solve the structural model presented above. By substituting the second equation into the first, we obtain

\[
Y_{1t} = \alpha_1 + \alpha_2 (\beta_1 + \beta_2 Y_{1t} + \beta_3 X_{2t} + U_{2t}) + \alpha_3 X_{1t} + U_{1t},
\]

or

\[
Y_{1t} = \frac{1}{1 - \alpha_2 \beta_2} \left[ \alpha_1 + \alpha_2 (\beta_1 + \beta_3 X_{2t} + U_{2t}) + \alpha_3 X_{1t} + U_{1t} \right],
\]

which reduces to

\[
Y_{1t} = \frac{1}{1 - \alpha_2 \beta_2} \left[ (\alpha_1 + \alpha_2 \beta_1) + \alpha_3 X_{1t} + \alpha_2 \beta_3 X_{2t} + U_{1t} + \alpha_2 U_{2t} \right].
\]

Substituting the final expression for \(Y_{1t}\) into the second structural equation yields

\[
Y_{2t} = \beta_1 + \beta_2 \frac{1}{1 - \alpha_2 \beta_2} [ (\alpha_1 + \alpha_2 \beta_1) + \alpha_3 X_{1t} + \alpha_2 \beta_3 X_{2t} + U_{1t} + \alpha_2 U_{2t} ] \\
+ \beta_3 X_{2t} + U_{2t},
\]

which reduces to

\[
Y_{2t} = \frac{\beta_1 + \alpha_1 \beta_2}{1 - \alpha_2 \beta_2} + \frac{\alpha_3 \beta_2}{1 - \alpha_2 \beta_2} X_{1t} + \frac{\beta_3}{1 - \alpha_2 \beta_2} X_{2t} + \frac{\beta_2}{1 - \alpha_2 \beta_2} U_{1t} + \frac{1}{1 - \alpha_2 \beta_2} U_{2t}.
\]
Recovery of Structural Coefficients from Reduced-Form Coefficients

It seems natural to ask, can we uniquely recover the structural parameters from the reduced form? If so, then we need only estimate the reduced form. If we can uniquely recover the parameters for a given structural equation from the reduced form, then that structural equation is said to be exactly identified. If we cannot, then the structural equation is said to be overidentified. It is quite possible, and indeed common, to have a structural model in which some equations are exactly identified and some equations are overidentified. If all structural equations are exactly identified, then estimation of the reduced form and algebraic recovery of the structural parameters, termed *Indirect Least Squares*, is consistent and efficient. To determine if an equation is exactly identified, we rely on the order condition, which is expressed for each equation as

**Order Condition:** If the number of predetermined variables in the system equals the number of slope coefficients in the equation, then the parameters of the equation are (if the rank condition is satisfied) exactly identified. If the number of predetermined variables in the system exceeds the number of slope coefficients in the equation, then the parameters of the equation are overidentified. (If there are fewer predetermined variables than slope coefficients, the parameters in the equation are not identified.)

For each of the two equations in our structure there are 2 slope coefficients. As there are 2 predetermined variables in the system, both equations are exactly identified. Solving the system of reduced-form parameter equations yields the expressions for the structural parameters

\[
\begin{align*}
\alpha_1 &= \eta_1 - \frac{\eta_1}{\theta_3} \eta_3, \\
\alpha_2 &= \frac{\eta_2}{\theta_3} \eta_3, \\
\alpha_3 &= \eta_2 - \frac{\eta_2}{\theta_3} \eta_3, \\
\alpha_4 &= \eta_4 - \frac{\eta_4}{\theta_3} \eta_3, \\
\beta_1 &= \theta_1 - \frac{\eta_1}{\theta_2} \theta_2, \\
\beta_2 &= \frac{\eta_2}{\theta_2} \theta_2, \\
\beta_3 &= \theta_3 - \frac{\eta_3}{\theta_2} \theta_2, \\
\beta_4 &= \theta_4 - \frac{\eta_4}{\theta_2} \theta_2.
\end{align*}
\]

If the parameters of a structural equation are overidentified, then there are several ways of constructing the structural parameter from the reduced-form parameters. The overidentifying restrictions can be tested. If the structural parameter estimates constructed from different reduced-form parameter functions are significantly different, then there is evidence against the adequacy of the structural model.

