In previous years, students have often asked me about practice problems in addition to the problem sets. Here is a collection. Some will be assigned for the weekly problem sets. I hope the others are useful for practice.

**Request**: Please tell me about errors or ambiguities. Many of the questions below are old exam problems. As I am updating the course over the years, the notation, references, and sequencing has changed, which means that some old problems may have lost educational value without me noticing immediately. So if a problem seems obscure to you, please let me know. (Your incentive: problem sets might shrink if you convince me that a problem is unclear.)

**Part 1:**

1.1. Consider the two-period consumption model: Individuals have initial assets $A$, earn interest $r$ on assets, and earn wage income $(w_1, w_2)$. They maximize utility $U = u(c_1) + \beta u(c_2)$.

a. Assume $u(c) = \ln(c)$.

i. Solve for optimal consumption and period-1 asset holdings as functions of wage income, the interest rate, and the time-discount factor $\beta$. Discuss under what conditions a marginally higher interest rate reduces consumption. [Discuss means: Interpret the solution. Conditions may be exact, or necessary, or sufficient. Hint: Distinguish cases with $w_2 = 0$ vs. $w_2 > 0$.]

ii. Show that the dependence of period-1 consumption on $(w_1, w_2)$ can be expressed in terms of permanent income.

b. Assume $u(c) = 1_{1-\gamma} \frac{1}{1-\gamma} c^{1-\gamma}$ where $\gamma > 0, \gamma \neq 1$.

Do the same as in (a). In the discussion, identify which results apply for all $\gamma$, and which ones only for $\gamma$ greater or less than one.
1.2. Economists sometimes use the marginal propensity to consume (MPC) to express the effects of income changes on consumption: MPC is defined as the ratio $\Delta c_t/\Delta w_t$ of a change $\Delta c_t$ in consumption triggered by some change $\Delta w_t$ in the current wage income that may or may not be accompanied by changes in future wages. This question will ask you to compute the MPC for several scenarios.

(Hint: Use a spreadsheet for calculation.)

Assume the permanent income model holds with planning horizon of n periods. Assume $\beta = 1/(1+r)$ with $r=3\%$ per year. Assume zero initial assets ($A=0$).

a. Assume the time horizon is $n=50$ years (interpretation: roughly the life expectancy at age 30).

Determine the MPC from
   i. a one-year wage increase;
   ii. an increase in wages that lasts 5 years;
   iii. a wage increase that last for 35 years (intuition: until about retirement);
   iv. a tax reduction for one year followed by a tax increase of the same size in the next year.

Discuss: How do the results compare across cases? Do you find them surprising? Why or why not? What is the economic intuition?

b. Determine the impact of a one-year wage increase for consumers with alternative planning horizons of, respectively: $n=1$ year; $n=2$ years; $n=10$ years; $n=50$ years; the limiting case of $n=\text{infinity}$.

Discuss: How do the results compare across cases? Do you find them surprising? Why or why not? What is the economic intuition?

1.3. [Midterm 2008] Consider an individual with utility function

$$U = \ln(c_1) + \beta \ln(c_2) + \beta^2 \ln(c_3).$$

The person has labor incomes $w_1$, $w_2$, $w_3$, in periods 1-3, respectively. Let

$$a_t = (1+r)a_{t-1} + w_t - c_t$$

denote assets carried into the next period. The interest rate $r$ is constant. Assets $a_0>0$ are given.

a. Specify the budget equations, derive the intertemporal budget constraint, and derive first order conditions for optimality. Be explicit about any terminal conditions that may be required, and make sure the number of optimality conditions and constraints matches the number of variables.

b. Suppose $\beta = 1/(1+r)$, $w_t=w$ is constant for all $t$. Solve for period-1 consumption as function of the labor incomes and of initial assets. Define the individual’s permanent income. Determine the marginal propensity to consume (MPC) in period 1 from period-1 labor income; compute the MPC value for $\beta = 0.9$.

c. Again assuming $\beta = 1/(1+r)<1$ and constant $w$, show that assets $a_t$ are a declining sequence (that is: $a_3<a_2<a_1<a_0$) and that $a_t/a_{t-1}$ is also decreasing over time. Can you explain why this makes sense
economically? Use the same economic argument to make a conjecture about the behavior of \( a_t/a_{t-1} \) in problems with more than \( T=3 \) periods, notably for \( T \to \infty \).

1.4. [Midterm 2009] Consider an individual with utility function \( U = u(c_1) + \beta u(c_2) \) over consumption, where \( 0 < \beta < 1 \) and \( u(\cdot) \) is increasing and concave. Labor incomes \( w_1 \) and \( w_2 \) are given. Initial assets are zero. The individual can save or borrow between periods 1 and 2 at a given interest rate \( r \).

a. Specify the budget equations, derive the intertemporal budget constraint, and derive first order conditions for optimal consumption.

b. Suppose \( u(c) = \frac{1}{1-\theta} c^{1-\theta} \) is a power function with parameter \( 0 < \theta \neq 1 \). Derive equations for \( c_1 \) and \( c_2 \) as functions of the preference parameters and the exogenous variables.

c. Define the elasticity of \( c_i \) with respect to \( (1+r) \) by \( \epsilon_i = \frac{dc_i}{d(1+r)} \cdot \frac{1}{c_i} \) (i=1,2). Show that

\[
\epsilon_1 = \frac{w_1 - c_1 - \frac{w_2}{1+r} c_1/(1+r)}{w_1 + w_2/(1+r)} \quad \text{and} \quad \epsilon_2 = \epsilon_1 + \frac{1}{\theta}.
\]

Explain in economic terms why for small \( \theta \), \( \epsilon_1 < 0 < \epsilon_2 \), whereas for large \( \theta \) both elasticities are positive for savers and negative for borrowers.

[Hint: If you have trouble deriving the formulas, take them as given and focus on the interpretation.]
Part 2:

2.1. Suppose an economy has a production function \( y_t = k_t^\alpha \) (in efficiency units), a savings rate \( s > 0 \), a population growth rate \( n \), and a depreciation rate of \( \delta \).

a. Suppose \( \alpha = 1/3 \), \( s = 0.2 \), \( n = 1\% \), \( g = 1\% \), \( \delta = 4\% \). What are the steady state value of the capital-labor ratio, output per efficiency unit, and consumption per efficiency unit?

For parts b-e, assume the economy starts in the steady state derived in (a).

b. Suppose an earthquake destroys 10% of the capital stock. Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

[To clarify: Per-capita means per actual worker, not in efficiency units.]

c. Suppose savings are increased to \( s = 0.22 \). What is the impact effect on consumption? What are the new steady state values of the capital-labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

d. Suppose population growth is increased to \( n = 2\% \). What are the new steady state values of the capital labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

e. Suppose productivity growth is increased to \( g = 2\% \). What are the new steady state values of the capital labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

2.2. Suppose an economy has a production function \( y_t = 3 \cdot k_t^{0.5} \) and a savings rate of 30%, a population growth rate of 5%, and a depreciation rate of 10%. Productivity is constant.

a. What are the steady state value of the capital-labor ratio, output per worker, and consumption per worker?

b. How do the values in (a) change if the savings rate is 40%?

c. How do the values in (a) change with 8% population growth (still 30% savings)?

2.3. This question is about economic growth with exogenous savings rate (s); notation as in Romer unless noted. Assume production is Cobb-Douglas with capital share \( 0 < \alpha < 1 \) and depreciation \( \delta \). Population L grows at rate \( n \). Assume total factor productivity \( A \) grows at an exogenous rate \( g \). For (b) and (c), assume the economy at \( t=0 \) is on a balanced growth path.

a. Derive the steady state capital stock per efficiency unit of labor. Derive steady state output per efficiency unit. Derive a formula for per-capita output along the steady state growth path. [Derive means: Show your work; no credit for memorized formulas.]

b. Suppose at time \( t=0 \), a genial discovery makes productivity \( A \) jump up by 100%. Productivity growth then continues at the original rate \( g \).
i. Determine how the change affects the steady state output and capital stock per efficiency unit of labor. Graph the time path.

ii. Determine how the change affects the output and capital stock per worker. Graph the time path.

c. Suppose at time \( t=0 \), the productivity growth rate increases from \( g=g_1 \) to \( g=g_2 > g_1 \). There is no jump in productivity levels.

   i. Determine how the change affects the steady state capital stock per efficiency unit of labor.
      Graph the time path.

   ii. Determine how the change affects the output and capital stock per worker. Graph the time path. Is everyone better off with higher productivity growth?

2.4. Assume production is Cobb-Douglas, 
\[
Y = K^{\alpha}(AL)^{1-\alpha},
\]
with capital share \( 0 < \alpha < 1 \) and depreciation \( \delta \). Population \( L \) grows at rate \( n \). Assume productivity growth depends on research labor \( L_A = s_R \cdot L \) and on existing productivity: \( \dot{A} = \gamma \cdot L_A \cdot A^\phi \), where \( \gamma > 0, \ 0 < \phi < 1, \) and \( 0 < s_R < 1 \) are parameters. Production labor is \( L_Y = (1-s_R) \cdot L \).

   a. Derive the steady state growth rate of productivity. Show that per-capita output grows at the same rate.

   b. Suppose at time \( t=0 \), the share of research labor is increased permanently to \( \hat{s}_R > s_R \). Graph the time paths of productivity and of per-capita output. Is everyone better off with more research labor? Explain.

2.5. This question is about a continuous-time Solow growth model with human capital (H) as additional factor of production. Assume production is Cobb-Douglas:
\[
Y = K^{\alpha}H^{\beta}(AL)^{1-\alpha-\beta}
\]
A is total factor productivity (TFP). L is raw labor. Capital K has factor share \( \alpha > 0 \), human capital H has share \( \beta > 0 \), and \( 1-\alpha-\beta > 0 \). A fixed share \( s_k > 0 \) of output is invested in physical capital, a share \( s_h > 0 \) is invested in human capital. Assume zero depreciation. Raw labor supply grows at a fixed rate \( n > 0 \). TFP grows at a fixed rate \( g > 0 \).

   a. Derive the steady state capital stock, steady state human capital stock, and steady state output, all per efficiency unit of labor.

   b. Derive the quantitative impact on steady state output per efficiency unit of labor of a marginal increase in changes (i) \( s_k \) and (ii) \( s_h \). Describe under what conditions \( s_k \) has a greater, smaller or equal output effect as \( s_h \). [Hint: You are seeking a simple condition relating savings rates to production parameters.]
c. Consider the limiting case $\alpha + \beta \rightarrow 1$. Explain why and in what sense growth becomes endogenous in the limit. Derive the economy’s growth rate for the case $\alpha + \beta = 1$ and $g=0$. What are the determinants of the growth rate?

2.6 [Midterm 2008] Consider a Solow growth model with exogenous savings rate $s$, population growth $n$, and productivity growth $g$. Output is $Y = K^\alpha (AL)^{1-\alpha}$ with $0<\alpha<1$. Capital depreciates at rate $\delta$.

a. Derive a formula for the state capital-labor ratio $k^*$. Assume the economy is in steady state. Suppose at some date $t_1$, population growth increases from $n=n_1$ to $n=n_2>n_1$. Determine how $k^*$ changes. Sketch the time path for $k$ to the new $k^*$ value.

b. Suppose $n_1=1\%$, $n_2=1.1\%$, $s=20\%$, $g=1\%$, $\delta=3\%$, and $\alpha=1/3$. Compute the percentage change in $k^*$. Rounded numbers suffice.

c. Determine how the growth rate of $Y/L$ (income per actual worker) changes at time $t_1$. That is, determine the growth rate just before and just after time $t_1$.

d. Suppose productivity growth is endogenous and linked to population growth, $g = \lambda n$ for some $\lambda>0$ (e.g. motivated by a research production function). If $n$ increases at time $t_1$, are there $\lambda$ values so that growth in $Y/L$ accelerates? Will $Y/L$ grow faster or more slowly in the long run?

e. Now suppose $Y=F(K,AL)$ is a general constant-returns-to-scale technology. Derive a formula for the percentage change in $k^*$ in response to a marginal increase in $n$ (no change in $g$). Compute the percentage change in $k^*$ for the same numbers as in (b), assuming the capital share at $k^*$ is again $1/3$. [Hint: Check if your answer is consistent with (b).]

