Overlapping Generations & Dynamic Inefficiency

Motivating example: why inefficiency may be empirically relevant and deserves analysis.

- Assume log-utility, Cobb-Douglas production, depreciation $\delta > 0$: $r_{t+1} = f'(k_{t+1}) = \alpha \cdot k_{t+1}^{\alpha - 1} - \delta$
- Steady state: $k^* = \left[ \frac{\beta/(1+\beta)}{(1+n)(1+g)} (1 - \alpha) \right]^{1-\alpha}$
  
  $\Rightarrow r^* = \alpha \cdot \left[ \frac{\beta/(1+\beta)}{(1+n)(1+g)} (1 - \alpha) \right]^{-1} - \delta = (1 + n) \cdot (1 + g) \cdot \frac{\alpha}{1-\alpha} \cdot \frac{1+\beta}{\beta} - \delta$

- Suppose most capital depreciates ($\delta \approx 1$), small time preference ($\beta \approx 1$): $r^* \approx (1 + n) \cdot (1 + g) \cdot \frac{\alpha}{1-\alpha} \cdot 2 - 1$
  
  If $\alpha < \frac{1}{3}$ then $\frac{\alpha}{1-\alpha} \cdot 2 < 1$ $\Rightarrow 1 + r^* < (1 + n) \cdot (1 + g)$

Illustration in capital market diagram: $f^{-1}(r + \delta)$ and $a(w(k), r)$ may intersect at “low” $r$-value

[Caution: high initial $k$ implies low $r$. Inefficiency only if $r$ is too low at $f^{-1}(r + \delta) = k^* = a(w(k^*), r).$]

Evidence

- Long-run U.S. Data: GDP-growth $\sim 3%$/p.a. ($n \sim 1%$/p.a., $g \sim 2%$/p.a.)
  
  Real return on T-bills $r \sim 1%$; return on capital $r \sim (5%-8%)$. Which return is relevant?

- Abel-Mankiw-Summers-Zeckhauser: (Capital income) $>$ (Gross investment) implies dynamic efficiency.

  Intuition: test of efficiency condition $\alpha > s$, eliminates noise from fluctuations in asset values.

  Evidence: inequality holds for all G7 countries throughout post-WWII period.

Implications: Return on capital matters for dynamic efficiency. Low safe interest rates due to risk aversion or frictions in financial intermediation/banking.
Why does Pareto-efficiency impose a lower bound on \( r \)?

- **Pareto efficiency**: Maximize utility of some cohort s.t. constraint that other cohorts’ utility is not reduced
  
  => Lagrangian with cohort utility + sum of other cohort utilities * multipliers
  
  => Equivalent to maximizing a weighted average of utilities

- **Social planning problem**: Maximize welfare function s.t. resource constraint. Suppose:
  
  - Period \( t=1 \) old are endowed with \( K_1 \).
  
  - Period \( t=T \) young all die at the end of period \( T \). (Finite endpoint to start.)
  
  - Planner discounts future generations’ per-capita utility with factor \( \gamma \) (unit weight on \( t=0 \))
  
  - Abstract from productivity growth (to avoid clutter).

- **Maximize welfare function**: \( W = L_0 \beta u(C_{2t}) + \sum_{t=1}^{T-1} \gamma^t L_t [u(C_{1t}) + \beta u(C_{2t+1})] + \gamma^T L_T u(C_{1T}), \) s.t.
  
  \[
  L_t \cdot C_{1t} + L_{t-1} \cdot C_{2t} + K_{t+1} = F(K_t, L_t) + (1 - \delta) \cdot K_t, \text{ or equivalently}
  
  C_{1t} + \frac{1}{1+n} \cdot C_{2t} + (1 + n) \cdot k_{t+1} = f(k_t) + k_t
  
  \]

