On Competitive and Welfare Effects of Cross-Holdings with Product Differentiation

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October 2, 2017

Abstract

Competitive implications of cross-holdings have been extensively analyzed in the literature. While the literature has generally focused on cases with homogenous products, this paper is the first attempt to analyze competitive and welfare effects of cross-holdings allowing for product differentiation. We find that the standard results found with homogenous products are also valid with differentiated products. However, both the strength of the competitive implications and the conditions resulting in these implications critically depend on the degree of product differentiation. Furthermore, cross-holdings generally increase social welfare when products are complementary, but decrease social welfare when products are substitutable. Our analysis has useful empirical and policy applications.

Keywords: Cross-holdings, Oligopoly, Product differentiation, Cournot equilibrium

JEL Classification: D43, L13

*I would like to thank Professor Ted Bergstrom, Ted Frech, and Cheng-Zhong Qin for advise and support. I would also like to thank Daniel Arnold, Conor Carney, Monica Carney, and Jenna Stearns for useful comments.

1 Introduction

Cross-holding takes place when a firm acquires passive ownership in another firm, which entitles the acquiring firm a share in the acquired firm's profit but not in decision making. This practice is commonly observed in the real world. Microsoft, for example purchased $150 million of nonvoting preferred stock of Apple Computer in 1997. Gillette, the international and U.S. leader in the wet shaving razor blade market acquired 22.9% of the nonvoting stock of Wilkinson Sword, one of its largest rivals. Other examples include cross-holdings in the automobile (Alley, 1997), telecommunication (Brito et al, 2014), airline (Airline Business, 1998), and banking (Dietzenbacher et al., 2000) industries.

A substantial literature on the anti-competitive effects of cross-holdings has emerged. Reynolds and Snapp (1986) were the first to examine the effects of cross-holdings in an oligopoly with a homogenous product. Their findings suggested that, when maximizing profits, firms take into account the effects their actions have on their competitors. They showed that cross-holdings will lead to a lower aggregate output level and higher prices than when there are no cross-holdings. Farrell and Shapiro (1990) explored when a firm might rationally increase its initial ownership share in a rival. They concluded that joint profit falls if a big firm (in terms of market share) acquires partial ownership in a small firm, but it is profitable for a small firm to acquire partial ownership in a big firm. They also found that joint ownership arrangements could well improve industry performance, even when they cause the price to rise. Flath (1991) demonstrated that, if two firms start without cross-holding, neither firm will initiate long ownership positions in the setting considered by Farrell and Shapiro. Dietzenbacher et al. (2000) conducted empirical studies using data from the Dutch financial sector to analyze consequences of cross-holdings. They showed that competition is reduced for both Cournot and Bertrand oligopolistic firms. There have also been studies that look at the competitive effects in repeat settings. Malueg (1992) and Gilo et al. (2006, 2009) found that cross-holding arrangements can facilitate collusion in infinitely repeated oligopoly models. However, we are not interested in the repeated competition. We refer the reader to Malueg (1992), Gilo et al. (2006, 2009) for further discussion.

The literature has largely focused on competitive implications of cross-holdings in oligopolies with homogenous products. Homogeneity is too extreme, as products in practice are mostly differentiated. It is therefore worth investigating the extent to which results established with product homogeneity are valid with product differentiation. This paper is the first attempt to analyze competitive and welfare effects of cross-holdings allowing for product differentiation. We find that the standard results found in markets with homogenous products are robust to the inclusion of product differentiation.

We conduct our analysis within the standard model of oligopoly with product differentiation as introduced in Spence (1976), Dixit (1979) and Singh and Vives (1984). In this model, product differentiation is derived from a representative consumer's preference and is summarized by the slopes of firms' demand curves with respect to own and competitors' quantities, respectively. Singh and Vives (1984) took the square of the ratio of the others-quantity slope to the slope of own-quantity as the degree of product differentiation. In a recent paper, Amir et al. (2015) provided a thorough exploration of the microeconomic foundations for the multivariate linear demand function for differentiated products. They found that strict concavity of the quadratic utility function is critical for the demand system to be well defined and imposes significant restrictions on the range of complementarity of the products.

The results of this paper can be summarized as follows. For the case with substitutable products, the total industry output decreases when one or more firms increase ownership shares in Cournot competitors. One implication of this finding is that industries with cross-holdings tend to be more concentrated. This anti-competitive effect of cross-holdings is more likely to be observed, when products are less substitutable.

For the case with complementary products in Cournot competition, all firms expand their productions if one or more firms raise ownership shares in other firms, under certain conditions which become weaker the less complementary the products are. With product complementarity, firms' choices have positive effects on other firms. Because of this, our result implies that cross-holdings make firms more cooperative.

Furthermore, we conduct welfare analysis of cross-holdings. With quantity competition, when a big firm acquires passive ownership in rivals, social welfare decreases in industries with substitutable products. When products are complementary, social welfare increases as cross-holdings increase. Lower degree of product differentiation makes the welfare applications easier to be achieved. Thus, from a social point of view, our analysis suggests that cross-holdings should be promoted in industries with complementary products, but discouraged in industries with substitutable products.

As a final point, we analyze the competitive and welfare effects of cross-holdings with product differentiation in Bertrand competition. When products are substitutable, as one or more firms raise ownership shares in other firms, all firms increase the prices for their products and social welfare falls. When products are complementary, the total price decreases as cross-holdings increase. Social welfare rises as a big firm increases ownership share in firms who produce complementary products.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 and Section 4 present the results in Cournot competition with substitutable and complementary products, respectively. Section 5 presents the results in Bertrand competition. Section 6 concludes. Proofs of results are organized in an Appendix.

2 The Model

Consider an oligopolistic industry with n firms producing differentiated products. We assume that each firm i's technology is of constant returns to scale for simplicity. Denote firm i's constant marginal cost is c_i. Let us adopt the notational convention that vectors and matrices are denoted by bold letters and scalars by italic. Vives (2000) considers an n-firm ($n \geq 2$) differentiated goods oligopoly model that is a direct generalization of the duopoly model developed by Dixit (1979). We will present the details of this quadratic and strictly concave utility function in the welfare analysis section. Vives (2000) derived demand functions from a representative consumer's preference. In this model, the inverse demand function for firm i's product is given by

$$p_i(q) = \alpha - \beta_i q_i - \sum_{j \neq i}^{n} \gamma_{ij} q_j \tag{1}$$

where $\mathbf{q} = (q_1, \cdots, q_n)$ with q_i as the quantity produced by firm i, $\alpha, \beta_i, \gamma_{ij}$ are constant such that $\alpha > c_i \geq 0$, $\beta_i > 0$ and $\gamma_{ij} = \gamma_{ji}$, $i = 1, 2, \cdots, n$, $j \neq i$. The parameter γ_{ij} can be positive, negative or zero depending on whether the goods are substitutable, complementary or independent. All γ_{ij} have the same sign so that the products are either all substitutable or all complementary. We define $Q = \sum_{i=1}^{n} q_i$ and $P = \sum_{i=1}^{n} p_i$ as the total output and price, respectively. The strict concavity of the quadratic utility function requires that $\gamma_{ij}^2 \leq \beta_i \beta_j$ for $j \neq i$. Notice that, when γ_{ij} in (1) is close to β_i, firm i's and j's products are homogenous. As γ_{ij} approaches 0, these two firms' products are independent. Consequently, γ_{ij} is an index of product differentiation. Singh and Vives (1984) take the ratio $\gamma_{ij}^2/(\beta_i \beta_j)$ as the degree of product differentiation between firm i's and firm j's products, ranging from zero when the goods are independent to one when the goods are perfect substitutes.[1] We will follow their approach.

[1]When $\beta = -\gamma$ the demand system may not be well defined.

Firms can be linked through a web of cross-holdings. Let δ_{ij} be firm i's ownership share in firm j and let $\mathbf{\Delta}$ be the $n \times n$ matrix of cross-holdings

$$\mathbf{\Delta} = \begin{bmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \dots & \delta_{nn} \end{bmatrix}.$$

The ith row is the vector of ownership share held by firm i in firm j, $j = 1, 2, \cdots, n$, while the jth column is the vector of ownership share held by firm j, $j = 1, 2, \cdots, n$, in firm i. Normalize the total ownership share in each firm to be 1. Thus, the sum of each column in $\mathbf{\Delta}$ is strictly equal to 1.