2.7 [Midterm 2009] Consider a Solow model with general production function $Y = F(K,AL)$. Notation and assumptions about $F$ are as in Romer. The depreciation rate is $\delta>0$. Population $L$ grows at the rate $n$. Total factor productivity $A$ grows at the exogenous rate $g$.

a. Derive an equation for the steady state capital-labor ratio $k^*$ (in efficiency units).

b. Derive a formula for the change in $k^*$ as result of a marginal increase in the productivity growth rate from $g=g_1$ to a higher value $g=g_2$.

c. Suppose the increase from $g=g_1$ to $g=g_2$ takes place at time $t=0$ and is obtained by diverting a fraction $x$ of workers into research; that is, productive labor supply for $t>0$ is $(1-x)AL$, where $0<x<1$. Assume the economy was in steady state for $t<0$. Describe the time paths of $k = K / (AL(1-x))$ and of $\ln(K/L)$, in words and with a time-series chart for each series. Determine how the growth rate of $K/L$ changes at $t=0$. 
2.8 [Midterm 2009] Consider the Solow model with general production function \( Y = F(K, AL) \).

Romer’s notation applies.

a. Derive an equation for the steady state capital-labor ratio \( k^* \) (in efficiency units).

b. Derive a formula for the change in \( k^* \) as result of a marginal increase in the population growth rate \( n \). Determine the sign of \( \frac{dk^*}{dn} \). What is the approximate percentage change in \( k^* \) if \( n \) increases by 1%, the capital share is 1/3, and \( n + g + \delta = 6\% \)?

c. Starting on a balanced growth path with \( n = n_1 \), suppose population growth increases at time \( t = 0 \) to a higher value \( n = n_2 \). Show graphically how this change affects the time paths of \( k(t) \), \( \ln(K(t)) \), and \( \ln(Y(t)) \). Each graph should show clearly how the new time path compares to the path that the series would have followed without the change, how the actual path compares to the new balanced growth path (if any), and how (if) the slope changes at \( t = 0 \). Explain.

2.9 [Midterm 2009] This question is about endogenous, research-driven growth. Population \( L \) is divided in to \( L_Y = (1 - s_R) \cdot L \) and \( L_A = s_R \cdot L \), where \( 0 < s_R < 1 \) is given. Population grows at rate \( n \). Production is Cobb-Douglas, \( Y = K^\alpha (ALY)^{1-\alpha} \), with capital share \( 0 < \alpha < 1 \). Productivity depends on research labor and on existing productivity: \( \dot{A} = \gamma \cdot L_A \cdot A^\phi \), where \( \gamma > 0 \) and \( 0 < \phi < 1 \).

a. Derive the steady state productivity growth rate. Explain how it depends on the parameter \( \phi \).

b. Suppose at time \( t = 0 \), the share of research labor is increased to \( \hat{s}_R > s_R \). Explain how productivity growth changes at time \( t = 0 \) and why it returns to the steady state.

c. Suppose population is fixed (\( L \) constant, \( n = 0 \)), and suppose \( \phi = 1 \). What determines productivity growth? What happens to productivity growth if research labor is increased to \( \hat{s}_R > s_R \)?

2.10 [Midterm 2010] Consider a Solow model with general production function \( Y = F(K, AL) \).

Notation is as in Romer unless noted. Population \( L \) grows at a rate \( n \). The savings rate \( s \) is constant. The depreciation rate is \( \delta = \delta_0 > 0 \). Productivity \( A \) grows at an exogenous rate \( g = g_0 > 0 \).

a. Derive an equation for the steady state capital-labor ratio \( k^* = k_0^* \).

b. Suppose at time \( t = t_1 \), a new way of organizing research is discovered that accelerates productivity growth but at the expense of making capital obsolete more quickly. Specifically, assume the economy after the discovery has the parameters \( g = g_1 > g_0 \) and \( \delta = \delta_1 > \delta_0 \). Derive the new steady state capital-labor ratio \( k^* = k_1^* \). Compare \( k_0^* \) and \( k_1^* \).

c. Suppose \( k(t_1) = k_0^* \), so the economy is in the original steady state at time \( t = t_1 \).

d. Show that the changes described in (b) reduce the aggregate capital stock (\( K \)) relative to its previous path for some time interval, and that the changes raise \( K \) in the long run.

e. Derive a condition under which per capita consumption is increased for all \( t \), so the changes described in (b) would make individuals unambiguously better off.
2.11 [Midterm 2010] Consider a Solow model with production function \( Y = F(K, AL) = ALf(k) \).
Assume \( A=1 \) to abstract (for now) from productivity growth. Assume there is a critical per-capita income value \((Y/L)_c>0\) so that \( s=s_L \) for \( Y/L<(Y/L)_c \) and \( s=s_H \) for all \( Y/L>(Y/L)_c \), where \( s_H>s_L \). (The motivation is that low-income individuals have greater needs to consume and cannot save as much.) Define \( k_c = f^{-1}((Y/L)_c) \) as the capital stock needed to attain \( (Y/L)_c = f(k_c) \) and assume 
\[ s_H f(k_c) > (n + \delta)k_c > s_L f(k_c). \]

**a.** Show that the economy has two steady states. Explain how the economy would converge from any given initial capital stock \( K/L \).

**b.** How would the analysis change if productivity started to grow at a constant rate \( g>0 \)?

2.12 [Midterm 2011] Consider a Solow model with production function \( Y = F(K, AL) = AL \cdot f(k) \).
Notation is as in Romer unless noted. Population \( L \) grows at a rate \( n=n_0>0 \). The depreciation rate is \( \delta>0 \). Productivity \( A \) grows at an exogenous rate \( g>0 \). The savings rate equal \( s=s_0>0 \). The capital-labor ratio follows the differential equation 
\[ \frac{dk}{dt} = sf(k) - (n + g + \delta)k. \]

**a.** Consider a marginal increase in the population growth rate \( n \). Determine \( dk^*/dn \) and \( dy^*/dn \), where \( k^* \) is the steady state capital stock. Draw a Solow diagram to illustrate your findings.

**b.** Suppose the economy is initially on a balanced growth path with population growth rate \( n_0 \). At some time \( t=t_1 \), population growth increases to \( n=n_1>n_0 \). Explain how the time path of aggregate output \( Y(t) \) compares to output along the original balanced growth path. Illustrate your findings in a graph of \( \ln Y \) against time. Determine how the growth rate of aggregate output changes at time \( t_1 \).

**c.** Assume a constant capital share \( \alpha \) (that is, Cobb-Douglas production). Consider a marginal increase in the savings rate \( s \) (given \( n=n_0 \)). Show that steady state consumption \( c^* \) increases if \( s_0<\alpha \) and \( c^* \) declines if \( s_0>\alpha \).

**d.** Suppose the economy is initially in a steady state with \( s_0>\alpha \). Comment on the following argument: “We know that higher population growth implies a lower capital-labor ratio. Hence a substantial increase in population growth should make the economy dynamically efficient.” Is this true, false, or uncertain? Explain why. (Math argument and/or diagram required for full credit.)
2.13. [Midterm 2012] Consider an endogenous growth model with Cobb-Douglas production, fixed savings rate \( s>0 \), and fixed population growth rate \( n>0 \). Population is used for production and research, \( L = L_t + L_A \). Output is \( Y = K^{\alpha}(AL)^{1-\alpha} \). Research labor raises productivity \( A \) by \( \frac{dA}{dt} = \delta \cdot L_A^{\lambda} \) where \( \delta > 0 \) and \( 0 < \lambda < 1 \) are constants. Assume \( s_R = L_A/L > 0 \) is exogenous and fixed.

a. Show how steady state growth rate of productivity \( g_A^* \) depends on population growth.

b. Suppose the economy is in steady state with population growth \( n=n_0 \). Then, starting at time \( t=t_1 \), population growth increases to \( n=n_1 > n_0 \).

c. Explain how this change alters the growth rate of productivity over time.

d. Show that per-capita income declines in the short run and increases in the long run. Provide an economic interpretation.

2.14. [Midterm 2013] Consider a Solow model with Cobb-Douglas production function \( Y = K^{\alpha}(AL)^{1-\alpha} \), where \( 0 < \alpha < 1 \). Population \( L \) grows at rate \( n>0 \). Productivity \( A \) grows at an exogenous rate \( g>0 \). The savings rate is \( s>0 \). Capital depreciates at rate \( \delta>0 \).

a. Solve for steady-state consumption in efficiency units \( c^* \) and explain briefly how \( c^* \) depends on model parameters. Show that \( c^* \) as function of \( s \) is maximized at \( s=\alpha \).

b. Suppose the economy starts in a steady state with \( s=s_1>\alpha \). At some time \( t_1 \) the savings rate drops to a value \( s=s_2<\alpha \). Describe the transition paths of \( k \) and \( c \) to their new steady state. Document your answer with diagrams for \( k(t) \) and \( c(t) \) over time. Show that \( c^* \) is higher in the new steady state if \( s_2 \) is sufficiently close to \( \alpha \), whereas \( c^* \) is reduced if \( s_2 \) is sufficiently far below \( \alpha \).

2.15. [Midterm 2013] Consider a Solow growth model with standard assumptions except for government. Assume the government taxes income at a constant rate \( \tau \) and uses the revenue to finance spending \( G(t)=\tau Y(t) \). Households save a constant fraction \( s \) of their after-tax income. Productivity is constant (\( A=1 \)).

a. Assume a Cobb-Douglas production function \( Y = K^{\alpha}L^{1-\alpha} \) with \( 0 < \alpha < 1 \). Derive a differential equation for the dynamics of the capital-labor ratio \( k \). [Hint: Government has no function here except to reduce disposable income.]

b. Solve for the steady state \( k^* \) and describe how it depends on the tax rate.

c. Now assume that government spending is productive in that it augments the production technology given by \( Y = K^{\alpha}G^\beta L^{1-\alpha-\beta} \) where \( 0 < \beta < 1 \) and \( 0 < \alpha < 1 - \beta \). Derive a differential equation for the dynamics of the capital-labor ratio \( k \).
d. Solve for the steady state $k^*$ as function of the tax rate $\tau$. Suppose the government’s objective is to maximize steady state consumption $c^*$. Solve for the optimal tax rate $\tau$ and discuss its interpretation.

2.16. [Prelim 2012] Consider the Solow growth model in continuous time. Production is $Y = ALf(K/AL)$, where $f$ satisfies the Inada conditions, $Y$ is aggregate output, $K$ capital, $L$ population, and $A$ is a productivity index. The savings rate $s$, population growth $n$, productivity growth $g$, and the depreciation rate $\delta$ are exogenous. Also, $dK/dt = sY - \delta K$.

a. Derive an equation for the dynamics of the capital-labor ratio $k=K/(AL)$. Derive an equation for the steady state $k^*$. Explain why $k$ converges to $k^*$ from any positive initial capital stock.

b. Suppose that, starting in steady state at time $t_1$, the population growth rate decreases permanently from an initial value $n=n_1$ to a lower value $n=n_2$. Show that the capital-labor ratio $k$ increases over time. Does the growth rate of aggregate output $Y$ change at time $t_1$? If so, by how much? Graph the time paths of $Y$ and of the growth rate of $Y$. [Hint: use a logarithmic scale for $Y(t)$]

c. Now consider a different change at $t_1$: A fraction $m$ of the population emigrates, where $0<m<1$, so population jumps down from $L(t_1)$ to $(1-m)L(t_1)$. Determine the impact of this change on the time paths of the capital-labor ratio $k$, of per-capita output $Y/L$, and of aggregate output $Y$. Graph these variables against time; use logarithmic scale if convenient. Provide algebraic answers whenever possible, e.g., about the impact of $m$ on growth rates at time $t_1$ and about the impact of $m$ on long-run values.
**Part 3:**

3.1. Consider a household with constant rate of time preference $\rho$ who faces a given path of future real wages $w(t)$ and a constant interest rate $r$. The household supplies one unit of work ($L=1$) and maximizes the utility function

$$\max_{C(t)} \int_{t=0}^{T} e^{-\rho t} u(C(t)) \, dt$$

where $u(C) = -\exp\{-\alpha \cdot C\}$ and $\alpha > 0$ is a constant parameter (This is known as the exponential utility function with constant absolute risk aversion $\alpha$.) Initial asset holdings $a_0$ are given. Asset holdings must be non-negative at time $T$.

a. Set up the Hamiltonian problem, apply the Maximum Principle, and derive an optimality condition for $dC/dt$.

b. Derive the intertemporal budget constraint and solve for the optimal consumption path $C(t)$.