- Consider Lagrangian with multipliers \( \lambda_t \) and take FOC:
  
  a. consumption of workers: \( \gamma^t L_t u'(C_{1t}) - L_t \lambda_t = 0 \)  
  
  \[\Rightarrow \lambda_t = \gamma^t u'(C_{1t})\]

  b. consumption of retirees: \( \gamma^{t-1} L_{t-1} \beta u'(C_{2t}) - L_{t-1} \lambda_t = 0 \)  
  
  \[\Rightarrow \lambda_t = \gamma^{t-1} \beta u'(C_{2t})\]

  c. capital accumulation: \( \lambda_t (F_K(K_t, L_t) + 1 - \delta) - \lambda_{t-1} = 0 \),

  where \( F_K(K_t, L_t) + 1 - \delta = 1 + f'(k_t) = 1 + r_t \)
• Combined the optimality conditions:

1. Period t workers & retirees: [Insert (a) in (b) at time t into (c)]
\[ \gamma^{t-1} \beta u'(C_{2t}) - \gamma^t u'(C_{1t}) = 0 \Rightarrow \beta u'(C_{2t}) = \gamma u'(C_{1t}) \]

   Social planner balances marginal utilities (consumption) of young and old.

2. Cohort t-1 as workers & retirees [Insert (a) for t-1 and (b) for t into (c)]:
\[ \gamma^{t-1} \beta u'(C_{2t})(1 + f'(k_i)) - \gamma^{t-1} u'(C_{1t-1}) = 0 \]
\[ \Rightarrow \quad u'(C_{1t-1}) = \beta u'(C_{2t})(1 + f'(k_i)) \]

   Social planner respects individual optimality conditions for consumption/savings.

3. Workers in periods t-1 and t [Insert (a) for t-1 and (a) for t into (c)]
\[ \gamma^t u'(C_{1t})(1 + f'(k_i)) - \gamma^{t-1} u'(C_{1t-1}) = 0 \]
\[ \Rightarrow \quad u'(C_{1t-1}) = \gamma u'(C_{1t})(1 + f'(k_i)) \]

   Condition characterizing consumption growth over time.

• Consider the infinite horizon limit: Steady state must satisfy $1 = \gamma (1 + f'(k^*))$ from [3]

  - Transversality condition:
  \[ \lim_{T \to \infty} \lambda_T K_{T+1} = \lim_{T \to \infty} \gamma^T u'(C_{1T}) K_{T+1} = k^* u'(C^*) \lim_{T \to \infty} \gamma^T (1 + n)^{T+1} = 0 \text{ requires } \gamma < \frac{1}{1+n}. \]
  - Conclude: $1 + r^* = 1 + f'(k^*) = 1 / \gamma > 1 + n$. If the planner discounts the future enough that the welfare problem is well defined, then $k^*$ is low enough that $r^* > n$. Implies dynamic efficiency.
Weird phenomena in dynamically inefficient economies

Leading example: Fiat money as store of value

- Money stock is introduced as transfer to the old at time $t=1$: $M$ units – fixed stock of money.
  \[ p_t = \text{Value of money in terms of goods} = \text{purchasing power} = \text{inverse of price level} \]
  \[ M \cdot p_t = \text{Aggregate value of money. Includes special case } p_t = 0 \text{ when money is not valued.} \]
  \[ m_t = M \cdot p_t / L_t = \text{Per-capita money holdings of the young (end of period)} \]
  \[ a_t = a_i^k + m_i = \text{Assets} = \text{Sum of money and claims on capital (a}^k) \]

- Question: Can we find an equilibrium with positive prices for fiat money?
  
  - Simplify: Assume no productivity growth.
  
  - Equilibrium condition with population growth $n$: $k_{t+1} = \frac{1}{1+n} a_i^k$.

- Budget constraints:
  
  - For the old generation in period 1, which receives fresh money:
    \[ C_{21} = (1 + r_1) \cdot a_0^k + M / L_0 \cdot p_1 \]
    where $a_0^k$ is invested in capital $\Rightarrow$ First generation benefits if $p_1 > 0$.