We consider cross-holdings that are passive. In the terminology of Bresnahan and Salop (1986), firms are assumed to have "silent financial interests" in rivals. This means that acquiring firms are not involved in decision-making of acquired firms. Most studies on cross-holdings assume this, such as Reynolds and Snapp (1986), Farrell and Shapiro (1990) and Flath (1991). With quantity competition each firm i decides on q_i independently. Let $p_i(q)q_i - c_i q_i$ denote firm i's operating earning, which is a measure of how much revenue will eventually become profit for firm i. Reynolds and Snapp (1986) assume that each firm keeps its operating earning net of those going to the acquiring firms. In addition, each firm receives financial interests in competitors' operating earnings. Following their approach, given cross-holding matrix, firm i's objective is to solve the following profit maximization problem:[2]

$$\max_{q_i} \quad \delta_{ii}[p_i(q) - c_i)]q_i + \sum_{j \neq i}^{n} \delta_{ij}[p_j(q) - c_j]q_j, \quad i = 1, 2, \cdots, n \tag{2}$$

[2]This profit formulation in the presence of cross-holdings was prosed in Reynolds and Snapp (1986) and was followed by Farrell and Shapiro (1990), and Alley (1997), among others.

taking other firms' quantities $q_{-i} = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_n)$ as given, where $\delta_{ii} = 1 - \sum_{j \neq i}^{n} \delta_{ji}$. The first-order condition for (2) is

$$2\beta_i \delta_{ii} q_i + \sum_{j \neq i}^{n} [\gamma_{ij}(\delta_{ii} + \delta_{ij}) q_j] - \delta_{ii}(\alpha - c_i) = 0 \tag{3}$$

Solving q_i from (3), firm i's reaction function is

$$q_i = -\sum_{j \neq i}^{n} [\frac{\gamma_{ij}}{2\beta_i}(1 + \frac{\delta_{ij}}{\delta_{ii}}) q_j] + \frac{\alpha - c_i}{2\beta_i}. \tag{4}$$

Observe that the slope of q_i's reaction function depends on the sign of γ_{ij}. When $\gamma_{ij} > 0$, products are substitutable and firm i's best response to an increase in another firm's output is to decrease its own output. In contrast, when $\gamma_{ij} < 0$, products are complementary and firm i's best response to an increase in another firm's output is to expand its production as well.

3 The Case with Substitutable Products

We begin our analysis with a simplified case, in which $\beta_1, \beta_2, \cdots, \beta_n$ are identical and $\gamma_1, \gamma_2, \cdots, \gamma_n$ are identical, and then extend our analysis to a more general case allowing for β_i be different and γ_{ij} be different, $i = 1, 2, \cdots, n$. With symmetry, we use β and γ to denote the common values, respectively. This symmetric case is commonly used, such as Hackner (2000), Amir (2001, 2015) and Zanchettin (2008). In these studies, β is taken as 1 and γ is regarded as the degree of product differentiation. To be consistent with our analysis of the asymmetric case in later subsection, we will not normalize β to be 1. With symmetry, (1) becomes

$$p_i(q) = \alpha - \beta q_i - \gamma \sum_{j \neq i}^{n} q_j, \tag{5}$$

and correspondingly the ratio γ^2/β^2 is regarded as the degree of product differentiation. Recall that the concavity of the utility requires $\gamma^2 \leq \beta^2$, implying that a change in own output has a larger effect on a firm's price than a change in any competing firm's output. The more substitutable (less differentiated) products are, the closer γ^2/β^2 is to 1. Firm i's reaction function in (4) now becomes

$$q_i = -\sum_{j \neq i}^{n} [\frac{\gamma}{2\beta}(1 + \frac{\delta_{ij}}{\delta_{ii}})q_j] + \frac{\alpha - c_i}{2\beta}.$$

Observe that the magnitude of the adjustment in firm i's output as a response to an increase in a competitor's output depends on the degree of product differentiation and the ratio of firm i's ownership share in this competitor to firm i's retained share. Firm i reduces its output more to an increase in a competitor's output when products are not differentiated much.

As usual, the existence of a Cournot equilibrium is not automatically guaranteed. The presence of cross-holdings may complicate the existence problem. Since cross-holdings in practice do not stop firms from producing, it is necessary to establish the existence of a positive Cournot equilibrium with cross-holdings and product differentiation. We then conduct comparative static analysis of total and individual firms' outputs with respect to the change of cross-holdings.

3.1 Cournot equilibrium

We can write firms' reaction functions together in vector notation as

$$\mathbf{q} = \mathbf{S}\mathbf{q} + \mathbf{m},$$

which is equivalent to

$$(\mathbf{I} - \mathbf{S})\mathbf{q} = \mathbf{m}, \tag{6}$$

where $\mathbf{m}$ denotes a column vector, with its transpose $\mathbf{m}' = (\frac{\alpha - c_1}{2\beta}, \cdots, \frac{\alpha - c_n}{2\beta})$, and $\mathbf{S} = (s_{ij})$ denotes a $n \times n$ matrix with $s_{ii} = 0$ and $s_{ij} = -\frac{\gamma}{2\beta}(1 + \frac{\delta_{ij}}{\delta_{ii}})$ for $j \neq i$. Notice that $\mathbf{I} - \mathbf{S}$ is a positive matrix because $\gamma > 0$. If $\mathbf{I} - \mathbf{S}$ is invertible, then a positive Cournot equilibrium can be solved from (6) as

$$\mathbf{q} = (\mathbf{I} - \mathbf{S})^{-\mathbf{1}}\mathbf{m}. \tag{7}$$

We will find conditions that can guarantee the equilibrium quantities in (7) be positive in the next subsection.

Lemma 1 below provides a sufficient condition for when $\mathbf{I} - \mathbf{S}$ is invertible.

Lemma 1. *Assume that*

$$\delta_{ii} > \frac{\gamma}{2\beta}, \quad i = 1, 2, \cdots, n. \tag{8}$$

Then, $\mathbf{I} - \mathbf{S}$ *is invertible.*

Condition (8) puts a lower bound on how much share each firm i needs to retain. Intuitively, each firm must have ownership share in its own firm. Otherwise, firms will not have incentives to produce. Observe that condition (8) is weaker as products become less substitutable. Recall that $\gamma \leq \beta$ and $\gamma > 0$ when products are substitutable. Consequently, this lower bound in (8) takes value from 0 to 1/2. In addition, condition (8) implies $\delta_{ij}/\delta_{ii} < 2\beta/\gamma - 1$ for all i. In that case, each firm's ownership share in other firms has an upper bound, which also depends on the degree of product differentiation. Specifically, the less substitutable the products are, the larger share a firm is allowed to hold in each rival under condition (8). For example, if $2\beta/\gamma = 3$, then firm i's ownership share in one of its competitors could be twice the share that firm i retains. It is worth noticing that in the case of an oligopoly without cross-holdings, condition (8) is automatically satisfied since $\delta_{ii} = 1$ for all i.

3.1.1 Technologies are Symmetric

When firms' technologies are symmetric, their marginal cost are the same. We use c to express this common value. Denote $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}} = (t_{ij})$. When firms have the same constant marginal cost, (7) reduces to

$$q_i = (\frac{\alpha - c}{2\beta})\sum_{j=1}^{n} t_{ij}, \quad i = 1, 2, \cdots, n.$$

Consequently, if a given cross-holding matrix satisfies $\sum_{j=1}^{n} t_{ij} > 0$ for all i, a strictly positive Cournot equilibrium exists. The following Proposition provides a condition for when $\sum_{j=1}^{n} t_{ij} > 0$.

Proposition 1. *Assume*

$$\delta_{ii} > \frac{\gamma}{2\beta} \quad \textit{and} \quad \frac{\sum_{j\neq i}\delta_{ij}}{\delta_{ii}} < \frac{2\beta}{\gamma} - 1, \quad i = 1, 2, \cdots, n. \tag{9}$$

Then, a positive Cournot equilibrium exists.

Condition (9) requires that the ratio of the sum of a firm's ownership shares in all other firms to its retained share has an upper bound. This condition is not as strong as it appears. For example, in the case of a duopoly, condition (9) is automatically satisfied because (9) reduces to $\delta_{ij}/\delta_{ii} < 2\beta/\gamma - 1$, $i = 1, 2$. In addition, in the case of oligopoly with a symmetric cross-holding structure condition (9) is automatically satisfied since $\sum_{j\neq i}\delta_{ij} = \sum_{j\neq i}\delta_{ji} = 1 - \delta_{ii}$. In a usual case of oligopolies without cross-holdings, condition (9) is also automatically satisfied because $\delta_{ij} = 0$ for $j \neq i$. In this case, $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}} = (t_{ij})$ is a $n \times n$ symmetric matrix with $t_{ii} = 1$ and $t_{ij} = \frac{\gamma}{2\beta}$ for $i \neq j$. It concludes that, without cross-holdings, a linear Cournot oligopoly with symmetric cost has a unique Cournot equilibrium, which is symmetric.

A direct application of Proposition 1 establishes

Corollary 1. *If*

$$\delta_{ii} > \frac{\gamma}{2\beta} \quad \textit{and} \quad \sum_{j \neq i} \delta_{ij} < 1 - \frac{\gamma}{2\beta}, \quad i = 1, 2, \cdots, n, \tag{10}$$

then (9) is satisfied.

The upper bound in (10) on the sum of each firm's ownership shares in rivals takes value between 1/2 and 1, and is weaker as products become less substitutable. The following example illustrates that, if condition (10) does not hold, a positive Cournot equilibrium may not exist.