[Hint: A differential equation of the form “$dx/dt=constant$” has the linear solution $x(t)=x(0)+t \cdot constant$.]

3.2. Consider an economy with utility-maximizing, infinitely lived households; notation as in Romer unless noted. Assume population ($L$) and the productivity index ($A=1$) are constant. Preferences are

$$\max_{C(t)} \int_{t=0}^{T} e^{-\rho t} u(C(t)) \, dt ,$$

where $u$ is increasing and concave and $0 < \beta < 1$. Capital accumulation is described by

$$\dot{k} = f(k) - \delta k - c .$$

a. Set up the Hamiltonian, apply the Maximum Principle, and derive a pair of equations that describe the dynamics of capital and consumption.

b. Suppose a hurricane destroys half the capital stock. Describe graphically how consumption and the capital stock will adjust over time.

c. Starting in a steady state with positive depreciation, suppose depreciation is suddenly eliminated, $\delta = 0$. Describe graphically how consumption and the capital stock will adjust over time. Is there a finite steady state?

3.3. Consider a continuous-time representative agent economy with constant population $L=1$ and constant productivity. Individuals have preferences

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \, dt ,$$

where $u$ is increasing and concave. The stock of capital for the aggregate economy evolves according to the equation $dK/dt = I - \delta K$. Government spending is a function of time $G(t)$. The National Income identity is given by $Y = F(K,L) = C + I + G$, where $F$ satisfies the Inada conditions. Taxes
are lump sum and equal to government expenditure in every period. The representative agent expects that government expenditure will be constant over time.

a. Set up the representative agent’s problem. Apply the maximum principle.
b. Derive the phase diagram and show that $K$ converges to a steady state value $K^*$ from any initial value $K_0$. Explain why $C$ is an increasing function of $K$ during the convergence process.
c. Suppose at time $t=0$, the government announces a tax-financed increase in government spending starting at $t=1$ and ending at $t=2$. Assuming the economy was at the steady state, show the dynamics of $C$, $K$ and the interest rate $r$. Illustrate these dynamics in the phase diagram and by sketching time-series charts for $C$, $K$, and $r$.

3.4. Consider an individual facing the following utility maximization problem:

$$\max_{c,l} \int_{t=0}^{\infty} e^{-\rho t} u(C(t),l(t)) dt$$

where $C$ is consumption and $l$ is leisure; utility is increasing and concave in both arguments. The individual is endowed with 1 unit of time and works: $n(t) = 1-l(t)$. The per-worker capital stock evolves according to

$$\frac{dk}{dt} = k^{\alpha} n^{1-\alpha} - c - G,$$

where $G$ is a constant level of per-capita government spending. There is no depreciation and the population is constant.

a. Assume for now that leisure $l$ is constant at some level, $0<l<1$. Set up the Hamiltonian and find the optimality conditions.
b. Solve for the steady state values $c^*$ and $k^*$.
   How do changes in $G$ affect $c^*$ and $k^*$? How do exogenous changes in $l$ affect $c^*$ and $k^*$?
c. Now let $l(t)$ be another choice variable. Set up the Hamiltonian and find the optimality conditions.
d. Assume utility has the form $U(c,l) = \ln(c) + \ln(l)$.
   Solve for the steady state values $c^*$, $k^*$, and $n^*$. How do changes in $G$ affect $c^*$, $k^*$, and $n^*$?

3.5. Suppose you are advising the government of a small country that just lost a war. The country must pay war reparations of a given amount $X$ per year. Assume production $F(K,L)$ has constant returns, satisfies the Inada conditions, and that depreciation is zero. Productivity is constant. The continuous-time GDP identity is

$$Y = F(K,L) = C \cdot L + X + \frac{dK}{dt}$$

a. Assume the population $L_t$ grows at rate $n>0$. Derive the budget equation linking per-capita consumption $C$ and the per-capita capital stock $k = K/L$. 

b. Assume that the reparations are in place forever. You want to maximize the utility of the representative agent, 
$$ \int_0^\infty e^{-\theta t} u(C(t)) L(t) dt $$
subject to the constraint derived in (a). $K_0$ is given.

(i) Set up the Hamiltonian and state the necessary conditions for optimality. What are the conditions for a steady state?

(ii) Use a phase diagram to explain the dynamics of the model.

c. Assume now that the war reparations are known to end at a finite date $T$ (X=0 for t>T). Use the phase diagram to explain how the path of $C$ and $K$ differs from (b).

3.6. Consider an economy with population $L$ that grows at the rate $n$. Output $Y$ is produced with physical capital $K$, labor $L$, and human capital $H$ according to the production function
$$ Y = K^\alpha H^\beta L^{1-\alpha-\beta}. $$
Consumption equals output minus investment in $K$ and $H$. Physical and human capital both depreciate at a common rate $\delta>0$. Assume $L=1$ is constant. (Also, disregard any non-negativity constraints on gross investment.)

a. To start, suppose individuals invest fixed fractions $s_K$ and $s_H$ of output in physical and human capital, respectively; $s_K,s_H>0$, $s_K+s_H<1$. Initial values $K_0$ and $H_0$ are given. Does the economy have a steady state in level and/or in per-capita values? Compute the steady state output and explain how it depends on the model parameters.

b. Now suppose individuals maximize 
$$ \int_0^\infty e^{-\rho t} u(C(t)) dt $$
where $u(C)$ is increasing and concave.

Define the choice variables, state variables, and constraints of the problem. Set up the Hamiltonian and derive the first order conditions for optimality. Compute the steady state output level and comment on how its determinants differ from the determinants in (a).

Can you determine the optimal steady state savings rates? What can you say about the relationship between $K(t)$ and $H(t)$?

3.7. [Midterm 2008] Consider a continuous time economy with households that maximize 
$$ U = \int_0^T \left[ e^{-\rho t} \frac{(C(t))^{1-\theta}}{1-\theta} \cdot \frac{L(t)}{h} \right] dt, $$
where $\theta>0$, $\theta\neq 1$. The notation is as in Romer unless noted. Output $Y=F(K,AL)$ is increasing in both factors and has constant returns to scale. Population $L$ grows at an exogenous rate $n$. Productivity $A$ grows at an exogenous rate $g$. The dynamics of the aggregate capital stock is 
$$ \frac{dK}{dt} = F(K,AL) - \delta K - C \cdot L. $$
$K(0)>0$ is given.
a. Set up the Hamiltonian, apply the Maximum Principle, and derive a condition for consumption growth as a function of the capital-labor ratio and of the model parameters. [Derive means: Show your work, no credit for a memorized Euler equation.]

b. Suppose the production function is the linear function \( F(K, AL) = \varphi_1 K + \varphi_2 AL \) with constants \( \varphi_1 > \delta \geq 0 \) and \( \varphi_2 > 0 \). To simplify, assume \( A = L = 1 \) are constant \( (n = g = 0) \). Can you solve for the optimal consumption path? [Hint: Explain why an intertemporal budget constraint applies.]

3.8. [Midterm 2010] Consider a continuous time economy with finite horizon \( T \) and constant population \( L = 1 \). Households maximize

\[
U = \int_0^T \left[ e^{-\rho t} \frac{(C(t))^{1-\theta}}{1-\theta} \right] dt ,
\]

where \( \theta > 0, \theta \neq 1 \) and \( \rho > 0 \). Notation is as in Romer unless noted. Each household earns a constant wage \( w \) and may borrow or invest at a constant interest rate \( r \). The asset position \( a(t) \) starts at time \( t=0 \) with \( a(0)=0 \).

a. Set up the Hamiltonian problem, apply the Maximum Principle, and derive a condition for optimal consumption growth as a function of the interest rate \( r \) and of the model parameters. [Derive means: Show your work, no credit for a memorized Euler equation.]

b. Explain why the terminal condition \( a(T)=0 \) must be imposed to determine optimal consumption. Solve for the optimal consumption path and explain under what conditions \( c(0) < w(0) \).

3.9. [Midterm 2011] Consider a continuous-time economy with infinitely lived optimizing households. Notation is as in Romer unless noted. The representative household (set \( H = 1 \)) maximizes

\[
U = \int_0^\infty \left[ e^{-\rho t} \ln(C(t))L(t) \right] dt ,
\]

where \( \rho > 0 \). Population grows at rate \( n \), where \( 0 < n < \rho \). Suppose the interest rate \( r > 0 \) is constant. The wage \( W(t) \) grows at a fixed rate \( g \), where \( 0 < g < r - n \). Household assets at time \( t \) are \( a(t) \). Assume \( a(0)=A>0 \) and \( L(0)=1 \) are given.

a. Set up the Hamiltonian problem, apply the Maximum Principle, and derive a condition for optimal consumption growth as a function of model parameters. [Derive means: Show your work. No credit for a memorized Euler equation.]

b. Derive the intertemporal budget constraint. Explain why a limit condition on assets is needed and explain its economic meaning.

c. Solve for optimal consumption \( c(0) \). Explain why the assumptions \( n < \rho \) and \( g < r - n \) are needed to derive \( c(0) \). Explain how a decline in \( g \) would change \( c(0) \).
3.10. [Midterm 2013] Consider an optimal growth model with power utility and linear production. Utility is
\[ U = \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} dt, \]
Productivity A and population L=1 are constant. Production Y=AK is linear in capital and requires no labor, so \( \frac{dK}{dt} = AK - \delta K - C \). Assume \( A>\delta>0 \) and \( \rho>(1-\theta)(A-\delta) \).

a. Set up the Hamiltonian problem, apply the Maximum Principle, and derive a condition for optimal consumption growth.
b. Explain why capital must satisfy a transversality condition. Is a No-Ponzi condition required?
c. Solve for optimal consumption \( C(0) \) as function of initial capital \( K(0) \). Show that optimal consumption in this model implies a constant savings rate.

3.11. [Final 2008] Consider an economy where individuals maximize utility
\[ \int_0^\infty e^{-\rho t} u(c(t)) dt \]
subject to the capital accumulation equation \( \dot{k} = f(k) - c - (\delta + n)k \), where \( k \) is the per-capita capital stock, \( c \) is per-capita consumption, \( n=n_1 \geq 0 \) is population growth, \( \rho>n_1 \), and \( \delta \geq 0 \) is the depreciation rate. Productivity does not grow.

a. State the Hamiltonian problem, apply the maximum principle, and derive the equations that describe the dynamics of consumption and capital.
b. Construct the phase diagram of this problem. Assume initial capital \( k(0)>0 \) is below the steady state \( k^* \). Explain why the saddle path has slope steeper than \( \rho-n_1 \) in a neighborhood of \( k^* \).
c. Suppose at time \( t_1>0 \), population growth unexpectedly declines to \( n=n_2<n_1 \). Determine the impact on the time-paths of consumption and the capital stock. (Hint: The economy is not in steady state.) What happens to the savings rate at time \( t_1 \)?
d. Now suppose individuals recognize at time \( t_0 \) that population growth will decline at time \( t_1>t_0>0 \). Again determine the impact on the time-paths of consumption and the capital stock.
e. To illustrate the differences between the answers in (c) and (d), sketch the time series for \( c(t) \) in one graph and the time series for \( k(t) \) in a second graph.

