The Solow Model: Agenda and Objectives

- Basics: Model assumptions & interpretation.
  - Key concepts: Steady state and balanced growth.
- Application #1: Derive the steady state and graph the balanced growth path.
  (a) General argument. (b) Graphical analysis. (c) Functional form example = Cobb-Douglas.
- Application #2: Determine the impact of changes in model parameters.
  (a) Graphical analysis. (b) General comparative statics. (c) Cobb-Douglas case.
- Application #3: Dynamic properties: Rate of convergence. General and Cobb-Douglas cases.

Practical applications: Growth accounting and growth projections. Implications for cross-country convergence: conditional vs. unconditional Implications for open economies: capital flows.

Main learning objective: Ability to do applications.

- Technical skill: Differential equations – because growth relates initial positions to changes.
  
  Problems sets for practice.

- Don’t memorize formulas – except for basic setup and certain “concise” results.
Context: Growth Theory

• Foundation: The Solow Model (Romer ch.1)
  - Basic version: Mechanics of production, saving, and capital accumulation.
  - Take technological progress as given. Take saving rate as given.
  - Extended versions: Add human capital; consider natural resources.

• Endogenous technical progress => New Growth theory
  - As applied to “leading edge” countries like US: Focus on innovation, incentives for research and development; or endogenous human capital. Romer ch.3.1-4; Jones ch.4-5.
  - As applied to less developed countries: Focus on technology transfer, barriers to growth.
  - All based on the Solow model. Here introduction only.

• Endogenous saving => Optimal Growth models (Ramsey – Romer ch.2).
  - Individuals maximize utility. Supply-side foundation: Solow model.
  - Optimal Growth = Framework for virtually all macro analysis => worth studying.
Solow Model: Main Equations

- Stripped-down model intended to focus on Capital Accumulation. Simplify everything else.

- **Production function**: \( Y = F(K, AL) \)
  - Positive marginal products. Concave.
  - Constant returns to scale: \( F(xK, xAL) = x \cdot F(K, AL) \) for any \( x > 0 \). Set \( x = \frac{1}{AL} \) then
    \[
    F(K, AL) = AL \cdot F(k, 1) \text{ where } k = K / (AL). \text{ Define } f(k) = F(k, 1).
    \]
  - Inada Conditions: \( F(0, 1) = 0, F_k(k, 1) \xrightarrow{k \to 0} \infty, F_k(k, 1) \xrightarrow{k \to \infty} 0 \)

- **Capital accumulation** (in continuous time): \( \frac{dK}{dt} = \dot{K} = I - \delta \cdot K \)
- **Constant saving rate** (= investment rate) \( I = S = sY \)
- **Exogenous labor force** (growth rate \( n \)): \( \frac{dL}{dt} = \dot{L} = n \cdot L \)
- **Exogenous technical progress** (growth rate \( g \)): \( \frac{dA}{dt} = \dot{A} = g \cdot A \)

- No government: interpret consumption & investment as including private & public.
- No international linkages: interpret as world model or as closed economy.
Time: Discrete or Continuous

• **Discrete-time:** Collect time into discrete periods (e.g., years). [Optional: Acemoglu ch.2.4]

\[
Y_t = F(K_t, A_t L_t)
\]
Production per period (Real$)

\[
K_{t+1} = K_t - \delta \cdot K_t + I_t
\]
Capital stock next period (Real$)

\[
I_t = s \cdot Y_t; \quad Y_t - I_t = \text{consumption.}
\]
Use of output per period (Real$)

• **Subdivide time into arbitrary time intervals** $\Delta t$ (fraction/multiple of the base period).

- **Dynamics:**
  \[
  K(t + \Delta t) = K(t) - \delta K(t) \Delta t + I(t) \Delta t
  \]
  
  - Implicit assumption: Investment becomes productive after time interval $\Delta t$.
  
