Introduction to Money

• How does money fit into modern macro models?
  - Money \( M = \) nominal units issued by the government. Price level \( p \). Purchasing power \( 1/p \).
  - Consider discrete periods: Household hold money and interest-bearing assets:
    \[
    c_t + a_{t+1} + M_{t+1} / p_t = w_t + (1 + r_t) a_t + M_t / p_t
    \]
    - Real rate of return on money holdings = \( p_t / p_{t+1} \).
    - Money is an inferior store of value whenever \( p_t / p_{t+1} < 1 + r_{t+1} \)
    - Equivalent: Positive interest rate on nominal bonds.

=> Two branches of monetary theory:
  a. Monetary models with a “transactions” motive for money
    - Accept that money is dominated by other assets in terms of return
    => Assume frictions in exchange processes as motive for holding money
  b. Monetary theory with money as a store of value:
    - Most popular: Overlapping-generations models with dynamic inefficiency

• How important is money for economic activity?
  - Even in models with motive for holding money, it may not “matter” economically.
  - Why should small frictions in exchange have large effects?
    => Theories why money is important. Most popular: Sticky prices.
• **Motives for holding money:**

   - Implicitly an imperfect information story: Impossible to verify credit at the points of purchase. Need government-issued money to certify purchasing power.
   - Value of consumption purchases limited by the money stock: \( p_t c_t \leq M_t \)
   - More elaborate versions: Constraint over subset of commodities; constraint over investment, etc.

2. Money in the utility function
   \[
   u = u(c_t, M_t / p_t) = u(c_t, m_t)
   \]
   - Shortcut for “money is useful.” Classic Paper: Sidrauski (1967)

3. Transactions cost models: Money saves time
   • Assume \( c_t \leq H(M_t / p_t, s_t) \), where \( s_t \) = shopping time
   \[
   \iff s_t \geq G(M_t / p_t, c_t), \text{ where } G(m, H(m, s)) = s
   \]
   • Time constraint: shopping reduces leisure time: \( n_t + s_t = 1 - l_t \)
   • Write \( u = u(c_t, l_t) = u(c_t, 1 - n_t - G(m_t, c_t)) = \tilde{u}(c_t, n_t, m_t) \Rightarrow \text{like money-in utility.} \)

4. Search models: Money is preferred to barter exchange (e.g., Lagos & Wright 2005)
   - Idea: Model with decentralized trading, many differentiated goods. Each agent produces a subset, must buy all others. Individuals meet randomly. Barter requires “double coincidence of wants.” Holding money allows purchases from sellers who don’t want the buyer’s goods.
   => Money facilitates trade. Technical analysis complicated due to multiple goods & randomness of search.
The Sidrauski Model

- Example of a continuous-time model: Application of Optimal Control.
- Preferences over money and consumption:  \[ \max_{t=0}^{\infty} \int e^{-\rho t} u(c(t),m(t))dt \]

where \( m = M/p \) = real money balances.

- Budget constraint: Agents hold money & claims on capital

\[ c + \frac{dk}{dt} + \frac{1}{p} \frac{dM}{dt} = w + r \cdot k + \chi \]

where \( \chi(t) = \) series of lump-sum transfers from the government

- Differentiate \( m=M/p \):

\[ \frac{dm}{dt} = \frac{1}{p} \frac{dM}{dt} - m \frac{dp}{dt} = \frac{1}{p} \frac{dM}{dt} - m \cdot \pi \]

where \( \pi(t) = \frac{1}{p} \frac{dp}{dt} \) = inflation rate

\[ \Rightarrow \quad c + \frac{dk}{dt} + \frac{dm}{dt} = w + r \cdot k - \pi \cdot m + \chi \]

- Define total wealth \( a = k + m \):

\[ \Rightarrow \quad c + \frac{da}{dt} = w + r \cdot a - (r + \pi) \cdot m + \chi \]

- Insight: Opportunity cost of money = Nominal interest rate
- Transversality condition: \( \lim_{T \to \infty} a(T) \cdot e^{\int_0^T r(\nu) d\nu} = 0 \)
Hamiltonian:
\[
H(c,m,a,\lambda,t) = e^{-\rho t} u(c,m) + \lambda \cdot [w - c + r \cdot a - (r + \pi) \cdot m + \chi]
\]

Maximum principle:

1. FOC for choice variables (c,m):
\[
e^{-\rho t} u_c(c,m) = \lambda \quad \text{and} \quad e^{-\rho t} u_m(c,m) = \lambda \cdot (r + \pi) \implies \frac{u_m(c,m)}{u_c(c,m)} = r + \pi
\]