3.12. [Final 2008] Consider an economy with capital \( k \) and real estate \( h \) (consisting of houses of various sizes). Individuals value housing \( h \) (per capita) as well as regular consumption \( c \). They maximize a separable utility function
\[ \int_0^\infty e^{-\rho t} [u(c(t)) + \omega \cdot u(h(t))] dt \]
where \( u \) is increasing and concave and \( \omega \) measures the weight on housing in utility. Assume productive capital can be converted into housing, and vice versa, instantaneously and at no cost. Productivity and population are constant.

a. Suppose a representative individual owns the capital and real estate. Let \( a = k + h \) denote total wealth, and assume

\[
\dot{a}(t) = f(k) - c(t) - \delta_k k - \delta_h h.
\]

where \( f(k) \) is output per unit labor as a function of \( k \), \( \delta_k \geq 0 \) and \( \delta_h \geq 0 \) are depreciation rates. State the Hamiltonian problem. Apply the maximum principle. Derive conditions for steady state consumption, capital, and housing. (Hint: Explain why \( h \) can be treated as choice variable.)

b. Assume individuals can borrow and lend at an interest rate \( r(t) \) and rent housing from/to each other at a rental rate \( R(t) \). Determine how \( r(t) \) and \( R(t) \) relate to the time-\( t \) capital stock.

c. Suppose \( u \) is a power function: \( u(c) = c^{1-\theta} \) and \( u(h) = h^{1-\theta} \) with parameter \( 0 < \theta < 1 \). Determine the effect of an increased depreciation rate of housing (higher \( \delta_h \)) on steady state consumption, capital, and housing.

3.13. [Final 2009] Consider the following economy with exogenous government spending. Population and productivity are constant (\( L = A = 1 \)). Households maximize utility

\[
\int_0^\infty e^{-\rho t} u(C(t)) \, dt
\]

subject to the capital accumulation equation

\[
\dot{k} = f(k) - C - G - \delta_k k,
\]

where \( k \) is the capital stock, \( C \) is consumption, \( G \) is government expenditure; \( u(C) \) and \( f(k) \) satisfy the usual conditions.

a. State the Hamiltonian problem, apply the maximum principle, and derive the Euler equation for consumption. Assuming \( G \) is constant, construct the phase diagram, explain why the economy converges to a steady state and why \( c \) is an increasing function of \( k \) during the convergence process.

b. Suppose the government announces at time \( t = 0 \) that at a future date \( t = t_1 > 0 \), government spending will increase permanently, from \( G = G_0 \) for \( t < t_1 \) to \( G = G_1 \) for \( t > t_1 \). Assume the economy at \( t = 0 \) is at the steady state associated with \( G = G_0 \). Explain how optimizing households will respond to the government announcement. Describe the economy's trajectory in the phase diagram and sketch the time paths of \( C(t) \) and \( k(t) \).

c. Suppose at \( t = t_1 \) the government unexpectedly cancels its spending plans and keeps spending constant at \( G = G_0 \). Explain how optimizing households will respond to the new announcement. Describe the resulting trajectory in the phase diagram and sketch the time paths of \( C(t) \) and \( k(t) \).
3.14. [Final 2009] Consider an infinite horizon representative agent economy with constant population and constant productivity. The government finances constant spending \( G(t) = G \) with lump sum taxes \( T(t) \) and with a sales tax that is levied at rate \( \tau(t) \geq 0 \) on consumption.

Individuals maximize utility

\[
\int_0^\infty e^{-\rho t} u(c(t)) dt \quad \text{subject to} \quad \dot{k} = f(k) - (1 + \tau(t)) \cdot c - T - \delta \cdot k ,
\]

where \( \rho > 0 \) and \( \delta > 0 \). Individuals take taxes as given when solving their optimization problem.

The government chooses taxes to satisfy its intertemporal budget constraint.

a. For parts (a-b), disregard the sales tax (assume \( \tau = 0 \)) so \( G \) is financed with lump-sum taxes.

State the Hamiltonian problem, apply the maximum principle, and (briefly) derive the equations that describe the dynamics of consumption and capital. (For the dynamics, start with \( k(0) < k^* \).)

b. Explain why changes in \( T(t) \) do not affect consumption, provided \( G \) is held constant and the government satisfies its intertemporal budget constraint.

c. Now assume \( \tau = \tau(t) \) is a continuous and differentiable known function of time. Show how the Hamiltonian problem is modified and derive a consumption Euler equation.

d. More specifically, suppose \( \tau(t) \) equals zero until time \( t_1 \), gradually increases between \( t_1 \) and \( t_2 \), and then stays constant at a positive value. Assume \( T(t) \) is varied to maintain a balanced budget; that is, \( T(t) = G - \tau(t) \cdot c(t) \). How does this sales tax affect consumption and capital accumulation? (Compare to lump-sum taxes.) Can you give an economic interpretation?

3.15. [Final 2010] Consider an economy with growing productivity and constant population (\( L = 1 \)). Efficiency \( A \) grows at rate \( g > 0 \). Assume the government lets spending \( G \) grow at the same rate as productivity, so \( G = \gamma \cdot A \) is proportional to \( A \) with proportionality constant \( \gamma \). Assume \( G = T \) is financed by lump sum taxes \( T \). Individuals maximize utility

\[
\int_0^\infty e^{-\rho t} \left( \frac{1}{1-\theta} (C(t))^{1-\theta} \right) dt ,
\]

where \( C \) is per-capita consumption; assume \( \rho > 0 \) and \( \theta > 0, \theta \neq 1 \). Capital \( K \) accumulates according to

\[
\dot{K} = F(K, L) - C - G - \delta K ,
\]

where \( F \) satisfies standard assumptions and \( \delta > 0 \).

a. State the Hamiltonian problem, apply the maximum principle, and derive the Euler equation for per-capita consumption.

b. Derive the dynamics of consumption and capital in efficiency units \((c,k)\), and the steady state conditions. Draw the phase diagram of this problem and explain why the economy converges along a positively sloped saddle path.

c. Suppose the economy is in steady state at time \( t_1 \). Then productivity growth unexpectedly increases (permanently) from \( g = g_1 \) to \( g = g_2 > g_1 \). Determine how capital and consumption respond,
instantly and over time. If some changes have ambiguous sign, first explain why; then explain
which case applies for a low elasticity of substitution between K and L (which means marginal
products are sensitive to changes in K/L).

d. Illustrate the dynamics of c and k in a phase diagram. Sketch the time paths of k(t) and c(t).
   Explain the economic intuition for your findings.

e. Again assume the economy is in steady state at time t1. Suppose a government advisor makes the
   following argument: “Because G is proportional to income Y=F(K,L) along the balanced growth
   path anyway, it would simplify the work of tax collectors if the lump-sum tax to finance G were
   replaced by a proportional tax on income that raises the same revenue. Then revenues would
   automatically grow with productivity so the lump sum tax, which would save politicians the
   trouble of increasing lump sum taxes all the time to keep up with productivity.” First verify that G
   is proportional to Y. Then comment on the proposed tax reform.

3.16. [Final 2011] Consider an economy with constant productivity and constant population (A=1,
L=1). Individuals maximize utility
\[
\int_0^\infty e^{-\rho t} \frac{1}{1-\theta} (C(t))^{1-\theta} \, dt,
\]
where C is per-capita consumption; assume \(\rho > 0\) and \(0 > \theta, \theta \neq 1\).

Capital K accumulates according to
\[
\dot{K} = F(K,L) - C - \delta K + X,
\]
where F satisfies standard assumptions, \(\delta > 0\) is given. The variable X is exogenous and refers to
revenue from extracting natural resources (say, oil), as specified below.

a. Take \(K(0)>0\) as given. Assume \(X(t)=X^+>0\) is a positive constant for \(0 \leq t \leq t_1\) and \(X(t)=0\) for \(t>t_1\).
   (That is, the natural resource is exhausted at time \(t_1\).) State the Hamiltonian problem, apply the
   maximum principle, and derive an Euler equation for per-capita consumption.

b. Explain why capital and consumption will converge to steady state values, \(K^*\) and \(C^*\), in the long
   run. Provide the steady state conditions.

c. Suppose \(K(0) = K^+\). Construct the phase diagram (with explanation). Then use the phase
diagram to determine the dynamics of capital and consumption. Sketch the time series of K(t) and
C(t). [Hint: Clearly distinguish \(t<t_1\) and \(t>t_1\).]

d. Compare the paths for K(t) and C(t) obtained in (a-c) to the paths you would have obtained: (i) if
   \(X(t)=0\) for all \(t\); (ii) if \(X(t)=X^+>0\) for all \(t\).

e. Consider the scenario of (a-c). Now suppose a foreign company makes the following proposal:
The company will give the representative household \(K^+=t_1*X^+\) units of capital at time \(t=0\) in
exchange for the resource. That is, under the proposed trade, the country would operate as if
\( K(0) = K^* + t_i X^* \) and then \( X(t) = 0 \). Should the household accept? Always, never, or only under certain conditions? Explain your reasoning.

3.17. [Final 2012] Consider an economy with constant productivity and constant population \((A=1, L=1)\). Individuals maximize utility

\[
\int_0^\infty e^{-\rho t} \frac{1}{1-\theta} (c(t))^{1-\theta} \, dt,
\]

where \( C \) is per-capita consumption; \( \rho > 0 \) and \( \theta > 0, \theta \neq 1 \). Capital \( k \) accumulates according to

\[
\frac{dk}{dt} = f(k) - c - \delta k - G,
\]

where \( F \) satisfies standard assumptions, \( \delta > 0 \) is given. Government spending \( G \) is exogenous and unless stated otherwise, financed with lump-sum taxes.

Simplifying assumption: whenever qualitative answers below depend on \( \theta \), you may assume a high elasticity of intertemporal substitution.

a. Derive the Euler equation and explain how the time preference and intertemporal substitution parameters influence consumption behavior.

b. Suppose \( G \) is constant forever at a value \( G = G_L \). Provide steady state conditions and construct the phase diagram. Assume initial capital \( k(0) \) is less than \( k^* \). Explain how the economy converges, using the phase diagram.

c. Suppose at some time \( t = t_0 \), the government enacts a major spending program, but the program will not take effect for several years. That is, the “low” value \( G = G_L \) applies until some time \( t = t_1 > t_0 \). Thereafter, \( G = G_H > G_L \) is “high.” Determine the effects of this change on consumption and capital, both steady state and convergence process. Use a phase diagram and sketch the time series for \( c(t) \) and \( k(t) \) to explain your answers. [Note: You may assume \( t_0 \gg 0 \), so \( k(t_0) \) is approximately equal \( k^* \).]

d. Consider the same spending program as in (c), but now suppose the government announces at \( t = t_0 \) that a capital income tax will pay for the new program. (That is, \( G_L \) is still financed by lump sum taxes. Starting at \( t = t_1 \) the increment \( G_H - G_L \) is financed by a capital income tax.) Assume households earn interest \( r = f'(k) - \delta \) on capital supplied to firms. Starting at \( t = t_1 \), interest income is taxed at rate \( \tau > 0 \), so the after-tax interest rate is \( (1 - \tau) \cdot r \).

i. Determine the of paths \( c(t) \) and \( k(t) \) and compare them to your findings in (c). That is, explain the effects of using income taxes instead of lump sum taxes, taking as given that \( G = G_L \) until \( t = t_1 > t_0 \) and \( G = G_H \) thereafter.

ii. Compare the paths of \( c(t) \) and \( k(t) \) here to the steady in (b). That is, determine the overall effects the spending program with income-tax financing as compared to keeping \( G = G_L \) forever and no new taxes,
3.18. [Final 2013] Consider an economy where households have preferences over per-capita private consumption \( C \) and per-capita government-provided goods \( G \):

\[
\int_0^\infty e^{-\rho t} L(t) u(C(t),G(t)) dt.
\]

where \( u(C,G) = \ln(C + \gamma_1 G) + \gamma_2 \ln(G) \)

with parameters \( 0 \leq \gamma_1 < 1 \) and \( 0 < \gamma_2 < (1 - \gamma_1)/\gamma_1 \).