Example 1. Consider a Cournot oligopoly with 3 firms. Suppose $\alpha = 30$, $c = 2$, $\beta = 1.5$ and $\gamma = 1$. Next, let cross-holding matrix be

$$\mathbf{\Delta} = \begin{bmatrix} 0.3 & 0.6 & 0.2 \\ 0.4 & 0.4 & 0 \\ 0.3 & 0 & 0.8 \end{bmatrix}.$$

Notice that the sum of firm 1's ownership share in firm 2 and 3 is 0.8, which is greater than $1 - \gamma/2\beta = 2/3$. It follows that condition (10) is not satisfied. The inverse of $\mathbf{I} - \mathbf{S}$ is given by

$$(\mathbf{I} - \mathbf{S})^{-1} = \begin{bmatrix} -3.62 & 4.65 & 0.63 \\ 2.85 & -2.4 & -1.07 \\ 0.97 & -1.81 & 1.09 \end{bmatrix}.$$

Observe that the sum of the second row in $(\mathbf{I} - \mathbf{S})^{-1}$ is $2.85 - 2.4 - 1.07 < 0$. In that case, the first-order condition yields a negative quantity for firm 2.

□

As we mentioned before, without cross-holdings a linear Cournot oligopoly with symmetric cost has a unique Cournot equilibrium which is symmetric. With cross-holdings, however, firms' Cournot equilibrium outputs are not necessarily symmetric.

For instance, with the cross-holdings matrix in Example 1, all three firms' outputs are different because the sums of each row are different. This example illustrates that cross-holdings can have a large effect on firms' performances.

3.1.2 Asymmetric Technologies

Firms' technologies are asymmetric when their marginal costs are different. Firms with better technologies have lower marginal costs. Recall that the first-order conditions in (3) are independent of costs. Thus, a positive Cournot equilibrium that allows firms to have different technologies can also be solved from (6). Assume the condition in Lemma 1 is satisfied, then $(\mathbf{I}-\mathbf{S})$ is invertible. In that case, a positive Cournot equilibrium exists if and only if

$$q_i = \sum_{j=1}^{n} \frac{(\alpha - c_i)t_{ij}}{2\beta} > 0, \quad i = 1, 2, \cdots, n.$$

Unlike the symmetric case, the proceeding sufficient conditions for a strictly positive Cournot equilibrium depends on firms's cost structure. As such, even if condition (10) is satisfied, solutions of (6) can still be negative, as the following example illustrates.

Example 2. Consider a Cournot oligopoly with 3 firms. Suppose that $\alpha = 30$, $c_1 = 5$, $c_2 = 10$, $c_3 = 20$, $\beta = 0.15$ and $\gamma = 0.1$. Let cross-holding matrix be

$$\mathbf{\Delta} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.8 & 0 \\ 0.1 & 0 & 0.8 \end{bmatrix}.$$

It is clear that condition (10) is satisfied. Simple calculation shows in Cournot equilibrium, $q_1 = 208$, $q_2 = 110$. However, the first-order condition yields a negative quantity $q_3 = -15$ for firm 3. It shows that, a strictly positive Cournot equilibrium does not exist in this example.

□

Finding conditions that guarantee positive solutions of (6) for a duopoly is possible, as Example 3 below illustrates. However, due to the asymmetric cost structures, it is not clear how to extend the result to a more general case.

Example 3. Consider a Cournot duopoly. Let cross-holdings matrix be

$$\mathbf{\Delta} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$$

Under condition (8),

$$det(\mathbf{I} - \mathbf{S}) = 1 - \frac{\gamma^2}{4\beta^2}(1 + \frac{\delta_{12}}{\delta_{11}})(1 + \frac{\delta_{21}}{\delta_{22}}) > 0.$$

Consequently,

$$(\mathbf{I} - \mathbf{S})^{-\mathbf{1}} = \frac{1}{1 - \frac{\gamma^2}{4\beta^2}(1 + \frac{\delta_{12}}{\delta_{11}})(1 + \frac{\delta_{21}}{\delta_{22}})} \begin{bmatrix} 1 & -\frac{\gamma}{2\beta}(1 + \frac{\delta_{12}}{\delta_{11}}) \\ -\frac{\gamma}{2\beta}(1 + \frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix}.$$

It follows that, a strictly positive Cournot equilibrium for a duopoly exists if and only if

$$\frac{\gamma}{2\beta}\frac{\delta_{11} + 1 - \delta_{22}}{\delta_{11}} < \frac{\alpha - c_1}{\alpha - c_2} < \frac{2\beta}{\gamma}\frac{\delta_{22}}{\delta_{22} + 1 - \delta_{11}}. \tag{11}$$

In a duopoly without cross-holdings, (11) reduces to

$$\frac{\gamma}{2\beta} < \frac{\alpha - c_1}{\alpha - c_2} < \frac{2\beta}{\gamma}, \tag{12}$$

which only depends on the cost structure and degree of product differentiation. Comparing (11) with (12), it is clear that cross-holdings complicate the existence of a strictly positive Cournot equilibrium.

□

In reality, cross-holdings do not stop firms from producing. To be consistent with this empirical observation, we will assume a strictly positive Cournot equilibrium exists in the next subsection.

3.2 Competitive Effects

We now analyze the competitive effects of cross-holdings, beginning with the case of a single firm raises ownership share in one of its competitors. We establish conditions under which the total output decreases as one firm acquires ownership in one of its competitors. By decomposing the total change into a chain of individual changes, we can show that total output decreases as more firms simultaneously increase ownership shares in their competitors .

To determine the effect of a small increase of firm 1's ownership share in firm k on total output, we totally differentiate firms' individual quantities characterized by the first-order conditions in (3) to get [3]

$$(\mathbf{I} - \mathbf{S}) \cdot \mathbf{dq} = \mathbf{k}, \tag{13}$$

where $\mathbf{k}$ is a $n \times 1$ matrix with the first element as $-q_k \gamma d\delta_{1k}/(2\beta\delta_{11})$, kth element as $-\sum_{j \neq k} \delta_{kj} q_j \gamma \cdot d\delta_{1k}/(2\beta\delta_{kk}^2)$, and zero otherwise. By Lemma 1, $\mathbf{I} - \mathbf{S} = (t_{ij})$ is invertible under condition (8). In that case, (13) can be solved by

$$\mathbf{dq} = (\mathbf{I} - \mathbf{S})^{-\mathbf{1}} \mathbf{k}. \tag{14}$$

Lemma 2 below summarizes some properties that are useful to determine the total output change due to a small increase of firm 1's ownership share in firm k.

[3]To determine the effect of a small increase from one acquiring firm's ownership share in a acquired firm on total output, without loss of generality we assume firm 1 increases its share in firm k, $k \neq 1$.

Lemma 2. *Assume that*

$$\delta_{ii} > \frac{\gamma}{2\beta}, \quad i = 1, 2, \cdots, n.$$

Then,

$$t_{ii} > 0 \quad \text{and} \quad \sum_{j=1}^{n} t_{ji} > 0.$$

Under condition (8), the sum of each column in $\mathbf{I} - \mathbf{S}^{-1}$ is positive. (14) characterizes each firm's output reaction to a small increase of firm 1's ownership share in firm k. Adding each firm's output reaction together yields the total industry output reaction below.

$$\frac{\partial Q}{\partial \delta_{1k}} = -\frac{\gamma}{2\beta}[\frac{q_k}{\delta_{11}}(\sum_{i=1} t_{i1}) + \frac{\sum_{j \neq k} \delta_{kj} q_j}{\delta_{kk}^2}(\sum_{j=1} t_{jk})].$$

By Lemma 2, the sum of each column in $\mathbf{I} - \mathbf{S}^{-1}$ is positive. It follows that

$$\frac{\partial Q}{\partial \delta_{1k}} < 0. \tag{15}$$

(15) indicates that the total industry output falls when a single firm increases ownership share in one of its competitors. The following example provides an illustration.

Example 4. Consider a Cournot duopoly. Assume cross-holding matrix satisfies condition (8). (14) reduces to

$$\begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \frac{1}{1 - \frac{\gamma^2}{4\beta^2}(1 + \frac{\delta_{12}}{\delta_{11}})(1 + \frac{\delta_{21}}{\delta_{22}})} \begin{bmatrix} 1 & -\frac{\gamma}{2\beta}(1 + \frac{\delta_{12}}{\delta_{11}}) \\ -\frac{\gamma}{2\beta}(1 + \frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix} \begin{bmatrix} -\frac{q_2}{\delta_{11}} \frac{\gamma}{2\beta} d\delta_{12} \\ -\frac{\delta_{21} q_1}{\delta_{22}^2} \frac{\gamma}{2\beta} d\delta_{12} \end{bmatrix}.$$

Consequently, the total output change, due to firm 1's increasing ownership share in firm 2, is given by

$$\frac{\partial Q}{\partial \delta_{12}} = -\frac{(\frac{2\beta}{\gamma}\delta_{22} - \delta_{22} - \delta_{21})q_2 + (\frac{2\beta}{\gamma}\delta_{11} - \delta_{22} - \delta_{21})\frac{\delta_{21} q_1}{\delta_{22}}}{\frac{4\beta^2}{\gamma^2}\delta_{11}\delta_{22} - (\delta_{11} + \delta_{12})(\delta_{22} + \delta_{21})}$$

Recall that condition (8) implies $\delta_{12}/\delta_{11} < 2\beta/\gamma - 1$ and $\delta_{21}/\delta_{22} < 2\beta/\gamma - 1$. As a result, $\partial Q/\partial\delta_{12} < 0$. In other words, total output decreases when firm 1 acquires passive ownership in firm 2.