  - For generations $t \geq 1$:
    \[ C_{1t} + a_i^k + m_i = C_{1t} + a_i = W_t \]
    \[ C_{2t} = (1 + r_t) \cdot a_{t-1}^k + p_t / p_{t-1} \cdot m_{t-1} = (1 + r_t) \cdot a_{t-1} - (1 + r_t - p_t / p_{t-1}) \cdot m_{t-1} \]
• Optimality condition for holding money (arbitrage condition):
  - For individuals to hold both money and capital, the returns on money and capital must be equal
    
    
    
    
    
    - Note that if \( r > n \), this requires deflation and increasing per-capita money holdings.
    - Implies dynamics of per-capital money holdings:
      
      \[
      m_{t+1} = \frac{1+r_{t+1}}{1+n} \cdot m_t
      \]
      
      - Steady state with \( m^* > 0 \) requires \( r^* = f'(k^*) - \delta = n \); and deflation at rate \( r^* \): \( \frac{p_{t+1} - p_t}{p_t} = -r^* \).

• FOC for savings: Given the arbitrage condition, savers care only about total assets
  - FOC is:
    
    \[
    u'(W_t - a_t) = \beta \cdot (1 + r_{t+1}) \cdot u'((1 + r_{t+1}) \cdot a_t)
    \]
    
    - defines the savings function \( a_t = a(W_t, r_{t+1}) \), where \( W_t = w(k_t) \) and \( r_{t+1} = f'(k_{t+1}) - \delta \)

• Equilibrium condition on the capital market can be written as:
  
  \[
  (1 + n)k_{t+1} + m_t = a(W_t, r_{t+1}) = a[w(k_t), f'(k_{t+1})]
  \]
  
  - Defines implicit function \( k_{t+1} = \hat{K}(k_t, m_t) \) with derivatives \( \frac{dk_{t+1}}{dk_t} = \frac{a_w w_k}{1 + n - a_r f''} \) and \( \frac{dk_{t+1}}{dm_t} = -\frac{1}{1 + n - a_r f''} \).

• Assume that the economy without money displays monotone convergence (in relevant range for \( k \))
  - Requires: \( 0 < a_w w_k < 1 + n - a_r f'' \). Implies: \( 0 < \hat{K}_k < 1 \) and \( \hat{K}_m < 0 \).
• **Construct an equilibrium** in which money is valued:
  - Necessary conditions for steady state: constant \( m = m^* \) and \( k = k^* \).
    => Monetary dynamics \( m_{t+1} = \frac{1+r_{t+1}}{1+n} \cdot m_t \) imply that \( r^* = n \).
    => Capital stock must be \( k^* = (f')^{-1}(n+\delta) = k_G \) = Golden Rule level
  - Capital market equilibrium at \( r = n \) requires that \( k^* = \frac{1}{1+n} [a(w(k^*),n) - m^*] \)
    => \( m^* = a(w(k_G),n) - (1+n) \cdot k_G \)
    => Steady state with \( m^* > 0 \) requires that \( a(w(k_G),n) > (1+n) \cdot k_G \)

• For comparison, let \( (k_d, r_d) \) denote the steady state in the economy without money (\( d = \text{Diamond} \)).
  - Equilibrium condition in steady state: \( a(w(k_d),r_d) = (1+n) \cdot k_d \)
    If \( a(w(k_G),n) > (1+n) \cdot k_G \) then \( r_d < n \) and \( k_d > k_G \)
    => Economy without money must be dynamically inefficient.

• Claims to prove:
  1. An equilibrium with fiat money exists if and only if the economy w/o money is dynamically inefficient;
     specifically, the configuration \( (k^* = k_G, m^*, r^* = n) \) constitutes a steady state.
  2. If \( m^* > 0 \), the for any initial value \( k_1 \), there is a unique value \( m_1 \) so that the economy converges to \( (k_G, m^*) \)
  3. The monetary equilibrium is not unique. For a range of (smaller) values \( m_1 \) there are perfect foresight paths that all converge to \( k^* = k_d \) and \( m \to 0 \).
    - Argument uses phase diagrams – constructive.
Diagrams from Blanchard&Fischer (Lectures on Macroeconomics)