[Hints: Interpret the \( \gamma_1 \)-part as public spending that can substitute for private spending and the \( \gamma_2 \)-part as spending unrelated to private consumption. Use \( \tilde{C} = C + \gamma_1 G \) to simplify expressions.]

There is an exogenous price \( P(t) \geq 1 \) that describes the real cost of producing \( G \) at time \( t \). PG is financed by lump sum taxes. Capital \( K \) accumulates according to

\[
\dot{K} = F(K,L) - L \cdot C - L \cdot PG - \delta K,
\]

where \( F \) satisfies standard assumptions and \( \delta > 0 \). Population \( L \) grows at rate \( n \). Productivity is constant.

a. State the Hamiltonian problem. Apply the maximum principle. Show that the ratios \( G/\tilde{C} \) and \( G/C \) at any time \( t \) depend only on \( P(t) \) and on preference parameters.

b. Now assume \( P \) is constant. Derive the dynamics of per-capita consumption and of capital in efficiency units, \( k \). Derive the steady state conditions. Draw a phase diagram for this problem and explain why the economy converges along a positively sloped saddle path.

c. Suppose the economy is in steady state at time \( t_0 \) with constant \( P=P_0 \). Then households learn that public spending will become more costly in the future—specifically, that at some time \( t_1 \) (where \( t_1 > t_0 \)) \( P \) will jump to a higher value \( P=P_1 \). Determine how \( k \), \( C \), and \( G \) respond to the increase in \( P \). [Hint: results may be surprising—try to find an economic interpretation.]

d. Now suppose \( u(C,G) = \frac{1}{\tau - \theta}(C + \gamma_1 G)^{1-\theta} + \frac{\gamma_2}{\tau - \theta} G^{1-\theta} \) with \( \theta \neq 1 \). How would your answers in (a)-(c) change?

3.19. [Prelim 2010] This question is about the effects of an increase in population growth. You may assume zero productivity growth to simplify the problem.

a. Consider the Solow growth model with Cobb-Douglas production: \( Y = K^\alpha L^{1-\alpha} \), where \( K \) is capital, \( L \) population, and \( Y \) aggregate output. The capital share \( \alpha \), the savings rate \( s \), the population growth rate \( n \), and the depreciation rate \( \delta \) are exogenous. Derive an equation for the steady-state capital-labor ratio \( k^* \) as a function of population growth and other relevant model parameters. Show that a marginal increase in the population growth rate reduces the capital-labor ratio. Does it also reduce consumption? Explain the intuition for your results.

b. Consider the continuous-time optimal growth model with general production and utility functions. A representative household maximizes
\[
\int_0^\infty e^{-\rho t} u(c(t))L(t)dt,
\]
subject to \( \dot{k} = f(k) - (n + \delta)k - c \), where \( f(k) \) is output per capita and \( u(c) \) is utility over per-capita consumption \( c \); both functions are increasing and concave. Set up the Hamiltonian to derive an equation for optimal consumption growth. Then determine the effects of a marginal increase in population growth on the steady state values \( k^* \) and \( c^* \). Explain the intuition for your results.

c. Explain in economic terms why the models in (a) and (b) yield different results. For the comparison, you may assume \( f(k) = k^\alpha \).

3.20. [Prelim 2010] Consider the following continuous time representative agent problem. An agent maximizes utility

\( \int_0^\infty e^{-\rho t} u(c(t))dt \)

subject to the capital accumulation equation \( \dot{k} = f(k) - c - G - \delta k \), where \( k \) is capital stock, \( c \) is consumption, \( G \) is government expenditure; \( u(c) \) and \( f(k) \) are increasing and concave. Assume \( G=G_0 \) is constant unless noted. The economy starts at time \( t=0 \) with a given capital stock \( k(0)>0 \).

a. State the Hamiltonian problem, apply the maximum principle and derive equations that describe the dynamics of consumption and capital. Explain the intuition for these equations.

b. Construct the phase diagram for this problem. Suppose government spending increases to \( G_1>G_0 \) for a time interval \([t_0,t_1]\), where \( t_1>t_0>0 \). The change occurs unexpectedly at time \( t_0 \). Describe the impact of the unexpected change on consumption, on capital, and on interest rates over time. Explain the intuition for your results.

c. Suppose \( f(k) - \delta k - G = f_1 \cdot k + f_0 \) is linear with constant coefficients \( f_0 > 0 \) and \( f_1 > 0 \). Also assume \( u(c) = \ln(c) \). Derive the intertemporal budget constraint and solve for the optimal consumption path. Explain why a transversality condition is needed and what it means in economic terms.


\( Y = ALf(K/AL) \), where \( f \) satisfies the Inada conditions, \( Y \) is aggregate output, \( K \) capital, \( L \) population, and \( A \) is a productivity index. The savings rate \( s \), population growth \( n \), productivity growth \( g \), and the depreciation rate \( \delta \) are exogenous. Also, \( dK/dt = sY - \delta K \).

a. Derive an equation for the dynamics of the capital-labor ratio \( k = K/(AL) \). Derive an equation for the steady state \( k^* \). Explain why \( k \) converges to \( k^* \) from any strictly positive initial capital stock.
b. Suppose that, starting in steady state at time \( t_1 \), the productivity growth rate increases permanently from its initial value \( g = g_1 \) to a higher value \( g = g_2 \). Show that the capital-labor ratio \( k \) declines over time.

c. If \( s = 0.20 \), \( n = 1\% \), \( g_1 = 1\% \), \( \delta = 4\% \), and the capital share is \( 1/3 \), what are the approximate percentage changes in \( k^* \) and in \( y^* \) in response to a marginal increase in \( g \) by 0.1%?

d. Show that higher productivity growth starting at \( t_1 \) reduces output per efficiency unit of labor, \( Y/(AL) \), and increases output per person, \( Y/L \). Does the growth rate of \( Y/L \) change at time \( t_1 \)? If so, by how much? Graph the time paths of \( Y/(AL) \) and of \( Y/L \); use a logarithmic scale for \( Y/L \).

3.22. [Prelim 2011] Consider the optimal growth model in continuous time. Production is \( Y = ALf(K/AL) \), where \( f \) satisfies the Inada conditions. \( Y \) is aggregate output, \( K \) capital, \( L \) population, and \( A \) is a productivity index. Population grows at rate \( n > 0 \), productivity grows at rate \( g > 0 \). Households maximize \( U = \int_0^\infty e^{-\rho t} u(C(t))L(t)dt \), where \( \rho \) is the rate of time preference, \( C \) is per-capita consumption, and \( u(C) = \frac{1}{1-\theta}C^{1-\theta} \), \( \theta > 0 \), \( \theta \neq 1 \). Capital accumulation is \( dK/dt = Y - LC - \delta K \) with \( \delta > 0 \).

a. Use the Hamiltonian and the maximum principle to derive equations describing the dynamics of the capital-labor ratio \( k = K/(AL) \) and of consumption per effective labor unit, \( c = C/A \). Also, derive conditions for a steady state.

b. Suppose, starting in a steady state at time \( t_1 \), productivity growth increases permanently by \( \Delta g \) and population growth decreases by same amount, so \( \Delta n = -\Delta g \) and the sum \( n + g \) remains unchanged. What is the impact of these changes on \( c^* \) and \( k^* \)? Show in a phase diagram how \( c \) and \( k \) move on impact and over time. Graph the time series of \( c \) and \( k \).

c. Use a phase diagram to explain how your answers in (b) would change if the increase in \( g \) were not accompanied by a change in \( n \). Could \( c \) decline at time \( t_1 \)? (Note: Diagram and economic reasoning suffice; no math expected in this part.)

d. For the experiment in (b), graph per-capita consumption \( C \), using a logarithmic scale. Explain the main features of the graph (e.g., slopes, jumps, curvatures). Can you prove that, for all \( t > t_1 \), the level and growth rate of \( C \) are increased?

3.23. [Prelim 2013] Consider the following continuous time economy with infinitely lived households and a government. A representative household maximizes utility

\[
U = \int_0^\infty [e^{-\rho t} u(C(t))L(t)]dt
\]
where $L$ is the household population and $C$ is per-capita consumption. Population grows at a constant rate $n$, where $0 < n < \rho$. Total factor productivity is constant. Each household member supplies a unit of labor and must pay per-capita lump-sum taxes $T(t)$.

The household starts with a capital stock $K(0)>0$ at time $t=0$ and can save by making capital investments $I(t)$. The household’s capital stock evolves over time according to $dK/dt = I - \delta K$.

Output is $Y = F(K,L)$, where $F$ satisfies the Inada conditions.

The government spends an amount $G(t)$ per-capita and it imposes taxes to balance the budget, so $T(t)=G(t)$ for all $t$. Output is used for consumption, government spending, and aggregate investment, so $Y = C \cdot L + G \cdot L + I$.

a. State the Hamiltonian problem, apply the maximum principle and derive equations that describe the dynamics of per-capita consumption $C$ and the capital-labor ratio $k=K/L$. Explain the economic intuition for these equations.

b. Construct the phase diagram for this problem, assuming $G(t)=G^*$ is constant. Show that $k(t)$ converges to a steady state value $k^*$. Explain why $C$ is an increasing function of $k$ during the convergence process.

c. Now suppose government spending is not constant, but proportional to the capital stock: $G = G(k) = g \cdot k$, where $g$ is a constant. Households continue to treat taxes as lump sum. Explain how the new assumption changes the problem as compared to part (b). (Notably, what changes in the phase diagram and in the convergence process?) For the comparison, assume that government spending here and in part (b) are equal when $k$ equals the steady state capital stock in part (b); that is, $G(k^*) = G^*$.

d. Now suppose the government changes the method of financing its spending: To pay for $G$, household capital holdings are taxed at a tax rate equal to $g$. Explain how this tax policy changes the problem as compared to part (c).

3.24. [Prelim 2013] This question is about the effects of an increase in productivity growth.

a. Consider the Solow growth model with aggregate production function $Y = F(K, AL)$, where $F$ satisfies the Inada conditions, $A$ is a productivity trend, $K$ is capital, and $L$ population. Productivity $A$ grows at an exogenous rate $g>0$. Output is used for consumption and capital investment. The savings rate $s$, the population growth rate $n$, and the depreciation rate $\delta$ are exogenous and constant.

Show that the capital-labor ratio $k=K/(AL)$ follows a differential equation. Derive an equation for the steady-state $k^*$ and explain how it depends on model parameters. Show that an increase in productivity growth (g) would reduce the capital-labor ratio. Does it also reduce per-capita consumption? Explain.
b. Consider the continuous-time optimal growth model. The assumptions about production are the same as in (a). A representative household maximizes
\[ \int_0^\infty e^{-\rho t} u(C(t))L(t)dt, \]
where \( u(C) \) is an increasing and concave utility over per-capita consumption \( C \).
Set up the Hamiltonian to derive equations for the capital-labor ratio \( k \) and for \( c=C/A \) (consumption in effective units). Determine how an unexpected increase in productivity growth (\( g \)) at some time \( t=t_0 \) would change the steady state values \( k^* \) and \( c^* \).

c. In the optimal growth model, explain why \( C(t) \) may either increase or decrease at time \( t=t_0 \).
Explain in economic terms under what conditions—regarding preferences and production—are likely to trigger higher consumption in response to higher productivity growth. Explain why the same conditions motivate a decline in the savings rate.