□

Now we are ready to analyze what effect increasing cross-holdings would have on the total output.

Theorem 1. *Suppose that cross-holding matrix satisfies*

$$\delta_{ii} > \frac{\gamma}{2\beta}, \quad i = 1, 2, \cdots, n.$$

Then, total industry output decreases as cross-holdings increase.

Simultaneous changes in cross-holdings can be achieved through a sequence of unilateral changes, with only one firm changing its ownership holding in one other firm. According to Lemma 2, when a single firm raises ownership share in a rival, the total industry output decreases. Applying Lemma 2 to each step in this sequence can establish Theorem 1. This effect arises because cross-holdings link the fortunes of competitors, producing a positive correlation among their profits. A well known implication of this finding is that industries with cross-holdings will tend to be more concentrated.

This is a result found in oligopolies with homogenous products. Reynolds and Snapp (1986) showed that cross-holding arrangements could result in less aggregate output and higher prices than when there are no cross-holdings, even if the ownership shares are relatively small. Farrell and Shapiro (1990) found that when firm raises its shares of a rival in which it previously had no financial interest, total output falls. We have extended this result to the inclusion of product differentiation. Our result critically depends on the degree of product differentiation. In other words, the condition resulting in this result is weakened as products become less substitutable. Another implication is that the anti-competitive effect of cross-holdings is more likely to be observed, as the products are less substitutable. The following example illustrates a case that cross-holdings decrease the total output by 6%, compared with the total output without cross-holdings.

Example 5. Consider a Cournot oligopoly with 3 firms. Suppose that $\alpha = 1000$, $\beta = 0.4$, $\gamma = 0.2$, $c_1 = 200$, $c_2 = 400$, $c_3 = 600$. Without cross-holdings, in Cournot equilibrium $q_1 = 833$, $q_2 = 500$, $q_3 = 167$ and $Q = 1500$. Let cross-holding matrix be

$$\mathbf{\Delta} = \begin{bmatrix} 0.7 & 0.2 & 0.3 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0 & 0.6 \end{bmatrix}.$$

Under this cross-holding structure, in Cournot equilibrium firm 1 produces 802 units, firm 2 produces 493 units, firm 3 produces 109 units and the total output is 1404 units. It concludes that cross-holdings decrease the total output by 96, which is 6.4%.

□

It has been shown in literature that when a firm raises ownership share in rivals, this acquiring firm will compete less aggressively while all other firms expand productions in oligopolies with homogenous products. We will examine how this result may depend on the degree of product differentiation.

Firm 1's output change due to a small change of its ownership share in firm k is given by [4]

$$\frac{\partial q_1}{\partial \delta_{1k}} = -[q_k t_{11} + \frac{p_k - \beta q_k - c_k}{\gamma} t_{1k}]. \tag{16}$$

As shown in (16), the sign of $\partial q_1 / \partial \delta_{1k}$ depends not only on the cross-holding structure of the target firm, but also relies on each firm's output. This shows that the acquiring firm

[4] From firm k's first-order condition, it is true that $\frac{p_k - \beta q_k - c_k}{\gamma} = \frac{\sum_{j \neq k} \delta_{kj} q_j}{\delta_{kk}}$.

might not alway reduce its output when it acquires passive ownership in its competitors. An illustration is provided in Example 6 below.

Example 6. Consider a Cournot duopoly. Suppose that $\delta_{11} = 0.7$, $\delta_{22} = 0.8$, $\beta = 2$ and $\gamma = 1$. Under condition (8), (16) becomes

$$\frac{\partial q_1}{\partial \delta_{1k}} = \frac{-\frac{2\beta}{\gamma}\delta_{22}q_2 + (\delta_{11}+\delta_{12})\frac{\delta_{21}q_1}{\delta_{22}}}{\frac{4\beta^2}{\gamma^2}\delta_{11}\delta_{22} - (\delta_{11}+\delta_{12})(\delta_{22}+\delta_{21})}.$$

An increase in firm 1's share in firm 2 causes its output to decrease if and only if [5]

$$\frac{\alpha - c_1}{\alpha - c_2} < \frac{\frac{2\beta}{\gamma}\delta_{22}^2 + \frac{\gamma}{2\beta}\frac{1}{\delta_{11}}(\delta_{11} + 1 - \delta_{22})^2(1-\delta_{11})}{1 + \delta_{22}^2 - \delta_{11}^2} = 2.3. \tag{17}$$

□

As Example 6 illustrates, firm 1's output reaction to its increasing ownership in a rival is not definite. The theorem below characterizes the conditions that are sufficient for the acquiring firm to produce less when it acquires passive ownership in its competitors.

Theorem 2. *Let cross-holding matrix be given and k be the target firm. Then, $\frac{\partial q_i}{\partial \delta_{ik}} < 0$ under any following conditions*

$$(i)\quad \delta_{kh} = 0, \quad \forall h \neq k; \tag{18}$$

$$(ii)\quad \sum_{j\neq k}\delta_{ij} + (n-2)\delta_{ii} > \frac{2\beta}{\gamma}(\delta_{ii} + \delta_{ik}). \tag{19}$$

[5]If $\delta_{11} - \delta_{22} + 1 - \delta_{11}\delta_{22}\frac{4\beta^2}{\gamma^2} > 0$, firm 1 will always decrease its output in a positive Courot equilibrium since $\frac{\frac{2\beta}{\gamma}\delta_{22}^2 + \frac{\gamma}{2\beta}\frac{1}{\delta_{11}}(\delta_{11}+1-\delta_{22})^2(1-\delta_{11})}{1+\delta_{22}^2-\delta_{11}^2} > \frac{2\beta}{\gamma}\frac{\delta_{22}}{\delta_{22}+1-\delta_{11}}$, which is the condition for a positive Cournot equilibrium for a duopoly in Example 3.

Under condition (18), firm k does not hold any passive ownership in rivals. Its objective is just to maximize its own operating earning. The first-order condition that characterizes its optimal output becomes $p_k - \beta q_k - c_k = 0$. In that case, equation (16) reduces to

$$\frac{\partial q_1}{\partial \delta_{1k}} = -\frac{q_k}{\delta_{11}} t_{11}$$

By Lemma 2, t_{11} is positive. Under condition (18), this result is similar to the findings in Qin et al. (2016) for cases where cross-holdings are "radiation type", in the sense that one firm holds shares in some other firms but none of other firms holds any shares in their rivals. They showed that each time when the acquiring firm raises its share in one of its competitors, it will be more conservative in production.

Unlike condition (18), (19) extends this result by allowing for the target firm to hold partial ownership in its competitors. Under condition (19), t_{1k} from (16) is positive. The left side of (19) depends on the sum of firm i's ownership shares in its competitors beside firm k, the number of firms and the share firm i retains. The right side relies on the degree of product differentiation, the ownership share firm i retains and the share firm i holds in firm k. Notice that, (19) is weaker as γ increases. In this sense, the more substitutable the products are, the more likely the acquiring firm compete less aggressively. When the sum of this acquiring firm's shares in other firms beside this target firm, is much larger than this acquiring firm's share in this target firm, (19) is easier to be achieved.

Under conditions in Theorem 2, when a firm increases its ownership share in rivals, this acquiring firm reduces its output. Intuitively, when a single firm increases its ownership in one of its rivals, this acquiring firm is induced to take into consideration the effect of its output decision on the acquired firm's profit. It realizes that increasing its output reduces the profit it earns on its ownership share of the acquired firm. This consideration makes the acquiring firm compete less aggressively, because in doing so it can augment profit at the acquired firm and hence its share in the acquired firm's profit.

In the literature it is well established that, with homogenous product and homogenous technology, when a firm increases its ownership share in its competitors, all other firms choose to produce more. Reynolds and Snapp (1986, p. 145) found that when the acquiring firm decreases its output all other firms only expand output until marginal revenue equals marginal cost, so they will never fully replace the output contraction. But with heterogeneous technology and product differentiation, this result is not guaranteed. The following example provides an illustration.