The above model is exactly identified, so indirect least squares avoids the need for a second stage. Yet it is more common to have a model in which each equation is not exactly identified. We consider a simultaneous equation model for which
each equation is identified, but the first equation is overidentified
\[
Y_{1t} = \alpha_1 + \alpha_2 Y_{2t} + \alpha_3 X_{1t} + U_{1t}, \\
Y_{2t} = \beta_1 + \beta_2 Y_{1t} + \beta_3 X_{2t} + \beta_4 X_{3t} + U_{2t}.
\]

The reduced form is
\[
Y_{1t} = \frac{\alpha_1 + \alpha_2 \beta_1}{1 - \alpha_2 \beta_2} + \frac{\alpha_3}{1 - \alpha_2 \beta_2} X_{1t} + \frac{\alpha_2 \beta_3}{1 - \alpha_2 \beta_2} X_{2t} + \frac{\alpha_2 \beta_4}{1 - \alpha_2 \beta_2} X_{3t} + \frac{U_{1t} + \alpha_2 U_{2t}}{1 - \alpha_2 \beta_2}, \\
Y_{2t} = \frac{\beta_1 + \alpha_1 \beta_2}{1 - \alpha_2 \beta_2} + \frac{\alpha_3 \beta_2}{1 - \alpha_2 \beta_2} X_{1t} + \frac{\beta_3}{1 - \alpha_2 \beta_2} X_{2t} + \frac{\beta_4}{1 - \alpha_2 \beta_2} X_{3t} + \frac{\beta_2 U_{1t} + U_{2t}}{1 - \alpha_2 \beta_2}.
\]

Unique estimates of the exactly identified second equation are obtained from
\[
\beta_2 = \frac{\theta_2}{\eta_2} \text{ and } \beta_i = \theta_i - \frac{\theta_2}{\eta_2} \eta_i \text{ for } i = 1, 3, 4.
\]

One set of estimates for the overidentified first equation are obtained from
\[
\alpha_2 = \frac{\eta_3}{\theta_3}, \alpha_1 = \eta_1 - \frac{\eta_3}{\theta_3} \theta_1 \text{ and } \alpha_3 = \eta_2 - \frac{\eta_3}{\theta_3} \theta_2.
\]

An alternate set of estimates are obtained from
\[
\alpha_2 = \frac{\eta_4}{\theta_4}, \alpha_1 = \eta_1 - \frac{\eta_4}{\theta_4} \theta_1 \text{ and } \alpha_3 = \eta_2 - \frac{\eta_4}{\theta_4} \theta_2.
\]

Finally, we consider a system in which each equation is not identified. To see the outcome for such a system, we return to the general structure introduced last time
\[
Y_{1t} = \alpha_1 + \alpha_2 X_{1t} + \alpha_3 X_{3t} + U_{1t}, \\
Y_{2t} = \beta_1 + \beta_2 Y_{3t} + \beta_3 X_{1t} + \beta_4 X_{2t} + U_{2t}, \\
Y_{3t} = \gamma_1 + \gamma_2 Y_{1t} + \gamma_3 X_{1t} + \gamma_4 X_{3t} + U_{3t}.
\]

From our examination of the rank condition, we know that the first two equations are identified, while the third is not. As we again have 3 predetermined regressors, the second equation is exactly identified, while the first equation is overidentified. The reduced-form equations are
\[
Y_{1t} = \eta_1 + \eta_2 X_{1t} + \eta_3 X_{2t} + \eta_4 X_{3t} + V_{1t}, \\
Y_{2t} = \theta_1 + \theta_2 X_{1t} + \theta_3 X_{2t} + \theta_4 X_{3t} + V_{2t}, \\
Y_{3t} = \kappa_1 + \kappa_2 X_{1t} + \kappa_3 X_{2t} + \kappa_4 X_{3t} + V_{3t}.
\]
As there are no jointly endogenous regressors in the first structural equation, the reduced-form coefficients for \(Y_{1t}\) are given directly: \(\eta_1 = \alpha_1, \eta_2 = \alpha_2, \eta_3 = \alpha_3\) and \(\eta_4 = 0\). The reduced-form equation for \(Y_{2t}\) is obtained as:

\[
Y_{2t} = \beta_1 + \beta_2 (\gamma_1 + \gamma_2 Y_{1t} + \gamma_3 X_{1t} + \gamma_4 X_{3t} + U_{3t}) + \beta_3 X_{1t} + \beta_4 X_{2t} + U_{2t}
\]

\[
= \beta_1 + \beta_2 \gamma_1 + \beta_2 \gamma_2 (\alpha_1 + \alpha_2 X_{1t} + \alpha_3 X_{3t} + U_{1t})
\]