*Marginal rate of substitution = Nominal interest rate*

2. Derivative with respect to the state variable:
\[
\frac{\partial \lambda}{\partial t} = -\frac{\partial H}{\partial a} = -r \cdot \lambda \quad \implies \quad \frac{1}{\lambda} \frac{\partial \lambda}{\partial t} = -r
\]

- Define \( \hat{\lambda} = \lambda \cdot e^{\rho t} \): Then \( u_c(c,m) = \hat{\lambda} \) and
\[
\frac{1}{\hat{\lambda}} \frac{\partial \hat{\lambda}}{\partial t} = \frac{1}{\lambda} \frac{\partial \lambda}{\partial t} \cdot e^{\rho t} + \rho \frac{\hat{\lambda}}{\lambda} \cdot e^{\rho t} = \frac{1}{\lambda} \frac{\partial \lambda}{\partial t} + \rho
\]
\[
\implies \quad \frac{\partial \hat{\lambda}}{\partial t} = -\hat{\lambda} \cdot (r - \rho)
\]

- Insight: Steady state has constant (c,m) => constant \( \hat{\lambda}^* = u_c(c^*,m^*) \) \( \Rightarrow r^* = \rho \)

*Real interest rate = Rate of time preference*  
[Model without growth, otherwise adjust]
• Equilibrium:
  - Interest and wages derived from production: \( f'(k) - \delta = r, \ f(k) - k \cdot f'(k) = w \)
  - Government transfer = seignorage: \( \chi = \frac{1}{p} \frac{dM}{dt} \). Taken as given by individuals.

Define: Seignorage = government revenue from issuing money

• Monetary policy is defined by the growth rate of money supply: \( \sigma = \frac{1}{M} \frac{dM}{dt} \)
  - Note that \( \chi = \frac{1}{p} \frac{dM}{dt} = \frac{M}{p} \sigma = m\sigma \)

• Steady state conditions with constant money growth:
  1. Constant capital-labor ratio \( k^* \): Implies \( r^* = \rho = f'(k^*) - \delta \)
      
      **The real interest rate does not depend on monetary factors.**

  2. Constant real money holdings \( m \): Otherwise MRS between \( c \) and \( m \) would vary, so \( c \) would vary.
      \[
      \frac{1}{m} \frac{dm}{dt} = \frac{1}{M} \frac{dM}{dt} - \frac{1}{P} \frac{dP}{dt} = \sigma - \pi. \text{ So } \frac{dm}{dt} = 0 \Rightarrow \pi^* = \sigma^*
      \]
      
      **Money growth rate determines the inflation rate.**

  3. Budget constraint: \( c = w + r \cdot a - (r + \pi) \cdot m + \chi = w + r \cdot (a - m) - \pi \cdot m + \sigma \cdot m \)
     In steady state: \( c^* = w^* + r^* \cdot k^* - \pi^* \cdot m + \sigma^* \cdot m = w^* + r^* \cdot k^* \)
• Definitions:

**Neutrality** = Changes in M have no real effects

**Super-neutrality** = Neither changes in M nor changes in $\sigma$ have real effects

• Result: In the Sidrauski model, money is super-neutral in steady state.

• Welfare effects of money growth: $\sigma \uparrow \Rightarrow m^* \downarrow \Rightarrow u \downarrow$

  $$[\sigma \uparrow \Rightarrow m^* \downarrow \text{ from } \frac{u_m(c^* m^*)}{u_c(c^* m^*)} = r^* + \sigma. \; u \downarrow \text{ because } u = u(c,m) \text{ is increasing in } m]\$$

  $\Rightarrow$ Optimal policy is to set $\sigma = -r^* \Rightarrow r^* + \pi^* = 0$

• Friedman’s Optimal Quantity of Money (“The Friedman rule”):

  Optimal supply policy means making money available at a zero opportunity cost. $\Rightarrow$ Nominal interest rate = zero. Argument: Optimal to equate private and social cost. Printing money is essentially costless.

• Result: In the Sidrauski model, the Friedman rule is optimal.

  - Note that the Friedman rule requires (a) deflation; (b) negative seignorage, which is implicitly financed by lump-sum taxes.
Learning Objectives on Money

• Conceptual:
  - Know about alternative ways of introducing money into macro models.
  - Know why money “does not matter much” unless there are additional frictions.
  - Know Friedman’s optimal quantity of money rule.

• Applied: The Sidrauski model as example of optimal control.