Example 7. Consider a Cournot duopoly. Suppose that $\alpha = 4000$, $c_1 = 1000$, $c_2 = 2000$, $\delta_{11} = 0.9$, $\delta_{22} = 0.7$, $\beta = 2$ and $\gamma = 1$. In Cournot equilibrium, $q_1 = 671$ and $q_2 = 276$. When firm 1 raises its ownership share in firm 2 by 0.1, in the new Cournot equilibrium firm 1 produces 663 units while firm 2 produces 272 units. It is clear that both firms reduce productions when firm 1 acquires ownership in firm 2. The intuition is this. When firm 1 increases its ownership share in firm 2, firm 1 will reduce its output to augment firm 2's operating earning. As a result, firm 1's reduction in output will decrease its own operating earning and hence the financial return that goes to firm 2. However, if firm 2 reacts to expand its output, the increasing amount of its operating earning due to its expansion in output is less than the reduction from firm 1's financial return because firm 2 has higher marginal cost. It turns out that firm 2 would rather to produce less to compensate the reduction from firm 1's output.

□

Our proceeding result shows how other firms' reactions may depend on the existing cross-holding matrix and specific cost structure. The following example illustrates that for cases where cross-holdings are "radiation type" with product differentiation, all other firms react by expanding their productions when the acquiring firm raises its ownership shares in the acquired firms.

Example 8. Consider an oligopoly with n firms. Suppose cross-holding matrix is

$$\mathbf{\Delta} = \begin{bmatrix} 1 & \delta_{12} & \dots & \delta_{1n} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Notice that condition (18) is satisfied. In other words, firm 1 reduces its output when it raises ownership share in of these non-acquiring firms. Under condition (8), by Theorem 1 total output decreases as cross-holdings increase. Other firms' output responses to firm 1's small increasing ownership shares firm k is determined by

$$\frac{\partial q_j}{\partial \delta_{1k}} = -\frac{1}{\frac{2\beta}{\gamma} - 1}\frac{\partial Q}{\partial \delta_{1k}}, \quad j \neq 1$$

According to Lemma 2, $\partial Q/\partial \delta_{1k} < 0$. It follows that

$$\frac{\partial q_j}{\partial \delta_{1k}} > 0.$$

All non-acquiring firms react to any reduction in the acquiring firm's output, and these indirect effects are felt throughout the industry. Given the stabilized Cournot equilibrium and downward sloping reaction functions, all other firms will react to output contractions by expanding their own productions. At the new equilibrium, however, the total output decreases. With this cross-holding structure, our result is consistent with the findings in Reynolds and Snapp (1986), and Farrell and Shapiro (1990).

□

We have showed the competitive effects of cross-holdings, holding the degree of product differentiation constant. Now we will analyze the effect of a small change of the degree of product differentiation on total output, holding cross-holdings constant. Consumers have a taste for variety and consume a variety of brands. For example, most consumers prefer to frequent a variety of restaurants, rather than eating at the same Chinese, Italian, or Mexican restaurant over and over again. If the market offers a variety of soft drinks, the total quantity of soft drinks sold in the market may be larger than if only one type were available. Perloff (2004) states, "Consumers value having a choice, and some may greatly prefer a new brand to existing ones." Now we will examine whether consumers' preference for variety is valid with cross-holdings.

Recall that in Example 5, $Q = 1500$ with $\gamma = 0.2$. If γ decreases to 0.15, the total output increases to 1716 units, holding everything else constant. The next Proposition reveals the connection between total output and the degree of product differentiation.

Proposition 2. *Assume that a cross-holdings matrix* $\mathbf{\Delta}$ *satisfies*

$$\delta_{ii} > \frac{\gamma}{2\beta}, \quad i = 1, 2, \cdots, n.$$

Then, equilibrium total industry output is higher as products become less substitutable

When γ decreases, consumers' demand function shifts away from the origin. At the same price, the demand is higher when γ is smaller. As a result, total output increases when the products become less substitutable. Proposition 2 confirms that consumers' preference for variety is robust with cross-holdings.

Quantitatively, the magnitude of the effect of cross-holdings on total output depends on the degree of product differentiation as well. In Example 5, cross-holdings decrease the total output by 6.4%, compared with the same industry without cross-holdings, when $\gamma = 0.2$. When γ decreases to 0.15, holding everything else constant, cross-holdings now decrease the total output by 5%. It concludes that, cross-holdings have a smaller effect on the total industry output when firms' products are less substitutable.

3.3 Welfare Analysis

Any change in cross-holdings in a Cournot oligopoly has ramifications for the outputs chosen by all of the oligopolists. As we have seen repeatedly, these induced shifts in output will affect the industry performance. In this subsection we explore the changes in social welfare due to an increase in cross-holdings.

We present the details of the representative consumer's utility function here as we promised at the beginning of this paper. Dixit (1979) introduced the quadratic and strictly concave utility function in a duopoly. In this model the representative consumer's utility function is given by

$$U(q, I) = \alpha_1 q_1 + \alpha_2 q_2 - \frac{1}{2}(\beta_1 q_1^2 + \beta_2 q_2^2 + 2\gamma q_1 q_2) + I$$

where I is the total expenditure on other goods that are outside this duopoly.

Vives (2000) generalized the Dixit (1979) model to allow for an arbitrary number of firms. In that case, the representative consumer's utility function is given by [6]

$$U(q, I) = \boldsymbol{\alpha}\mathbf{q} - \frac{1}{2}\mathbf{q}'\mathbf{B}\mathbf{q} + I, \tag{20}$$

where $\mathbf{B} = (l_{ij})$ is a symmetric $n \times n$ positive definite matrix with $l_{ii} = \beta_i$ and $l_{ij} = \gamma_{ij}$ for $j \neq i$. Most studies interpret $\boldsymbol{\alpha}$ as quality vector for vertical product differentiation, such as Hackner (2000, 2005), Hsu and Wang (2005), and Zanchettin (2008). Since we only consider horizontally differentiated products in this paper, we let each firm have the same α. The inverse demand function for firm i's product (1) is derived from (20).

[6] Oligopoly pricing: old ideas and new tools by Xaviver Vives (2000), page 180.

The representative consumer maximizes the utility function (20) subject to a linear budget constraint of the form

$$\sum_{i=1}^{n} p_i q_i + I \leq M.$$

As usual, social welfare, denoted as $W(\mathbf{\Delta})$, is the sum of consumer surplus,

$$U(q, I) - (\sum_{i=1}^{n} p_i q_i + I),$$

and producer surplus,

$$\sum_{i=1}^{n} [p_i(q) q_i - c_i q_i].$$

To determine the social welfare change due to a small change of firm 1's ownership share in firm k, we totally differentiate social welfare, yielding

$$\frac{\partial W(\mathbf{\Delta})}{\partial \delta_{1k}} = \sum_{i=1}^{n} [(p_i - c_i) \frac{\partial q_i}{\partial \delta_{1k}}]. \tag{21}$$

As we mentioned at the beginning of this section, a symmetric version of (20) is commonly used (e.g., Hackner, 2000, Amir, 2001, 2015). We shall continue to analyze the cross-holding effect on social welfare with the symmetric case of (20) here. The analysis with asymmetric utility will be provided in the next subsection.

With symmetry, the representative consumer's utility function becomes

$$U(q, I) = \sum_{j=1}^{n} \alpha q_j - \frac{1}{2}[\beta \sum_{j=1}^{n} q_j^2 + \gamma \sum_{j=1}^{n} q_j (\sum_{s \neq j}^{n} q_s)] + I. \tag{22}$$

The inverse demand functions in (5) are derived from (22).

Notice that the social welfare reaction in (21), due to a small change of δ_{1k}, depends on the price-cost gaps and the output reactions to a small change of δ_{1k}. With these terms in place, we are ready to present our characterization of conditions for when social welfare increases or decreases with cross-holdings. Denote $p_i^\star$ as the equilibrium price for firm i, $i = 1, 2, \cdots, n$.

Theorem 3. *Let firm i be a firm such that*

$$p_i^\star - c_i = \max_{1 \le j \le n} (p_j^\star - c_j).$$

Suppose that either condition (18) or (19) is satisfied. Then,

$$\frac{\partial W(\mathbf{\Delta})}{\partial \delta_{ij}} < 0.$$

We rank firms by their price-cost gaps in equilibrium. In other words, the bigger firms have larger price-cost gaps. By Theorem 1, the total output decreases as one firm increases its ownership share in a rival. When the biggest firm acquires partial ownership in its rivals, the negative effect on social welfare from this firm's reduction in output dominates the total change of social welfare. As a result, overall social welfare decreases.

Recall that in oligopoly with a "radiation" cross-holding structure in Example 8, all non-acquiring firms react to produce more when the acquiring firm increases its ownership share in one of its rivals. At the new equilibrium, however, the total output decreases. In this case the social welfare effect depends on the interaction of two basic forces. First, social welfare decreases because of the reduction in output by the acquiring firm. The social welfare impact associated with this reduction in output depends upon the acquiring firm's price-cost gap. Second, social welfare increases due to the expansion in productions from all non-acquiring firms because these firms move along their reaction curves in response to the reduction in the acquiring firm's output. It concludes that, if the acquiring firm is a big firm, the negative social welfare effect is larger than the positive social welfare effect. As a result, overall welfare decreases when the acquiring firm raises its share in its competitor. However, welfare may well rise if the acquiring firm is small. In that case the negative social welfare effect from the acquiring firm is smaller than the positive social welfare effect from all non-acquiring firms.