  - Transform:
    \[
    \frac{K(t+\Delta t)-K(t)}{\Delta t} = I(t) - \delta K(t).
    \]

  - **Take the limit** $\Delta t \to 0$:
    \[
    \frac{dK(t)}{dt} = I(t) - \delta K(t).
    \]
Net investment per time unit.

Distinguish stocks vs, flows: Stocks = Real$. Flows = Real$/time (often expressed as annual).


- Examples of stock variables: Capital $K$, labor force $L$, technical knowledge $A$.

- Production function relates *stocks* of factors to a *flow* of output.

• **Choice of time unit is a matter of analytical convenience** (if conversions are done correctly).

  - Typical: Growth – continuous; fluctuations – discrete. $\Delta t \sim 30$ years in OG models.
Key Technique: Transformation to Stationary Variables

- Exponential growth in productivity: if \( A(0) = A_0 \) then \( A(t) = A_0 \cdot e^{gt} \) is exponential.

  => Aggregate economy keeps growing: Difficult to graph and to capture analytically.

- Claims (which we will have to prove):
  1. Economic activity measured in effective units of labor \( AL \) (“efficiency units”) will eventually stabilize, regardless of where the economy starts.

  2. The capital-labor ratio \( k = K/(AL) \) determines all other model variables
     (a.k.a. capital in effective units, capital-labor ratio in effective units).

  => Convenient to describe the economy in effective units, focusing on the capital-labor ratio.

    - Notation: Small letters = effective units. Capital letters = aggregate units (except C=per capita)
    - Find: Growing variables can be written as product of efficiency units times a common growth trend.

- Key concepts:
  - **Steady state**: Position such that variables remain constant. (Here: effective units.)
  - **Balanced growth path**: Trajectory along which variables grow at the same rate.

    Variables on a balanced growth path = Steady state values * Exogenous growth trend.
The dynamics of the capital-labor ratio

• Claim: The capital-labor ratio follows a first-order differential equation
  - Exploit constant returns to scale: \( Y = F(K, AL) = AL \cdot F(k, 1) \Rightarrow y = F(k, 1) = f(k) \)
  - Differentiate \( k = K / AL \) with respect to time, using product & ratio rules:
    \[
    \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{\dot{k}}{k} - g - n
    \]
  - Note that \( \dot{K} = sY - \delta K \Rightarrow \frac{\dot{K}}{K} = sY / K - \delta = sy / k - \delta \)
  - Combine: \( \frac{\dot{k}}{k} = s \frac{y}{k} - (\delta + g + n) = s \frac{f(k)}{k} - (\delta + g + n) \)
• Result:
  \[
  \frac{dk}{dt} = sf(k) - (\delta + g + n)k
  \]
  Note: Equation does not depend on other endogenous variables.

• Steady state condition: \( sf(k^*) = (\delta + g + n)k^* \)

  *Equation for \( \frac{dk}{dt} \) and steady state condition are worth memorizing!*

• Graphical analysis:
  - Solow diagram: \( sf(k) \leftrightarrow (\delta + g + n)k \) [Romer Fig. 1.2]
  - Phase diagram: \( \frac{dk}{dt} = sf(k) - (\delta + g + n)k \) [Romer Fig. 1.3]

  Both diagrams illustrate convergence to \( k^* \). Ignore \( k(0) = 0 \). Then \( k = 0 \) for all \( t \).

• Role of Inada conditions: ensure convergence to a unique steady states value \( k^* \)
  from any starting value \( k(0) > 0 \).
Implications for other variables

- Steady state for $k^*$ implies steady state values for output, consumption…
- Balanced growth for per-capita and aggregate variables…
- Convergence of $k(t)$ to $k^*$ implies convergence of all other variables to the balanced growth path

Qualitative Experiments

- Increase in the saving rate [Romer Fig.1.4-1.5]
- Increase in the depreciation rate …
- Increase in the population growth rate …
- Increase in the rate of technical progress … [left as exercises]
FIGURE 1.4 The effects of an increase in the saving rate on investment
Higher Saving: Implied time series

- Conclusions:
  - A permanently higher saving rate changes the level of output, but unchanged long run growth rate.
  - Growth is exogenous in the long run. Measured growth varies in the transition to a new steady state.
Comparative statics: Increase in the Saving Rate

- Example of how to examine marginal changes. General effect: Shift to a new steady state.