• Different notation: $b = m$, ("b" for bubble—reinterpretation to be discussed)

• Conditions for a steady state:

$$ m = a(w(k),n) -(1 + n) \cdot k $$

for any given $k$. Equivalently $k = \hat{K}(k,m)$, i.e., constant $k$ for given $m$.

$$ \Rightarrow \text{Line with inverted-U shape running through (0,0), (}k_G, m^*\text{), and (}k_d, 0\text{).} $$

$$ k^* = k_G = (f')^{-1}(n + \delta) $$

vertical line.

• Point $A$ = steady state w/o money; $E$ = steady state with money.
• Dynamics out of steady state: system of difference equations in \((m,k)\).

1. Write \(k_{t+1} = \hat{K}(k_t,m_t)\) as difference equation: \(k_{t+1} - k_t = \hat{K}(k_t,m_t) - k_t\).
   - Constant \(k\) requires: \(0 = \hat{K}(k_t,m_t) - k_t\). Above the line: \(m\) up \(\Rightarrow k_{t+1}\) down \(\Rightarrow \Delta k < 0\).

2. Write: \(m_{t+1} - m_t = \left(\frac{r_{t+1} - n}{1+n}\right) \cdot m_t = \frac{f'(\hat{K}(k_t,m_t)) - \delta - n}{1+n} \cdot m_t\).
   - Constant \(m\) requires: \(f'(\hat{K}(k_t,m_t)) - n = 0\). Recall \(\hat{K}_k > 0, \hat{K}_m < 0 \Rightarrow\) Phase arm has positive slope.
   - To the right: \(k\) up \(\Rightarrow \hat{K}\) up, \(f'(\hat{K})\) down \(\Rightarrow \Delta m < 0\). To the left: \(\Delta m > 0\)

• Combine phase arrows: there is a saddle path with positive slope though \((k_G, m^*)\). But multiple solutions:
   - Given initial \(k_1\), pick \(m_1\) on saddle path \(\Rightarrow\) economy converges to \((k_G, m^*) \Rightarrow\) equilibrium with money.
   - Given \(k_1\), pick initial \(m_1\) below the saddle path \(\Rightarrow\) phase arrows lead to \((k_d, 0) \Rightarrow\) equilibrium w/o money.
• Intuition:
  - Dynamical inefficiency means that individuals are “desperate” to save for retirement
    => They accept low or negative returns on savings
    => Capital does not suffice as store of value.
    => *Anything* can be a store of value provided the next generation is expected to value it, too.

• Examples:
  - **Fiat money**: Government can distribute the gains from money creation at time $t=1$.
  - **Paygo social security**: Social contract to pay the next generation.
  - **Ponzi schemes**: Private creation of a store of value.
  - **Speculative bubbles**: Agreement to value a return-producing asset above its “fundamental” value.

• Is dynamic efficiency more than a theoretical curiosity?
  - Existence of long-lived, yield-producing assets: Land
    (Exercise: Land value becomes infinite as $r \to 0$, removing any shortage of assets)
Learning Objectives for Part 4

• Problem Solving:
  - Compute optimal savings and capital accumulation in the OG model.
  - Determine the impact of changes in fiscal policy and other exogenous changes.
  - Graphical analysis: indifference curve; capital/credit market; dynamics of capital/45°-line.

• Conceptual:
  - Know and be able to explain key differences between OG and dynastic models.
  - Be able to explain why altruistic bequests imply Ricardian neutrality.
  - Know about the controversy surrounding Ricardian neutrality.
  - Know about the possibility of dynamic inefficiency and about its ramifications.

Conclusion to Econ 204A

• Dynastic and OG models are the main paradigms of macroeconomic modeling.
  - Both serve as laboratories for examining the dynamics of aggregate economic activity.
  - Rigorous analysis = Optimization subject to constraints (due to technology & policy).
• Sometimes express results in terms of traditional models/labels.
• This class: Basic models.