This result is consistent to the finding in Farrell and Shapiro (1990) in a oligopoly with a homogenous product. They found that if a small firm acquires partial ownership in a rival in which it previously had no financial interest, welfare may well rise. In their paper, firms' ranking is determined by their market shares. With a homogenous product, firms with lower marginal cost yields a larger market share in a linear demand function. Therefore, Farrell and Shapiro (1990)'s finding can be interpreted as when a firm, which has small price-cost gap, increases its ownership share in a rival in which it previously had no financial interest, welfare may well rise.

A policy implication is that, bigger firms should be prohibited from raising their ownership shares in their competitors. Nonetheless, antitrust concerns might not be raised when smaller firms acquire ownership in their rivals.

3.4 Asymmetric Utility Function

We have shown that the competitive and welfare effects of cross-holdings in a symmetric case with product differentiation. It is therefore worth investigating the extent to which results established under symmetry are valid under asymmetry. We begin with the case allowing for β_i to be different. Then, we will extend the analysis to the case introduced in (20).

3.4.1 The Case with Different β_i but Same γ

When γ remains identical, (20) becomes

$$U(q, I) = \sum_{j=1}^{n} \alpha q_j - \frac{1}{2}[\sum_{j=1}^{n}(\beta_j q_j^2) + \gamma \sum_{j=1}^{n} q_j(\sum_{s \neq j}^{n} q_s)] + I \tag{23}$$

Firm i's inverse demand can be derived from (23) as

$$p_i = \alpha - \beta_i q_i - \gamma \sum_{j \neq i} q_j. \tag{24}$$

Notice that each firm has different effect on the price of its own product but has the same effect on the price of other firms' products. In this case, a natural extension of (8) is

$$\delta_{ii} > \frac{\gamma}{2} \max\{\frac{1}{\beta_1}, \frac{1}{\beta_2}, \cdots, \frac{1}{\beta_n}\}. \tag{25}$$

Lemma 1 and 2 can be shown to hold under condition (25). Consequently, a strictly positive Cournot equilibrium can still be solved from (6). Under condition (25), the lower bound on how much share each firm i needs to retain depends on the largest ratio, of the others-quantity slope in inverse demand functions to the slope of own-quantity among all firms. For example, if two firms produce a homogenous product while others produce different substitutable products, the lower bound on the shares for each firm needs to retain is 50%. Recall that the ratio $\gamma^2/(\beta_i\beta_j)$ expresses the degree of product differentiation between firm i's and firm j's products. If the degree of product differentiation between any two firms' products is changing, it does not necessarily affect condition (25) since the lower bound depends only on the smallest of β_i.

Under condition (25), Theorem 1 and 3 can be shown to hold. Consequently, under condition (25) total industry output decreases as cross-holdings increase and social welfare decreases when a big firm raises its ownership share in its competitors. Under condition (18) and (25), as one firm acquires passive ownership in a firm which does not hold ownership in others, the acquiring firm competes less aggressively.

3.4.2 The Case with Different β_i and Different γ_{ij}

Recall that the inverse demand function for firm i's product can be derived from (20) as

$$p_i = \alpha - \beta_i q_i - \sum_{j \neq i} \gamma_{ij} q_j.$$

Notice that each firm has different effects on the price of one single product. Lemma 3 below provides the condition that are sufficient to ensure the results from Lemma 1 and 2 still hold.

Lemma 3. *Assume that*

$$\sum_{j \neq i} [\frac{\gamma_{ji}}{2\beta_j}(1 + \frac{\delta_{ji}}{\delta_{jj}})] < 1, \quad i = 1, 2, \cdots, n. \tag{26}$$

Then, $(\mathbf{I} - \mathbf{S})$ *is invertible,* $t_{ii} > 0$ *and* $\sum_{j=1}^{n} t_{ji} > 0$.

Observe that, $(\mathbf{I} - \mathbf{S})$ is a column diagonal matrix under condition (26). Under condition (26), the upper bound of firm i's ownership shares held by all other firms depend on the degrees of product differentiation between firm i's product and those of other firms. In that case, firm j is allowed to acquire more shares in firm i when these two firms' products are less substitutable. On the other hand, when firm j's retained share is not large, the ownership share that firm j holds in firm i has to be small. Condition (26) is weakened as products are less substitutable.

Under condition (26), Theorem 1 and 3 can be shown to hold. Under condition (18) and (26), the acquiring firm reduces its production as it raises ownership shares in a firm which does not have ownership shares in others. Recall that the competitive effect of cross-holdings on the total industry output is larger when products are more substitutable. The following example illustrates that the competitive effect of cross-holdings on the total industry output is larger when a firm raises ownership share in a target firm whose product is more substitutable to this acquiring firm's product, than the case when the acquiring firm acquires ownership in a target firm whose product is less substitutable to this acquiring firm's product.

Example 9. Consider a Cournot oligopoly with 3 firms. Assume that $\alpha = 4000, c_1 = 1000, c_2 = c_3 = 3000, \beta_1 = 4, \beta_2 = 1.5, \beta_3 = 1.3, \gamma_{12} = 1.1, \gamma_{13} = 1.1, \gamma_{23} = 1.2$, and cross-holding matrix is given by

$$\mathbf{\Delta} = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}.$$

Simple calculation shows that Q decreases by 29 units when firm 1 raises its share in firm 2 by 0.1 and Q decreases by 23 units when firm 1 raises its share in firm 3 by 0.1. $\square$

4 The Case with Complementary Products

In an industry with complementary products, firms might view other firms as more of partners than competitors, as in the case with substitutable products. Given this fundamental difference, one would expect that the effects of cross-holdings in an industry with complementary products will also be different from the previous section. We will follow a similar approach to the case with substitutable products, beginning our analysis with a simplified case using utility function (22). Then, we will extend our analysis to a more general case as introduced in (20) in the later subsection.

4.1 Cournot Equilibrium

Recall that if $(\mathbf{I}-\mathbf{S})$ is invertible, ignoring the positivity requirement, the Cournot equilibrium can be solved as (6). All off-diagonal elements in $\mathbf{S}$, $s_{ij} = -\frac{\gamma}{2\beta}(1+\frac{\delta_{ij}}{\delta_{ii}})$ for $j \neq i$, are now all positive because $\gamma < 0$ when products are complementary. Unlike the case with substitutable products, the complementary firms' products indicate that $\mathbf{S}$ is a non-negative matrix. Consequently, $\mathbf{I}-\mathbf{S}$ is an open *Leontief Matrix*. Takayama characterizes conditions to ensure the inverse of a *Leontief Matrix* to be non-negative in the theorem below. [7]

Theorem 4. *Given a Leontief Matrix* $(\mathbf{I}-\mathbf{S})$*, the following conditions are equivalent.*
(i) There exists an $\mathbf{x} \geq 0$ *such that* $(\mathbf{I}-\mathbf{S})\cdot\mathbf{x} > 0$*.*
(ii) For any $\mathbf{c} \geq 0$*, there exists an* $\mathbf{x} \geq 0$ *such that* $(\mathbf{I}-\mathbf{S})\cdot\mathbf{x} = c$*.*
(iii) The matrix $(\mathbf{I}-\mathbf{S})$ *is nonsingular and* $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}} \geq 0$*.*

Recall that $\mathbf{m} > 0$ in (6). In that case, a positive solution to (6) depends on an $\mathbf{x}$ that satisfies (i) in Theorem 4. A candidate for such an $\mathbf{x}$ is given by $[1, 1, \cdots, 1]'$. Consequently, a direct application of Theorem 4 establishes

Proposition 3. *Assume that a given cross-holding matrix* $\mathbf{\Delta}$ *satisfies*

$$\frac{\sum_{j\neq i}\delta_{ij}}{\delta_{ii}} < -\frac{2\beta}{\gamma} - (n-1), \quad i = 1, 2, \cdots, n. \tag{27}$$

Then, $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ *is non-negative.*

Observe that under condition (27), $(\mathbf{I}-\mathbf{S})$ becomes a row diagonal dominant matrix. Condition (27) requires that the ratio of the sum of a firm's ownership shares in all other firms to its retained share, has an upper bound, which depends on the degree of product differentiation and the number of firms. In a recent paper, Amir et al. (2015)

[7] Takayama, Mathematical Economics, Theorem 4.C.4, page 383

found that the strict concavity of the utility function imposes significant restrictions on the range of complementarity of the products. The valid parameter range for the complementarity cross-slope is $\gamma \in (\frac{\beta}{1-n}, 0)$. In that case, the upper bound in (27) is greater than $n-1$. We refer the reader to Amir et al. (2015) for further discussions regarding this significant restrictions on γ.