- **General technique: Take the total differential** of the steady state conditions. Start with \( k^\ast \).

  - Condition: \( sf(k^\ast) = (\delta + g + n)k^\ast \) (Convenient because it links \( s \) and \( k^\ast \)).

  \[
  f(k^\ast)ds + sf'(k^\ast)dk^\ast = (\delta + g + n)dk^\ast \Rightarrow \frac{dk^\ast}{ds} = \frac{f(k^\ast)}{(\delta + g + n) - sf'(k^\ast)} > 0
  \]

- Implications for other variables follow from \( dk^\ast \):

  - Output: \( y^\ast = f(k^\ast) \) implies \( \frac{dy^\ast}{ds} = f'(k^\ast) \frac{dk^\ast}{ds} = \frac{f(k^\ast) \cdot f'(k^\ast)}{(\delta + g + n) - sf'(k^\ast)} > 0 \)

  - Consumption: \( c^\ast = f(k^\ast) - (\delta + g + n)k^\ast \) implies \( \frac{dc^\ast}{ds} = [f'(k^\ast) - (\delta + g + n)] \frac{dk^\ast}{ds} \)

- **Interpretation: What do these derivatives mean? Conceptually and/or quantitatively?**
  - Answers generally depend on the context. (Be clever and use what you know!)
  - Here: Output and consumption effects do have substantive interpretations. Follow Romer.
Interpreting the Output Effect

• How big is the impact of saving on output?

- Define: capital share in output \( \alpha_k (k) \equiv f'(k) k / f(k) \). Depends on \( k \) except for Cobb-Douglas.

- Note that \( sf(k^*) = (\delta + g + n)k^* \Rightarrow \delta + g + n = s \cdot \frac{f(k^*)}{k^*} \)

\[
\Rightarrow \delta + g + n - s \cdot f'(k^*) = s \cdot \frac{f(k^*)}{k^*} - s \cdot \alpha_k(k) \frac{f(k)}{k} = s \frac{f(k)}{k} \cdot (1 - \alpha_k(k)) > 0
\]

- Write \( \frac{dy^*}{ds} = \frac{f(k^*) f'(k^*)}{(\delta + g + n - sf(k^*))} = \frac{f(k^*)}{s} \cdot \frac{\alpha_k f(k^*) / k^*}{f(k^*) / k^* [1 - \alpha_k(k^*)]} = \frac{y^*}{s} \cdot \frac{\alpha_k(k^*)}{1 - \alpha_k(k^*)} \)

Interpret \( \frac{dy^*}{y^*} = d\ln(y^*) \) as the percentage change in output.

- Find: \( \frac{d\ln(y^*)}{ds} = \frac{1}{s} \cdot \frac{\alpha_k(k^*)}{1 - \alpha_k(k^*)} \). Applied to U.S.: \( \alpha_k \sim 1/3, \ s \sim 1/6, \ \frac{d\ln(y^*)}{ds} = 3 \) (Example of Calibration.)

• Substantive Answer: Saving one percentage point more will raise output by about 3% in the long run.

• Optional exercise: Show that the same 1-percentage change will raise the capital stock by about 9%.

[Hint: Use the same steps for \( dk^*/ds \). Show that \( \frac{dy^*}{y^*} = \alpha_k(k^*) \frac{dk^*}{k^*} \)]
Interpreting the Consumption Effect

• How big is the impact of saving on consumption? Is it positive or negative?
  - Intuition: In the short run, more saving reduce consumption one-for-one – output is given
  
  => A positive long-run effect requires a sufficiently strong increase in output.