\[
+ (\beta_2 \gamma_3 + \beta_3) X_{1t} + \beta_4 X_{2t} + \beta_2 \gamma_4 X_{3t} + U_{2t} + \beta_2 U_{3t},
\]

the right side of which simplifies to

\[
(\beta_1 + \beta_2 \gamma_1 + \beta_2 \gamma_2 \alpha_1) + (\beta_2 \gamma_3 + \beta_3 + \beta_2 \gamma_2 \alpha_2) X_{1t} + \beta_4 X_{2t} + (\beta_2 \gamma_4 + \beta_2 \gamma_2 \alpha_3) X_{3t} + U_{2t} + \beta_2 U_{3t} + \beta_2 \gamma_2 U_{1t}.
\]

The reduced-form equation for \(Y_{3t}\) is obtained as:

\[
Y_{3t} = \gamma_1 + \gamma_2 (\alpha_1 + \alpha_2 X_{1t} + \alpha_3 X_{3t} + U_{1t}) + \gamma_3 X_{1t} + \gamma_4 X_{3t} + U_{3t},
\]

which simplifies to

\[
Y_{3t} = (\gamma_1 + \alpha_1 \gamma_2) + (\alpha_2 \gamma_2 + \gamma_3) X_{1t} + (\alpha_3 \gamma_2 + \gamma_4) X_{3t} + U_{3t} + \gamma_2 U_{1t}.
\]

Because \(\kappa_3 = 0\), which is the root cause of the identification problem, we have only 10 nonzero reduced-form coefficients. The 10 reduced-form coefficients form a system of 10 equations with 11 unknowns (the structural equation coefficients). We are unable to recover the structural parameters from the reduced-form parameters, as expected given the identification problems.

Two-Stage Least Squares

For the general case in which we cannot uniquely recover the structural coefficients from the reduced-form coefficients, how do we proceed? We use the instruments formed in the first stage as regressors in the second stage. Returning to our initial example, let \(Y_{1t}^P\) and \(Y_{2t}^P\) be the predicted values of the endogenous variables:

\[
Y_{1t}^P = H_1 + H_2 X_{1t} + H_3 X_{2t} + H_4 X_{3t},
\]

\[
Y_{2t}^P = T_1 + T_2 X_{1t} + T_3 X_{2t} + T_4 X_{3t}.
\]

The predicted values are used as instruments in the second stage regression

\[
Y_{1t} = \alpha_1 + \alpha_2 Y_{2t}^P + \alpha_3 X_{1t} + \alpha_4 X_{3t} + U_{1t}^*,
\]

\[
Y_{2t} = \beta_1 + \beta_2 Y_{1t}^P + \beta_3 X_{2t} + \beta_4 X_{3t} + U_{2t}^*.
\]
where $U_{1t}^* = U_{1t} + \alpha_2 (Y_{2t} - Y_{2t}^P)$. The 2SLS coefficient estimators are obtained by OLS on each equation. A computational note: standard OLS standard errors are not correct, as they do not account for the fact that the instrument is generated in a first-stage regression. Econometric software calculates the correct standard errors in the 2SLS estimation routine. Therefore, do not do the stages separately with OLS, rather use the 2SLS software routine.

The 2SLS estimators are consistent, but they remain biased. Why? Because the reduced-form coefficient estimators $(H_1, \ldots, H_4)$ are functions of $Y_{1t}$ (and the coefficient estimators of $T_1, \ldots, T_4$ are functions of $Y_{2t}$), so $Y_{1t}^P$ is still a function of $Y_{1t}$ and is correlated with $U_{2t}$. Of course, passing $Y_{1t}$ through the “$X$ filter” greatly reduces the correlation and the 2SLS estimators are much more accurate. (The reduction in bias is so large that the 2SLS estimators are typically biased downward, while the bias of the OLS estimators of the structural equations is generally substantially upward.)

The estimated $R^2$ from the reduced form equations gives some idea of the adequacy of the instrument. A very low value indicates a great loss in precision in estimating $\alpha_2$ as the instrument moves only slightly with the regressor. A very high value indicates such strong collinearity that one doubts if the instrument is truly exogenous. Finally, if the predetermined regressors are highly correlated, the 2SLS standard errors will be large as the instrument and other regressors are highly correlated. Of course, if no predetermined regressors are excluded from an equation, then the instrument and the predetermined regressors will be multicollinear. (Recall, this is the point of the identification condition that at least one predetermined regressor be excluded from each equation.)