3.25. [Prelim 2014] This question is about the effects of an increase in productivity growth. You may assume zero population growth and labor \( L=1 \) to simplify the problem.
Consider the Solow growth model with Cobb-Douglas production:
\[ Y = A^{1-\alpha}K^\alpha, \]
where \( K \) is capital, \( A \) is a productivity index, and \( Y \) aggregate output. The capital share \( \alpha \), the savings rate \( s \), the productivity growth rate \( g \), and depreciation \( \delta \) are exogenous. Per-capita consumption is
\[ C = (1-s)Y. \]

a. Derive an equation for the steady-state capital-labor ratio \( k^* \) as a function of productivity growth and other relevant model parameters.

b. Determine by how much a marginal increase in productivity growth (\( g \)) reduces the capital-labor ratio. Does higher productivity growth also reduce per-capita consumption? Explain the economic intuition for your results.

c. Consider the continuous-time optimal growth model with the same production function
\[ Y = A^{1-\alpha}K^\alpha. \] A representative household maximizes
\[ \int_0^\infty e^{-\rho t} u(C(t))dt, \]
where \( u(C) = \frac{1}{1-\theta}C^{1-\theta} \),
subject to \( \dot{K} = dK/dt = Y - \delta K - C. \) The parameters satisfy \( \theta > 0, \theta \neq 1 \) and \( \rho > g(1-\theta) \).
Define \( c = C/A \). Set up a Hamiltonian to derive an equation for optimal consumption growth.
Use a phase diagram to explain why \( c \) and \( k \) converge to steady state values \( (c^*, k^*) \).

d. Suppose productivity growth increases unexpectedly and permanently, starting in steady state at \( t=0 \). Determine in which direction \( c^* \) and \( k^* \) will change. Use the phase diagram to explain the process of convergence to the new steady state.

e. Express the steady state savings rate \( s^* \) as function of productivity growth. Determine under what conditions higher productivity growth will increase or decrease \( s^* \).
3.26. [Midterm 2012] Suppose you are stranded on a remote island. At time $t=0$, you have a stock of food $K(0)$. You know that you will be rescued at time $t=T>0$. Your preferences over food consumption are

$$U = \int_0^T e^{-\rho t} \ln(C(t)) dt,$$

where $\rho > 0$. There is no production. Stored food spoils at rate $\delta > 0$, so $\frac{dK}{dt} = -\delta K - C$.

a. Set up the Hamiltonian problem, apply the Maximum Principle, and derive a condition for optimal consumption growth. Show that the costate variable for $K$ is increasing over time; provide an economic interpretation.

b. Derive the intertemporal budget constraint with $K(T)=0$ and determine the optimal time path of consumption. [Hint: No need to simplify exogenous terms—suffices to show the determinants.]

c. Solve for optimal consumption and food stocks if $T = \infty$, i.e., if no rescue is expected. [Hint: Either re-derive the intertemporal budget constraint for the case $T = \infty$ or show that the limit for $T \to \infty$ of the solution in (b) satisfies the appropriate boundary conditions.]
Part 4:

4.1. Consider an overlapping generations economy. Individuals in generation $t$ maximize utility

$$U = \ln(c_{1t}) + \beta \cdot \ln(c_{2t+1})$$

They earn a wage $w_t$ when young, save an amount $a_t$, and earn interest at rate $r_{t+1}$ on their savings. Technology is Cobb-Douglas with capital share $\alpha$ and 100% depreciation, so output net of depreciation is

$$Y_t = K_t^{\alpha} L_t^{1-\alpha} - K_t.$$ 

Wage and interest are determined competitively. There is no population growth and no productivity growth; population size is $L$.

a. Set up the individual optimization problem for generation $t$; derive savings function; state the equilibrium condition; and determine the steady state capital-labor ratio $k^*$. 

b. Start in the steady state and assume dynamic efficiency. Suppose in some period $t_0$ the old generation discovers that a fraction $\phi > 0$ of their savings was “invested” in assets that prove worthless (say, unfinished houses in the desert), so that the useful capital stock is only $K_{t_0} = (1 - \phi) \cdot L \cdot a_{t_0-1}$. Describe how the economy will evolve over time after this loss of wealth. Sketch a time path for $k_t$.

c. Suppose the old generation in period $t_0$ proposes that the government should “bail them out” by taxing the young and making a transfer to the old to compensate for the loss. (Let $x$ denote the amount.) How would this proposal affect the path of $k_t$ and the utility levels of successive generations? Compare to the steady state and to the path in (b).

4.2. Consider an overlapping generations economy in which young individuals work full time (1 time unit) and old individuals work part-time, namely $\gamma$ time units, where $0 < \gamma < 1$. The economy has a Cobb-Douglas technology with capital share $\alpha$ and 100% depreciation. Individuals of generation $t$ maximize utility

$$\ln(c_{1t}) + \beta \cdot \ln(c_{2t+1}),$$

where $0 < \beta < 1$. There is no government, no population growth, and no productivity growth.

a. Derive individual asset holdings $a_t$ as function of $(w_t, w_{t+1}, r_{t+1})$. What are the effects of second period work effort ($\gamma$) on the level and the interest sensitivity of savings? Provide an economic interpretation of these effects. 

b. Define the capital-labor ratio $k_t$ as the capital stock divided by total labor supply. Explain why $k_{t+1} = a(w_t, w_{t+1}, r_{t+1})/(1 + \gamma)$. 

c. Compute the steady state capital stock and the steady state interest rate. Is the aggregate savings rate constant?
4.3. Consider an overlapping generations economy in which individuals receive exogenous endowment income rather than factor incomes. The emphasis in this problem is on distinguishing the income growth over generations (denoted m) versus income growth over an individual life cycle (denoted g). Individuals of generation t receive an income y\_t when young and y\_t(1+g) when old. Across generations, endowments grow at the rate m, y\_{t+1} = (1+m)\cdot y_t. Individuals maximize the utility function \ln(C_{1t}) + \beta\cdot \ln(C_{2t+1}). They can borrow and lend at a constant interest rate r. The population growth rate is a constant n.

a. How does an increase in the growth rate of individual incomes, g, affect individual savings?
b. How does an increase in m affect the aggregate savings rate, taking g as given? The aggregate savings rate is defined as the ratio of total savings to total endowments.
c. Assuming g = m, how does an equal increase in m and g together affect the aggregate savings rate?
d. Comment on the following claim: “A country with high income growth tends to have a high savings rate.”

4.4. This question is about an overlapping generations economy with a social security system. Individuals have a utility function U = \ln(c_{1t}) + \beta\cdot \ln(c_{2t+1}). They supply one unit of labor in the first period of their life. The production technology is Cobb-Douglas with capital share \alpha and 100% depreciation. Let w_t be the wage rate, r_t be the interest rate, and k_t be the capital stock per worker. The population growth rate is n.

a. Derive the savings function of young workers and then derive the steady state capital stock in this economy. Under what conditions is r_t greater than or less than n?
b. Suppose in a period t_0, the government unexpectedly introduces a pay-as-you-go social security system of the following form: Each period, a fraction \tau of the wage income is taxed. The tax receipts are immediately distributed to the old generation. Describe the impact of this social security system on the economy (on all variables that you consider relevant). Can you determine the new steady state? Is it important for welfare comparisons whether or not the economy without social security was dynamically efficient? Explain your findings.
4.5. Consider the following overlapping generations economy. Individuals in generation t maximize utility \( U = \ln(c_1^t) + \beta \cdot \ln(c_{2t+1}) \). They work one unit when young, earning a wage \( w_t \), and save an amount \( a_t \). Technology is Cobb-Douglas with capital share \( \alpha \) and 100% depreciation, so output net of depreciation is \( Y_t = K_t^\alpha L_t^{1-\alpha} - K_t \). The number of individuals in generation t is \( L_t \). The size \( L_t \) of generations t grows at a fixed rate \( n > 0 \), \( L_{t+1} = (1+n) \cdot L_t \).

There is also a government that operates a pay-as-you-go social security system: Each period, the young pay a tax \( T_t = \tau \cdot w_t \), \( 0 < \tau < 1 \). (Since labor supply is fixed, the tax is lump sum.) The receipts are given to the old as a transfer \( TR_t \).

a. Set up the individual optimization problem for generation t, derive the first order conditions, and derive the savings function. [Hint: Savings should depend on \( T_t \) and \( TR_{t+1} \).]

b. Explain why the ratio of transfers to wages is \( TR_t/w_t = \tau \cdot (1+n) \equiv b \).

c. Derive an equation linking next period’s capital labor ratio \( k_{t+1} = K_{t+1}/L_{t+1} \) to the current capital labor ratio \( k_t \), and determine the steady state.

For parts (d)-(f), suppose the economy is in steady state at some date \( t_0 \). Then population growth unexpectedly stops: In period \( t_0 \), everyone learns that \( L_t \) will be constant for all \( t \geq t_0 \).

d. Suppose tax rate \( \tau \) is held constant; transfers are varied, if necessary. Determine how the change in population growth affects the time path of the capital stock.

e. Consider the same change in population growth, but assume that ratio of transfers to wages, \( b \), is held constant. Again, determine how the change in population growth affects the time path of the capital stock.

f. Compare how the alternative policy responses in (d) and (e): How do the capital stocks differ? Can you tell which different generations are better off with one or the other policy?

4.6. Consider the following overlapping generations economy. Individuals in generation t maximize a power utility function

\[
U = \frac{1}{1-\theta} (C_1^t)^{1-\theta} + \beta \cdot \frac{1}{1-\theta} (C_{2t+1})^{1-\theta}
\]

with \( 0 < \beta < 1 \) and \( \theta > 0 \). Cohort size grows over time at some rate \( n_t \) that may vary over time, so \( L_t = L_{t-1} \cdot (1 + n_t) \). \( L_0 \) is given. Output is given by a linear production function \( Y_t = rK_t + wL_t \), where \( r \) and \( w \) are positive constants. In period 0, capital \( K_0 > 0 \) is held by the old. Let \( (T_{1t}, T_{2t}) \) denote period-t net taxes on the young and old. Let \( a_t \) denote individual assets. Depreciation is zero; productivity is constant; there is no government spending and no government debt.

a. Set up the optimization problem of a young individual in period t for arbitrary \( (T_{1t}, T_{2t}) \). Derive the optimality conditions. Explain how \( a_t \) depends on wages, taxes, and interest rates.

b. Consider a simple social security system: Taxes on workers are a constant \( \tau > 0 \), \( T_{1t} = \tau < w \).

Transfers to retirees are \( T_{2t} = \tau \cdot (1+n) \), where \( n_t = n \) is constant and \( n < r \).
Explain how social security affects asset accumulation. Does the economy have a steady state? How fast is convergence? Explain your findings.

c. Suppose population growth declines: \( n_t = n^* \) for \( t \leq t_0 \) and \( n_t = n'' < n^* \) for \( t \geq t_0 + 1 \). Assume taxes on workers are unchanged at \( \tau > 0 \), and let \( T_{2t} = \tau \cdot (1 + n_t) \) vary.

How does the decline in population growth affect \( a_t \)? (When does the impact start?) Which generations experience higher or lower utility than without reduced population growth?

4.7. This question is about an overlapping generations economy in which there are durable goods that produce earnings—to be specific, call them fruit trees. Except for the fruit trees, the assumptions are standard: Individuals have log-utility with time preference parameter \( \beta \). They consume and work in their first period of life and receive a wage \( w_t \). They consume but do not work in the second period. Firms produce output from capital and labor at constant returns to scale. The capital share is \( \alpha \), \( 0 < \alpha < 1 \) and the depreciation rate is 100%. To simplify, assume that there is no population growth and no government.

The number of fruit trees and their yield is exogenous. Trees are infinitely lived. The number of fruit trees equals the number of individuals per generation. Initially, each member of the old generation owns one tree. At the start of each period, a tree yields \( \varepsilon \) units of output (fruits) to its owner, where \( \varepsilon > 0 \) is a constant. After the harvest, a market opens where the old sell trees to the young. Let \( x_t \) be the number of fruit trees that each member of the young generation buys at time \( t \), where \( x_t \geq 0 \). Let \( p_t \) be the period-\( t \) market; assume \( p_t > 0 \). In period \( t+1 \), when the period-\( t \) young are old, each person receives \( \varepsilon \) per tree and then sells the tree at price \( p_{t+1} > 0 \) to the next generation.