  - Write \( \frac{dc^*}{ds} = \left[ f'(k^*) - (\delta + g + n) \right] \frac{dk^*}{ds} = \left[ \alpha_k(k^*) - s \right] \frac{f(k^*)}{k^*} \frac{dk^*}{ds} \)

• Both versions have an interpretation:
  
  1. Sign depends on how \( k^* \) compares to \( (f')^{-1}(\delta + g + n) \). Negative above, positive below.
  2. Sign depends on \( \alpha_k(k^*) - s \): Positive sign for \( s < \alpha_k(k^*) \), negative for \( s > \alpha_k(k^*) \), zero for \( s = \alpha_k(k^*) \).

• Substantive Answer: Long-run consumption is maximized at the saving rate \( s = \alpha_k(k^*) \). [Fact to remember]

• Define the Golden Rule capital stock by \( k^* = (f')^{-1}(\delta + g + n) \).

  - Call an economy dynamically inefficient if its capital stock exceeds the Golden Rule.

  - Inefficiency in the Pareto sense: Consumption could be increased without reducing consumption later.

  - All saving rates \( s \leq \alpha_k(k^*) \) are dynamically efficient: Tradeoff between current and future generations.

• Applied to the U.S.: \( s \sim 1/6 \) is far below \( \alpha_k \sim 1/3 \) => U.S. is dynamically efficient.

  Tradeoff: Increasing \( s \) by 1% would reduce \( c(0) \) by 1.2\% = 1/(1-s) and raise \( c^* \) by 1.5\%.
Graphical Illustration of the Golden Rule Capital Stock

(k* here = k_H* in Romer’s graphs)
Quantitative Application: Cobb-Douglas Production

- Empirical observation: Capital and labor shares are roughly constant over time (no systematic variations)
  - Suggest that output has constant elasticities with respect to K and L
  - Claim: Production function must be a power function:
    \[ Y = F(K, AL) = K^\alpha (AL)^{1-\alpha} \quad \text{or} \quad y = f(k) = k^\alpha \]
  - Where \( \alpha \) is the capital share in output (empirically: about 1/3). Called Cobb-Douglas.

- Confirm that Cobb-Douglas implies constant factor shares:
  \[ MPK = \frac{dY}{dK} = \alpha \cdot K^{\alpha - 1} \cdot (AL)^{1-\alpha} = \alpha \cdot Y / K \text{ is proportional to } Y/K \Rightarrow MPK \cdot K / Y = \alpha \]
  Similarly: \[ MPL = \frac{dY}{dL} = (1 - \alpha) \cdot Y / L \Rightarrow MPL \cdot L / Y = 1 - \alpha \]

- Proof (not on the exam) that constant factor share require Cobb-Douglas. Intuition: Key feature is log-linearity.
  - Define \( \kappa = \ln(k) \) and \( \ln(y) = \ln(f(e^\kappa)) = z(\kappa) \) for any arbitrary production function \( f \).
  - In general \( z'(\kappa) = \frac{1}{f(e^\kappa)} f'(e^\kappa) e^\kappa = \frac{k f'(k)}{f(k)} \) is the factor share of capital.
  - The capital share is constant if and only if \( z'(\kappa) = \alpha \Rightarrow z(\kappa) = z_0 + \alpha \cdot \kappa \) with integration constant \( z_0 \).
    \[ \Rightarrow \ln(y) = z_0 + \alpha \cdot \ln(k) \Rightarrow y = e^{z_0} k^\alpha. \] Normalize \( z_0 = 0 \) to simplify notation.
Cobb-Douglas Case: Explicit Solutions for Steady States and Transition Paths

- **Steady states:** Capital-labor ratio is again the key variable:

\[ \dot{k} = 0 \iff 0 = s \cdot k^{\alpha - 1} - (\delta + g + n) \iff k = k^* = \left( \frac{s}{\delta + g + n} \right)^{1/1-\alpha} \]

- Implied solutions for other variables. For example:

\[ y^* = \left( k^* \right)^\alpha \left( \frac{s}{\delta + g + n} \right)^{\alpha / (1-\alpha)} \quad \text{and} \quad c^* = (1-s) \cdot y^* = (1-s) \left( \frac{s}{\delta + g + n} \right)^{\alpha / (1-\alpha)} \]

- Formulas provide quantitative predictions when parameter change (not only marginal)

- Exercises (recommended): Confirm that c* is maximized at s=α. Confirm that k*- Golden matches s=α.