Condition (27) is not as strong as it appears. For example, in a case of an oligopoly without cross-holdings, condition (27) is automatically satisfied since $\delta_{ij} = 0$ for $j \neq i$. In this case, a positive Cournot equilibrium always exists when γ satisfies the concavity requirement. Under condition (27), firms whose retained shares are larger, are allowed to acquire more shares in other firms than firms whose retained shares are smaller. A simple corollary to Proposition 3 is the following.

Corollary 2. *Suppose a given cross-holding matrix* $\mathbf{\Delta}$ *satisfies*

$$\delta_{ii} > -\frac{\gamma(n-1)}{2\beta}, \quad i = 1, 2, \cdots, n. \tag{28}$$

Then, (27) is satisfied.

Condition (28) puts a lower bound on how much ownership share each firm i needs to retain. The same intuition to the case with substitutable products can be applied here. Each firm must have ownership share in its own firm. Otherwise, firms will not have incentives to produce. Recall that the strict concavity of the utility function requires that $\gamma \in (\frac{\beta}{1-n}, 0)$. Correspondingly, this lower bound in (28) takes value from 0 to 1/2. Condition (28) is weakened as products are less complementary. Therefore, a positive Cournot equilibrium is more likely to exist as products become less complementary. Observe that $\mathbf{m}$ in (6) does not require firms' technologies to be the same. Consequently, under condition (28), a positive Cournot equilibrium allows for firms to have different marginal costs. The following example provides an illustration.

Example 10. Consider a Cournot duopoly. $(\mathbf{I}-\mathbf{S})$ is given by

$$\begin{bmatrix} 1 & \frac{\gamma}{2\beta}(1+\frac{\delta_{12}}{\delta_{11}}) \\ \frac{\gamma}{2\beta}(1+\frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix}$$

Under condition (28),

$$det(\mathbf{I}-\mathbf{S}) = 1 - \frac{\gamma^2}{4\beta^2}(1+\frac{\delta_{12}}{\delta_{11}})(1+\frac{\delta_{21}}{\delta_{22}}) > 0.$$

Consequently,

$$(\mathbf{I}-\mathbf{S})^{-\mathbf{1}} = \frac{1}{1-\frac{\gamma^2}{4\beta^2}(1+\frac{\delta_{12}}{\delta_{11}})(1+\frac{\delta_{21}}{\delta_{22}})} \begin{bmatrix} 1 & -\frac{\gamma}{2\beta}(1+\frac{\delta_{12}}{\delta_{11}}) \\ -\frac{\gamma}{2\beta}(1+\frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix} > 0.$$

It follows that,

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = (\mathbf{I}-\mathbf{S})^{-\mathbf{1}} \begin{bmatrix} \frac{\alpha-c_1}{2\beta} \\ \frac{\alpha-c_2}{2\beta} \end{bmatrix} > 0.$$

□

4.2 Competitive Effects

A firm's individual decision has positive effects on other firms with product complementarity. Such positive externalities are not taken into consideration without cross-holdings. However, the positive externalities will be partially internalized with cross-holdings. Therefore, we should expect that cross-holdings increase productions.

Recall that the marginal changes of all firms' outputs due to a small change of firm 1's ownership share in firm k are given by (13)

$$(\mathbf{I}-\mathbf{S}) \cdot \mathbf{dq} = \mathbf{k},$$

where $\mathbf{k}$ is a $\mathbf{n} \times \mathbf{1}$ matrix with the first element as $-\gamma q_k \cdot d\delta_{1k}/(2\beta\delta_{11})$ and kth elements as $-\gamma(\sum_{j \neq i} \delta_{ij} q_j) d\delta_{1k}/(2\beta\delta_{kk}^2)$, zero otherwise. Observe that $\mathbf{k}$ is non-negative since $\gamma < 0$. By Corollary 2, under condition (28)

$$\mathbf{dq} = (\mathbf{I} - \mathbf{S})^{-\mathbf{1}} \mathbf{k} > 0.$$

It follows that when a single firm raises ownership share in one other firm, all firms produce more. Example 11 below provides an illustration.

Example 11. Consider a Cournot duopoly. Under condition (28), firm 1's and firm 2's output changes, due to firm 1's increasing ownership share in firm 2, are given by

$$\begin{aligned} \begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} &= \begin{bmatrix} 1 & \frac{\gamma}{2\beta}(1 + \frac{\delta_{12}}{\delta_{11}}) \\ \frac{\gamma}{2\beta}(1 + \frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\gamma q_2 \cdot d\delta_{12}/(2\beta\delta_{11}) \\ -\gamma(\sum_{j \neq i} \delta_{ij} q_j) d\delta_{1k}/(2\beta\delta_{kk}^2) \end{bmatrix} \\ &= (\mathbf{I} - \mathbf{S})^{-\mathbf{1}} \begin{bmatrix} -\gamma q_2 \cdot d\delta_{12}/(2\beta\delta_{11}) \\ -\gamma(\sum_{j \neq i} \delta_{ij} q_j) d\delta_{1k}/(2\beta\delta_{kk}^2) \end{bmatrix} > 0 \end{aligned}$$

Under condition (28), both firms expand their productions when firm 1 acquires partial ownership in firm 2.

□

Intuitively, when a single firm increases its ownership in one of its rivals, this acquiring firm is induced to take into consideration the effect of its output decision on the acquired firm's profit. It realizes that increasing its output raise the profit at the acquired firm and hence its share in the acquired firm's profit. This consideration makes this acquiring firm to expand its production. Recall that reaction functions are upward sloping when products are complementary. It implies that the initial output increase by the acquiring firm will be followed by outputs increase on all other firms. In the new

Cournot equilibrium, after full adjustment has taken place, all outputs in the industry will have risen, and all firms will be better off.

Now we are ready to analyze what effects cross-holdings would have on the total industry output when more firms simultaneously increase ownership shares in other firms.

Theorem 5. *Assume that*

$$\delta_{ii} > -\frac{\gamma(n-1)}{2\beta}, \quad i = 1, 2, \cdots, n.$$

Then, all firms increase their outputs as cross-holdings increase.

When more firms acquire partial ownership in complementary firms simultaneously, the changes in cross-holdings can be decomposed into a sequence of unilateral changes made by each single firm. In each step from this sequence, only one firm acquired ownership share in one complementary firm. According to Corollary 2, when a single firm raises share in one other firm, all firms expand their productions. Applying Corollary 2 to each step from this sequence can establish Theorem 5. This effect arises because a firm's individual decision has positive effects on other firms with product complementarity. With cross-holdings, the positive externalities will be partially internalized. The following example illustrates a case that cross-holdings increases the total output by 13%, compared to the total output without cross-holdings.

Example 12. Consider a Cournot oligopoly with 3 firms. Suppose $\alpha = 100$, $\beta = 1.25$, $\gamma = -0.5$, $c_1 = 20$, $c_2 = 40$, $c_3 = 60$. Without cross-holdings, simple calculation shows in equilibrium $q_1 = 47$, $q_2 = 40$, $q_3 = 33$ and $Q = 120$. Suppose cross-holding matrix is given by

$$\mathbf{\Delta} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0 \\ 0.2 & 0.2 & 0.9 \end{bmatrix}.$$

Under this cross-holding structure, firm 1 produces 52 units, firm 2 produces 44 units, firm 3 produces 40 units and the total output is 136 units. It follows that, cross-holdings increase the total output by 16 units, which is 13%.

$\square$

Condition (28) also points out the best strategy for firms to engage in cross-holdings. Firms should acquire share in a firm, whose retained share is large, not in a firm whose retained share is small. Otherwise, more cross-holdings may violate condition (28), as the following example illustrates.

Example 13. Consider a Cournot oligopoly with 3 firms. Suppose $\beta = 2$, $\gamma = -1$ $\alpha = 3$. Let the original cross-holding matrix be $\mathbf{\Delta}_1$.

$$\mathbf{\Delta}_1 = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.5 \\ 0.1 & 0.5 & 0.3 \end{bmatrix} \qquad \longrightarrow \qquad \Delta_2 = \begin{bmatrix} 0.7 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.5 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}$$

Notice that condition (28) is satisfied. Consequently, a positive Cournot equilibrium exists, as shown a positive $(\mathbf{I} - \mathbf{S})_{\mathbf{1}}^{\mathbf{-1}}$.

$$(\mathbf{I} - \mathbf{S})_{\mathbf{1}}^{\mathbf{-1}} = \begin{bmatrix} 4.55 & 4.8 & 4.66 \\ 5.81 & 8.12 & 7.29 \\ 5.88 & 7.7 & 8 \end{bmatrix} \qquad \longrightarrow \qquad (\mathbf{I} - \mathbf{S})_{\mathbf{2}}^{\mathbf{-1}} = \begin{bmatrix} -2.48 & -4 & -4.35 \\ -5.77 & -6.45 & -7.5 \\ -5.11 & -6.19 & -6 \end{bmatrix}$$

If both firm 1 and 3 raise their ownership shares in firm 2 by 0.1, holding everything else constant. The cross-holding matrix becomes $\mathbf{\Delta}_2$. It is worth noticing that condition (28) is violated. As a result, a positive Cournot equilibrium does not exist, as shown a negative $(\mathbf{I} - \mathbf{S})_{\mathbf{2}}^{\mathbf{-1}}$. However, if γ increases from -1 to -0.5, under the same cross-holdings matrix $\mathbf{\Delta}_2$, a positive Cournot equilibrium still exists. It concludes that, condition (28) is weakened as products become less complementary.