Let \( a^k \geq 0 \) be the amount of savings that each member of the young generation supplies to the capital market. Let \( a_t = x_t \cdot p_t + a^k_t \) be total individual savings. Individuals take wages, interest rates, and tree prices as given.

a. For reference, derive the steady state capital stock \( k^* \) and the steady state interest rate \( r^* \) for the economy without fruit trees, i.e., the standard OG model. Show that the economy is dynamically inefficient for some value of \( \alpha \).

b. Specify the individuals’ optimization problem with fruit trees. Show that individuals will not buy trees and supply saving to the capital market, unless the rate of return on capital \( r_{t+1} \) and equilibrium prices of fruit trees satisfy the condition

\[
(*) \quad p_t = (\varepsilon + p_{t+1})/(1 + r_{t+1}).
\]

c. Assuming (\( * \)) holds, derive the individual demand functions for \( a_t \), \( c_{1t} \) and \( c_{2t+1} \). Specify the economy’s equilibrium conditions.
d. Derive conditions for the steady state interest rate $r^*$, the steady state capital stock $k^*$, and the steady state price of fruit trees $p^*$. Can you show that this economy is always dynamically efficient? [Hint: Can you show that the equilibrium conditions imply $r^* > 0$?]

4.8. Consider the following overlapping generations economy. Individuals in generation $t$ maximize utility $U = \sqrt{c_{1t}} + \beta \sqrt{c_{2t+1}}$. They work one unit when young, earning the wage $w_t$, and save an amount $a_t$. Output is produced with a Cobb-Douglas technology $Y_t = K_t^\alpha \cdot L_t^{1-\alpha}$ and 100% depreciation. Population grows at the rate $n$, $L_{t+1} = (1+n) \cdot L_t$. Initially, the government is inactive.

a. Set up the individual optimization problem, derive the first order conditions, and derive the savings function.

b. Derive an equation linking next period’s capital labor ratio $k_{t+1} = K_{t+1}/L_{t+1}$ to the current capital labor ratio. Graph the demand and supply of capital as function of the interest rate. Is there a unique equilibrium?

c. Starting in period $t_0$, the government spends a fixed share $\gamma$ of output, so $G_t = \gamma Y_t$. If the young are taxed to finance this spending, what changes in the economy relative to (a)-(b)? (Qualitative answers suffice. Discuss the immediate as well as the long run effects.)

d. Assume spending as in (c), but assume the old are taxed to finance the spending. How are economic outcomes different than in (a)-(c)?

4.9 [Final 2008] Consider an overlapping generations economy where individuals in generation $t$ maximize utility $U = \ln(c_{1t}) + \beta \cdot \ln(c_{2t+1})$ They work one unit when young, earn a wage $w_t$, save an amount $a_t$, and earn interest rate $r_{t+1}$ on their savings. Output is produced with a Cobb-Douglas technology $Y_t = K_t^\alpha \cdot L_t^{1-\alpha}$ and 100% depreciation. The number of individuals in generation $t$ is $L_t$. The size $L_t$ of generations $t$ grows at a fixed rate $n > 0$, $L_{t+1} = (1+n) \cdot L_t$.

The government operates a pay-as-you-go social security system as follows: Each period, the young pay a tax $T_{1t} = \tau w_t$, where $0 < \tau < 1$. (Since labor supply is fixed, the tax is lump sum.) The receipts are given to the old as a transfer $(-T_{2t})$.

a. Set up the individual optimization problem for generation $t$, derive the first order conditions, and solve for asset accumulation as function of $T_{1t}$ and $T_{2t+1}$.

b. Explain why $(-T_{2t+1})/w_{t+1} = \tau (1+n)$. Show how next period’s capital-labor ratio $k_{t+1} = K_{t+1}/L_{t+1}$ relates to the current capital labor ratio $k_t$. [Hint: Simplify $w_{t+1}/(1+r_{t+1})$.]

c. Determine the steady state capital-labor ratio $k^*$. Show that $k^*$ is decreasing in $\tau$. 

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4.10 [Final 2009] Consider an overlapping generations economy with logarithmic utility and Cobb-Douglas production. Individuals in generation $t$ maximize utility

$$U = \ln(C_{1t}) + \beta \cdot \ln(C_{2t+1})$$

Individuals earn a wage $w_t$ when young, save an amount $a_t$, and earn interest $r_{t+1}$ on their savings. Output is $Y_t = K_t^{\alpha}L_t^{1-\alpha}$. Capital fully depreciates after one period, so $\delta=1$. Generation $t$ has $L_t$ members. Population grows at a fixed rate $n>0$, $L_{t+1} = (1+n)L_t$. The government may impose lump-sum taxes $T_{1t}$ on the young and $T_{2t}$ on the old (transfers if negative). The government budget equation

$$D_{t+1} = (1 + r_t) \cdot D_t - [L_t \cdot T_{1t} + L_{t-1} \cdot T_{2t}]$$

describes the dynamics of government debt.

a. Set up the individual optimization problem for generation $t$, derive the first order conditions, and solve for $a_t$ as function of $w_t$, $r_{t+1}$, $T_{1t}$ and $T_{2t+1}$.

b. Suppose the government is inactive, $T_{2t} = T_{1t} = D_t = 0$ for all $t$. Show how next period’s capital-labor ratio $k_{t+1}$ depends on the current value $k_t$. Show that the capital-labor ratio converges to a steady state $k^*$. Under what conditions about $\beta$ is the economy dynamically inefficient?

c. Suppose government debt carries into the next period that is proportional to the size of the young generation,

$$D_{t+1} = d \cdot L_t$$

for all $t$, where $d>0$. Suppose $T_{2t}=0$ and $T_{1t}$ is set so that the budget equation is satisfied. Show how $k_{t+1}$ depends on $k_t$ and $d$. Derive a condition for $k^*$. Show that a higher $d$ implies a lower $k^*$.

4.11 [Final 2009] Consider an overlapping generations economy. Individuals in generation $t$ maximize utility

$$U = \ln(C_{1t}) + \beta \cdot \ln(C_{2t+1})$$

they earn a wage $w_t$ when young, save an amount $a_t$, and earn interest at rate $r_{t+1}$ on their savings. Output is produced with a Cobb-Douglas technology $Y_t = K_t^{\alpha}L_t^{1-\alpha}$ and 100% depreciation. Wage and interest are determined competitively. There is no population growth and no productivity growth; population size is $L$.

a. Set up the individual optimization problem for generation $t$; derive savings function; state the equilibrium condition; and determine the steady state capital-labor ratio $k^*$.

b. Start in the steady state and assume dynamic efficiency. Suppose in some period $t_0$ the old generation discovers (unexpectedly), that a fraction $\phi>0$ of their savings was “invested” in items that prove worthless, so that the useful capital stock is only $K_{t_0} = (1-\phi) \cdot L \cdot a_{t_0}$. Describe how the economy will evolve over time after this “loss” of wealth. Sketch a time path for $k_t$.

c. The old generation in period $t_0$ proposes that the government should “bail them out” by taxing the young and making a transfer to the old to compensate for the loss. (Let $x$ denote the amount.)
How would this proposal affect the path of $k_t$ and the utility levels of successive generations? Compare to the steady state and to the path in (b).

d. The young generation in period $t_0$ proposes to finance the bailout to the old with public debt instead of taxes. The debt would be issued in period $t_0$ and used to compensate the old; it would remain outstanding in perpetuity. Starting in period $t_0+1$, each young generation would pay taxes to cover the interest payments. How would this proposal affect the path of $k_t$ and the utility levels of successive generations? Compare to the initial steady state and to the paths in (b) and (c).

Hint: In (b-d), most credit is given for qualitative comparisons. But try to be precise as quantitative answers do receive additional credit.

4.12 [Final 2010] Consider an overlapping generations economy where individuals in generation $t$ maximize utility

$$U = \ln(C_{1t}) + \beta \cdot \ln(C_{2t+1})$$

They work one unit when young, earn a wage $w_t$, save an amount $a_t$, and earn interest rate $r_{t+1}$ on their savings. Output is produced with a Cobb-Douglas technology and 100% depreciation. To simplify, assume productivity and cohort size are constant ($L=1$).

The government operates a pay-as-you-go social security system as follows: Each period, the young pay a tax $T_{1t} = \tau \cdot w_t$, where $0 < \tau < 1$. (Since labor supply is fixed, the tax is lump sum.) The receipts are given to the old as a transfer $-T_{2t+1}>0$.

a. Set up the individual optimization problem for generation $t$. Solve for optimal asset holdings as function of wages, taxes, transfers, and interest rates.

b. Express $k_{t+1}$ as function of $k_t$ and the tax rate $\tau$. Show that the capital-labor ratio converges to a steady state $k^*$ and that the economy is dynamically efficient for some (sufficiently high) values of $\tau$ [Hint: write $-T_{2t+1}/(1+r)$ in terms of $k$ and simplify.]

c. In some period $t=t_1$ the government proposes to terminate social security: Taxes and transfers are set to zero, effective immediately. Assume the economy was in steady state in period $t_1-1$. How would this policy affect the path of the capital-labor ratio in period $t_1$ and in the following periods?

d. The old in period $t_1$ propose an alternative: Let social security end in period $t_1+1$, so retirement savers get advance notice. Taxes and transfers in period $t_1$ would remain unchanged. Examine how this alternative would alter the path of the capital-labor ratio.

e. The young in period $t_1$ propose another alternative: Set taxes to zero in period $t_1$ (as in c) and issue government debt to keep transfers unchanged (as in d). In future periods, impose taxes on the young to keep the debt constant. Again examine how this alternative would change affect the path of the capital-labor ratio.
f. Compare the alternative policies proposed in parts (c), (d), and (e). How do the resulting capital stocks compare in the long run? Which generations are better or worse off in the alternative scenarios? [Hint: consider who is proposing what.]

4.13 [Final 2011] Consider the following overlapping generations economy. Individuals in generation $t$ maximize a power utility function

$$ U = \frac{1}{1-\theta} (C_{1t})^{1-\theta} + \beta \frac{1}{1-\theta} (C_{2t+1})^{1-\theta} $$

with $0<\beta<1$ and $0<\theta<1$. Output net of depreciation is produced from capital and labor with an increasing, concave, constant-returns-to-scale production function $F$. Total factor productivity is constant ($A=1$). Cohort size $L_t = L$ is constant. The government imposes lump-sum taxes $T_{1t}$ on the young and $T_{2t}$ on the old. Government spending is proportional to wage income, $G_t = \gamma \cdot L_t W_t$ where $0 \leq \gamma < 1$ is exogenous. Government debt follows the budget equation

$$ D_{t+1} = (1 + r_{t+1})D_t + G_t - L_t T_{1t} - L_{t-1} T_{2t} $$

where $r_{t+1}$ is the interest rate from $t$ to $t+1$. In equilibrium, generation-$t$ savers must hold all capital and all government bonds that are carried from period $t$ to $t+1$.

a. Set up the individual optimization problem for generation $t$. Solve for optimal asset holdings as a function of wages, taxes, transfers, and the interest rate.

b. For the special case $T_{2t+1}=0$, show that optimal savings can be written in the form

$$ a_t = s(r_{t+1}) \cdot (W_t - T_{1t}) $$. Determine $s(r)$ and show that it is increasing in $r$.

c. Assume $T_{2t}=0$ and $D_{t}=0$ for all $t$. Assume $T_{1t}$ is set to satisfy the government budget equation. Show that the capital-labor ratio converges to a steady state $k^*$ that depends on $\gamma$. Explain the economic effects of a permanent increase in $\gamma$ (say, starting in some period $t=t_0$.)

d. Assume $T_{2t}=0$ and $D_{t+1} = \varphi \cdot L_t W_t / (1 + r_{t+1})$ for all $t$, where $\varphi>0$ is an exogenous constant. Again, $T_{1t}$ is set to satisfy the government budget equation. Show that the capital-labor ratio has a steady state $k^*$ that depends on $\gamma$ and on $\varphi$.

e. Suppose the economy of part (d) is in steady state. Then in some period $t_0$, the government unexpectedly decides to default on the government debt. That is, $T_{1t}$ is set to finance government spending only; and the old generation, which holds the debt, receives nothing (no principal payment and no interest). No new debt is issued thereafter. Determine the economic effects in period $t_0$ and in subsequent periods.
4.14 [Final 2012] Consider the following overlapping generations setup. Individuals in generation $t$ maximize a power utility function

$$U = \frac{1}{1-\theta} (C_{1t})^{1-\theta} + \beta \frac{1}{1-\theta} (C_{2t+1})^{1-\theta}$$

with $0<\beta<1$ and $\theta>0$, $\theta\neq1$. For $\theta=1$, assume log-utility. Output is produced with Cobb-Douglas production function, with capital share $0<\alpha<1$ and full depreciation $\delta=1$. Population and productivity are constant ($L=A=1$). There is no government activity. For reference below, define the function

$$s(r) = \frac{1}{1 + \beta^{-1/\theta} \cdot (1 + r)^{1-1/\theta}} = 1 - \frac{1}{1 + \beta^{1/\theta} \cdot (1 + r_{t+1})^{1/\theta} - 1}.$$

a. Set up the individual optimization problem and show that asset holdings can be written as $a_t = s(r_{t+1}) \cdot W_t$.

b. Suppose the parameters $(\alpha, \beta)$ satisfy $(1 + \beta) = (1 - \alpha) / \alpha$. Show that the steady state capital stock $k^*$ does not depend on $\theta$.