- **Transition paths:** One can find closed form solutions. (Not possible in general.)

  - Claim:

\[ k(t) = \left[ \left( k^* \right)^{1-\alpha} + \left( (k(0))^{1-\alpha} - (k^*)^{1-\alpha} \right) \cdot e^{-\lambda t} \right]^{1/1-\alpha} \]

  where \( \lambda = (\delta + g + n)(1-\alpha) \) \ [Later interpret \( \lambda \) as rate of convergence]

  - Note: As t goes from 0 to infinity, k(t) moves from k(0) to k*; \( \lambda \) determines the weight on k(0) vs. k*

  - Formula can be evaluated for various times after a parameter change to map out dynamic responses.

    [Example: Raise s in an efficient economy: How long does it take for consumption to turn positive?]
Convergence Dynamics in the Cobb-Douglas case
(Proof that \( k(t) \) has the solution claimed above)

- Starting point: Basic differential equation for capital
  \[
dk / dt = sf(k) - (\delta + g + n)k = sk^\alpha - (\delta + g + n)k
  \]

- Main trick: Consider \( z = k^{1-\alpha} = k / y \) = the capital-output ratio. Note that
  \[
dz / dt = (1-\alpha)k^{-\alpha} \cdot dk / dt = [(1-\alpha)k^{-\alpha}] \cdot [sk^\alpha - (\delta + g + n)k]
  \]
  \[
  = (1-\alpha)s - (1-\alpha)(\delta + g + n)k^{1-\alpha} = (1-\alpha)s - \lambda z
  \]
  with \( \lambda = (\delta + g + n)(1-\alpha) > 0 \). This is a linear differential equation in \( z \), constant coefficients.

- Linear differential equations have closed-form solutions:
  - Math fact: \( dz / dt = -a \cdot z + b \) with constants (a,b) has solution \( z(t) = \frac{b}{a} + (z(0) - \frac{b}{a}) \cdot e^{-at} \).
  - Apply: \( z(t) = z^* + (z(0) - z^*) \cdot e^{-\lambda t} \) with \( z^* = \frac{(1-\alpha)s}{\lambda} = \frac{s}{\delta + g + n} \)
  - Note: \( z^* \)-formula matches \( k^* = \left(\frac{s}{\delta + g + n}\right)^{1/(1-\alpha)} \).

- Infer dynamics of capital: \( k(t) = z(t)^{1-\alpha} = [(k^*)^{1-\alpha} + \{(k(0))^{1-\alpha} - (k^*)^{1-\alpha}\} \cdot e^{-\lambda t}]^{1/(1-\alpha)} \). QED.

  - Applied: Initialize \( k(0) = y(0)^{\frac{1}{\alpha}} = \left(\frac{Y(0)}{L(0)A(0)}\right)^{\frac{1}{\alpha}} \) from data on output & labor, normalize A(0)=1.

  - Similar dynamics for output: \( y(t) = z(t)^{\frac{\alpha}{1-\alpha}} \) and consumption.
Convergence Dynamics in general

• **Main claim:** Convergence is slow. Disturbances have “half-lifes” on the order of decades.

• **Rate of convergence** = Ratio of $dk/dt$ to the distance ($k^*-k$) from steady state.

\[
\lambda(k) = \frac{dk/dt}{(k^*-k)} \quad \text{for } k \neq k^* \quad \text{[More general def. than in Romer]}
\]

- Answer to “What fraction of the distance to $k^*$ is eliminated per time unit?”

• **General relation:**

\[
dk/dt = sf(k) - (\delta + g + n)k
\]

- Compute $\lambda(k)$ in a neighborhood of $k^*$ by taking a linear approximation at $k^*$:

\[
dk/dt \approx [sf'(k^*) - (\delta + g + n)](k - k^*) \Rightarrow \lambda = \delta + g + n - sf'(k^*)
\]

- Recall: $sf'(k^*) = \alpha_k(k^*)(\delta + g + n) \Rightarrow \lambda = (1 - \alpha_k(k^*))(\delta + g + n)$

- Cobb-Douglas case: $\lambda = (1 - \alpha)(\delta + g + n)$ is the rate of convergence. [Fact to remember]

• **Reasonable calibration:** $\delta + g + n \approx 6\%$ per year; $\alpha = 1/3$. Implies $\lambda \approx 4\%$

  => Slow: After one year 96% of the gap remains. Takes about 17 years to eliminate 50%.

• **What about $\lambda(k)$ further away from $k^*$?** Consider a quadratic approximation:

\[
dk/dt \approx -\lambda(k - k^*) + \frac{sf''(k^*)}{2}(k - k^*)^2 \Rightarrow \lambda(k) \approx \lambda + \frac{-sf''(k^*)}{2}(k - k^*)
\]

- Because $f''<0$: $\lambda(k)<\lambda$ for $k<k^*$ and $\lambda(k)>,\lambda$ for $k>k^*$. Rate of convergence is not constant.
Analysis when time is discrete

(1) \[ Y_t = F(K_t, A_t L_t) \] Production per period (Real$)

(2) \[ K_{t+1} = K_t - \delta \cdot K_t + I_t \] Capital stock next period (Real$)

(3) \[ I_t = s \cdot Y_t; \ Y_t - I_t = \text{consumption.} \] Use of output per period (Real$)

• Define \( k_t = K_t / (A_t L_t) \) and \( y_t = f(k_t) = F(k_t, 1) \). Then \( K_{t+1} = (1 - \delta) \cdot K_t + s \cdot Y_t \)

\[ \Rightarrow K_{t+1} / A_t L_t = s \cdot Y_t / A_t L_t + (1 - \delta) \cdot K_t / A_t L_t \quad \text{where} \quad K_{t+1} / A_t L_t = K_{t+1} / A_{t+1} L_{t+1} \cdot A_{t+1} L_{t+1} / A_t L_t = k_{t+1} \cdot (1 + g)(1 + n) \]

\[ \Rightarrow \text{Dynamics of capital:} \quad k_{t+1} \cdot (1 + g)(1 + n) = s \cdot f(k_t) + (1 - \delta) \cdot k_t. \]

1. Difference equation for capital stocks:

\[ k_{t+1} = \frac{(1 - \delta) k_t + s f(k_t)}{(1 + g)(1 + n)} \]

Graphical analysis: plot \( k_{t+1} \) vs. \( k_t \), compare to \( 45^\circ \)-line.

2. Equation for change as function of level:

\[ \Delta k_{t+1} = \frac{s f(k_t) - ((1 + g)(1 + n) - (1 - \delta)) k_t}{(1 + g)(1 + n)} \]

\[ \Rightarrow \text{Steady state condition:} \quad \Delta k_{t+1} = 0 \iff s \cdot f(k^*) = ((1 + g)(1 + n) - (1 - \delta)) \cdot k_t, \text{ so} \]

\[ s \cdot f(k^*) = (\delta + g + n + gn) \cdot k^* \quad \text{[Product term} \ gn \text{is a nuisance -- normally small.]} \]

• Lessons:

1. New insight: express dynamics as difference equation for capital stocks.

2. Discrete-time analysis is similar to continuous time; less convenient for compound growth.
Learning Objectives

• **Applied: Ability to solve problems**
  - Compute steady states and dynamics; general and parametric cases; exact and linearized solutions. *Problem sets for practice.*

• **Conceptual:**
  - Know the model’s main assumptions, theoretical properties, and empirical implications.
  - Know and use key concepts, such as steady state, balanced growth, convergence.

• **Technical skills:**
  - Taking total differentials to obtain linear approximations to a model.