[Hint: Find $k^*$ for $\theta=1$ and show that the same value applies for all $\theta$.]

c. Consider two economies: Economy #1 has $\theta = \theta_1 < 1$. Economy #2 has $\theta = \theta_2 > 1$. Both have the same $(\alpha, \beta)$-parameters with $(1 + \beta) = (1 - \alpha) / \alpha$. Suppose in both economies a natural disaster destroys a fraction of the capital stock.

d. Describe and compare how capital adjusts over time in the two economies. Graphical analysis suffices. Explain the differences in economic terms.

4.15 [Final 2012] Consider an overlapping generations economy with the following special feature:

There are durable objects $X_t$ (say, art, gold, collectibles) that individual like to own. Specifically, suppose preferences are logarithmic in consumption and objects $X_t$:

$$U = \ln(C_{1t}) + \beta \ln(C_{2t+1}) + \gamma \ln(X_t)$$

with parameters $0<\beta<1$ and $0<\gamma<1$. Output is produced with Cobb-Douglas production function, with capital share $0<\alpha<1$ and full depreciation $\delta=1$. Population and productivity are constant, unless stated otherwise ($L=A=1, n=g=0$).

There is a fixed stock $\bar{X}$ of objects in existence, which is owned by the old generation. (If $\bar{X}$ is small, one might call them rare objects.) Let $p_t$ be the period-$t$ unit price of $X_t$. Generation $t$ buys $X_t$ at price $p_t$ in period $t$ and sells it at price $p_{t+1}$ in period $t+1$. The generation-$t$ budget equations are

$$C_{1t} + p_t X_t + a_t^k = W_t$$
$$C_{2t+1} = (1 + r_{t+1}) a_t^k + p_{t+1} X_t,$$

where $a_t^k = K_{t+1}$ provides capital for the next period. Equilibrium requires $X_t = \bar{X}$. 

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a. Assume \( p_t > p_{t+1}/(1 + r_{t+1}) \). Determine individual consumption, capital supply, and the demand for X as function of the wage, the interest rate, and the prices of X.
b. Explain why market equilibrium requires \( p_t > p_{t+1}/(1 + r_{t+1}) \) and \( p_t < \frac{\beta + \gamma}{\gamma + \beta + \gamma} W_t / \bar{x} \).
c. Provide conditions for the steady state values \( K^*, r^*, \) and, \( p^* \). [Hint: No closed-form solution required, three implicit functions suffice.]
  
  i. Explain why \( r^* \leq 0 \) is impossible.
  
  ii. Show that \( K^* \) and \( r^* \) do not depend on \( \bar{x} \). Can you provide an economic interpretation?
  
  (Note: The irrelevance of \( x \) is actually general, applies for all t, but you don’t have to show this.)
d. Now suppose productivity \( A_t = A_{t-1}(1 + g) \) is rising at a rate \( g > 0 \). Define \( k_t = K_t / A_t \) and write capital market equilibrium as \( k_{t+1} = \frac{1}{1 + g} a_t k_t / A_t \).
  
  i. Show that \( p_t \) must satisfy the same restrictions as in (b).
  
  ii. Derive steady state conditions (with balanced growth for \( p_t \)) and explain why \( r^* \leq g \) is impossible.

4.16. [Final 2013] Consider an overlapping generations economy with the following special feature: the old generation is alive for only a fraction \( \lambda_t \) of the second period, where \( 0 < \lambda_t \leq 1 \); \( \lambda_t \) may vary over time. Generation \( t \) knows \( \lambda_t \) when making decisions. The working-age population is constant. Interpret \( C_{2t+1} \) as consumption flow per unit time, so the costs and utility benefits of old-age consumption are proportional to \( \lambda_t \): Preferences are

\[
U_t = u(C_{1t}) + \lambda_t \beta u(C_{2t+1}).
\]

The budget constraints with taxes (T) and old-age transfers (TR) are

\[
C_{1t} + a_t = W_t - T_{1t}, \quad \lambda_t \cdot C_{2t+1} = (1 + r_{t+1})a_t + \lambda_t \cdot TR_{2t+1}.
\]

Taxes are \( T_{1t} = \tau_t W_t \) with tax rate \( 0 \leq \tau_t < 1 \), and they pay for transfers \( \lambda_{t-1}TR_{2t} = \tau_t W_t \). Transfers are flow per unit time, paid while the retiree is alive. Production \( F \) satisfies the standard assumptions.

a. Consider a household that takes factor prices and policy as given. Show that savings are an increasing function of \( \lambda_t \).

b. Suppose \( \lambda_t = \lambda_0 \) and \( \tau_t = \tau_0 \) are constant (starting at \( t=0 \)). Assume log-utility, Cobb-Douglas production, and \( \delta=1 \). Show that the economy converges to a steady state. Explain how \( k^* \) depends on \( \lambda_0 \) and on \( \tau_0 \). To what extent is dynamic efficiency implied by, or ruled out by, sufficiently high or low values of \( \lambda_0 \) and/or \( \tau_0 \)? (Discussion about high versus low \( \lambda_0 \) and \( \tau_0 \) suffices, no need for algebraic arguments about marginal effects.)

c. Suppose \( \lambda_t \) is increasing over time and converges to an upper bound \( \lambda_t = \bar{\lambda} \). Again assume log-utility, Cobb-Douglas production, and \( \delta=1 \). Consider two possible tax policies:
  
  i. Fixed taxes: \( \tau_t = \tau_0 > 0 \) is constant.
ii. Fixed benefits: $\tau \equiv b_0 \cdot \lambda_{t-1}$ increases with $\lambda_{t-1}$, so $TR_t/W_t = b_0 > 0$ is constant. Assume the two policies start off with the same taxes/transfers in $t=0$: $\tau_0 = b_0 \cdot \lambda_{-1}$. Describe and compare how the capital-labor ratio changes over time under each of the two policies. Explain which generations (early/late) are better or worse off under (i) or (ii).

4.17. [Prelim 2012] Consider the following overlapping generations economy. Individuals in generation $t$ maximize a logarithmic utility function $U = \ln(C_{1t}) + \beta \ln(C_{2t+1})$ over young- and old-age consumption, where $0<\beta<1$. Technology is Cobb-Douglas with capital share $\alpha$ and 100% depreciation. The wage $W_t$ and the interest rate $r_{t+1}$ are determined competitively. The number of young, $L_t$, grows over time at rate $n>0$, so $L_{t+1} = (1+n)L_t$. Each young supplies one unit of labor.

The government may introduce a simple social security system by imposing a per-capita lump-sum tax $T_{1t} \geq 0$ on the young and giving the receipts to the old as per-capita transfer $TR_t \geq 0$.

a. Set up the individual optimization problem with social security for generation $t$. Solve for optimal asset holdings as function of wages, taxes, transfers, and the interest rate. Show that asset holdings are an increasing function of the interest rate if and only if $TR_{t+1} > 0$.

b. For this part, assume no taxes and no transfers. Show that the period-$(t+1)$ capital-labor ratio $k_{t+1}$ is an increasing function of $k_t$. Show that the economy converges to a unique steady state $k^*$ from any positive initial capital stock. Solve for $k^*$ as function of the model parameters. Determine the steady state interest rate $r^*$. Under what conditions is $r^* > n$?

c. Suppose in some period $t_0 > 0$, the government starts a social security system with taxes $T_{1t} = \min(T, W_t)$ for all $t \geq t_0$, where $T > 0$ is a constant. That is, the young pay $T$ if their wage is at least $T$, and 100% of their wage if the wage is less than $T$. Assume $T < W_{t_0}$. Assume the old in period $t_0$ did not expect transfers when they made savings decisions. Show that the introduction of social security increases the interest rate and reduces the capital stock in period $t_0+1$. (Hint: A graphical argument is sufficient.)

d. Show that for small values of $T$ the economy describes in part (c) converges to a steady state $k^{**}$ that satisfies $0<k^{**}<k^*$. Provide an equation that characterizes $k^{**}$. Show that for high values of $T$, the capital stock will converge to zero. (Hint: Graphical arguments for convergence are sufficient.)

4.18. [Prelim 2014] Consider the following overlapping generations economy. Individuals in generation $t$ maximize a logarithmic utility function $U = \ln(C_{1t}) + \beta \ln(C_{2t+1})$ over young- and old-age consumption, where $0<\beta<1$. Technology is Cobb-Douglas with capital share $\alpha$ and 100% depreciation. Productivity is constant. The wage $W_t$ and the interest rate $r_{t+1}$ are determined competitively. The number of young, $L_t$, grows over time at rate $n>0$, so $L_{t+1} = (1+n)L_t$. Each
young supplies one unit of labor. Government may operate a simple social security system: each young person pays taxes $T_t = \tau_t \cdot W_t$ in proportion to wages. Each old person receives per-capita transfer $TR_t = (1 + n)T_t$. Assume $0 \leq \tau_t < 1$, which includes no social security as special case.

a. Set up the individual optimization problem for generation $t$. Solve for optimal asset holdings as function of $W_t, \tau_t, r_{t+1}$, and $TR_{t+1}$. Show that asset holdings are a strictly increasing function of $r_{t+1}$ provided $TR_{t+1} > 0$.

b. Suppose the economy at time $t=0$ is in a steady state without taxes. Determine $k^*$. Show that the economy is dynamically inefficient if $\alpha$ is small enough.

c. Unexpectedly, government at time $t=0$ introduces social security by setting $\tau_t = \tau > 0$. Determine the new steady state ($k^{**}$) and show how it depends on $\tau$. Show that for a sufficiently high tax rate, the economy is dynamically efficient. Explain the convergence process to $k^{**}$ and show graphically how $k$ changes over time.

d. A government consultant proposes following alternative plan to remedy the dynamic inefficiency without introducing social security; you are supposed to evaluate it.

The plan: The government shall print beautifully engraved “certificates”, give $M$ such certificates to each old person, and organize an exchange of certificates. On the exchange, a young person can make a “donation” to the old, and in exchange, the old will give a certificate to the young. (Fractional certificates are allowed. $M>0$.) Moreover, the government promises that such exchanges will operate in all future periods. Thus a young person who obtains a certificate will have a certificate in old age to reward “donations” by the next generation.

i. Could this plan work in the sense of removing the dynamic inefficiency? Is it sure to work? Explain.

ii. Is it possible that non-zero donations are made in every period, but the economy remains dynamically inefficient? Explain.

iii. If the plan works, is it equivalent to a social security system? Explain.