$\square$

When products are substitutable, the total output is negatively correlated with the degree of product differentiation. We will examine how this result may vary when products are complementary. In Example 13, $Q = 136$ with $\gamma = -0.5$. If γ decreases to -0.75, holding everything else constant, the total output increases to 413 units. The next Proposition reveals the connection between total output and the degree of product differentiation.

Proposition 4. *Suppose cross-holding matrix satisfies*

$$\delta_{ii} > -\frac{\gamma(n-1)}{2\beta}, \quad i = 1, 2, \cdots, n.$$

Then, equilibrium total output is lower as products become less complementary.

When products become less complementary, it will shift consumers' demand function toward to the origin. At the same price, the demand is higher when γ is smaller. Quantitatively, the magnitude of the effect caused by cross-holdings on total output also depends on the degree of product differentiation. In Example 13, cross-holdings increase the total output by 13% when $\gamma = -0.5$. If γ decreases to -0.75, holding everything else constant, cross-holdings now increase the total output by 14.2%. It concludes that cross-holdings have a smaller effect on total output the less complementary the firms' products are.

4.3 Welfare Analysis

When products are substitutable, industries tend to be more concentrated with cross-holdings. As a result, when a big firm raises share in its competitors, social welfare falls. However, when products are complementary, we have found that cross-holdings increase total output. We would expect that the effect of cross-holdings on social welfare will also be different from the previous section. Recall that the marginal change of social welfare due to a small change of firm 1's ownership share in firm k is given by (21)

$$\frac{\partial W(\mathbf{\Delta})}{\partial \delta_{1k}} = \sum_{i=1}^{n}[(p_i - c_i)\frac{\partial q_i}{\partial \delta_{1k}}]$$

By Theorem 5 each firm reacts to a small increasing of firm 1's ownership share in firm k by expanding its production. With these terms in place, the following Theorem reveals the effect of cross-holdings on social welfare.

Theorem 6. *Assume that*

$$\delta_{ii} > -\frac{\gamma(n-1)}{2\beta}, \quad i = 1, 2, \cdots, n.$$

Then, social welfare rises as cross-holdings increase.

Theorem 6 indicates that, in our setting, regulatory restrictions on cross-holdings always decrease social welfare. This result is in stark contrast to the common view that limiting cross-holdings enhances welfare. Economides and Salop (1992) found that joint ownership or integration by firms with complementary products raises welfare. Our analysis provides another way to enhance total welfare when products are complementary by encouraging firms to engage in cross-holdings. Condition (28) is weaker as products become less complementary. In this sense, the less complementary the products are, the more likely cross-holdings are socially preferable. Policy makers should encourage firms to acquire passive ownership in the firms whose retained shares are large, but discourage firms to raise ownership shares in the firms whose retained shares are small.

4.4 Asymmetric Utility Function

As we can see in section 3.4, the asymmetry complicates the conditions resulting the competitive and welfare effects of cross-holdings. We follow a similar approach to the case with substitutable products, beginning with the case allowing for β_i to be different. Then, we will extend the analysis to the case allowing for γ_{ij} to be different.

4.4.1 The Case with Different β_i but Same γ

When β is identical to all firms, the representative consumer maximizes utility given by (23). With (23), $(\mathbf{I}-\mathbf{S})$ is still a *Leontief Matrix.* In that case, $[1,1,\cdots,1]'$ is still a candidate for an $\mathbf{x}$ in Theorem 4. Consequently, a direct application of Theorem 4 establishes

Corollary 3. *Assume that a given cross-holding matrix* $\mathbf{\Delta}$ *satisfies*

$$\delta_{ii} > -\frac{\gamma(n-1)}{2}\max\left(\frac{1}{\beta_1},\frac{1}{\beta_1},\cdots,\frac{1}{\beta_n}\right), \quad i=1,2,\cdots,n. \tag{29}$$

Then, $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ *is non-negative.*

In order for firms have incentives to produce, condition (29) pus a lower bound for the ownership share each firm i needs to retain. This lower bound depends on the lowest β_i among all firms. Theorem 5 and 6 can be shown to hold under (29). In other words, as cross-holdings increase all firms choose to produce more and social welfare increases.

4.4.2 The Case with Different β_i and Different γ_{ij}

When the representative consumer's utility is (20), another application of Theorem 4 establishes

Corollary 4. *Suppose that a given cross-holding matrix* $\mathbf{\Delta}$ *satisfies*

$$\delta_{ii} > -\frac{n-1}{2}\max\left(\frac{\gamma_{i1}}{\beta_1},\frac{\gamma_{i2}}{\beta_2},\cdots,\frac{\gamma_{in}}{\beta_n}\right), \quad i=1,2,\cdots,n. \tag{30}$$

Then, $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ *is non-negative.*

Under condition (30), the upper bound of firm i's retained share depends on the largest ratio of firm i's effect on a firm's price to the slope of own-quantity of this firm. Theorem 5 and 6 can be shown to hold under condition (30). It follows that, as cross-holdings increase all firms expand their products and social welfare rises.

4.5 Discussion

The performance of a non-cooperative oligopoly model with cross-holdings depends on the interaction of two effects. First, cross-holdings allow linked firms to absorb a negative or positive externality. Second, cross-holdings elicit a spiral of responses from rival firms. When products are complementary, this response tends to be beneficial to all firms because reaction functions are upward sloping. In such an environment, firms are likely to have incentives to engage in cross-holdings. When products are substitutable, on the other hand, the response of all other firms tend to hurt the acquiring firm because in this environment, the reaction functions are typically downward sloping.

The welfare effects in the two situations are also very different. It is shown that cross-holdings generally increase social welfare when products are complementary, but decrease social welfare when the acquiring firm is big in an industry with substitutable products. Thus, from a social point of view, our analysis suggests that cross-holdings should be promoted in industries with complementary products, while they should be discouraged in industries with substitutable products.

There is a fundamental asymmetry between substitutable products and complementary products. For substitutability, the valid range for γ is indeed $(0, \beta]$, and this range is independent of the number of firms. However, for complementarity, the valid range is $(\frac{\beta}{1-n}, 0)$, which monotonically shrinks with the number of firms, and converges to the empty set as the number of firms goes to positive infinity. This fundamental difference between the two products, makes the competitive and welfare effects of cross-holdings different.

5 Bertrand Competition

Kreps and Scheinkman (1983) argued that whether firms compete in quantities or prices is ultimately an empirical question. In the real world, both Cournot and Bertrand behaviors are observed. For example, farmers set quantities at local farmers' markets, while restaurants set prices. These empirical observations indicate that the study of cross-holdings in Bertrand model with product differentiation is also necessary.

To get the analysis under price competition, we take advantage of the duality structure of Cournot and Bertrand competition in our differentiated product setting. This duality was first pointed out by Sonnenschein (1968) in a non-differentiated framework, and extended by Singh and Vives (1984). We will use the symmetric utility provided in (22) to analyze the competitive and welfare effects of cross-holdings under price competition. Under symmetry, firm i's demand function is

$$q_i = a - bp_i + g\sum_{j\neq i}^{n} p_j, \quad i = 1, 2, \cdots, n,$$

where

$$a = \frac{\alpha}{[\beta + (n-1)\gamma]}$$
$$b = \frac{\beta + (n-2)\gamma}{[\beta + (n-1)\gamma](\beta - \gamma)}$$
$$g = \frac{\gamma}{[\beta + (n-1)\gamma](\beta - \gamma)}$$

a, b and g are always positive for substitutable products. Recall that the strictly concavity of utility requires that $\gamma \in (\frac{\beta}{1-n}, 0)$ when products are complementary. In that case, a and b are positive while g is negative. Under Bertrand competition, firm i's first-order is given by

$$b(p_i - c_i) - \sum_{j \neq i} g(p_j - c_j) = \frac{g}{2} \sum_{j \neq i}^{n} \frac{\delta_{ij}}{\delta_{ii}} (p_j - c_j) + \frac{a + bc_i - g \sum_{j \neq i}^{n} c_j}{2}. \quad (31)$$

In our setting, it turns out that Cournot (Bertrand) competition with substitutable products is the dual of Bertrand (Cournot) competition with complementary products. This means that they share similar strategic properties. It is a matter of interchanging prices and quantities. A useful corollary is that one only needs to make computations or prove theorems for one type of competition (Cournot or Bertrand) or for one type of product (substitutable or complementary); the other cases follow by duality. Therefore, to each of the results of Section III and IV corresponds a dual theorem dealing with Bertrand competition.

5.1 The Case with Substitutable Products

Bertrand competition with substitutable products is the dual of Cournot competition with complementary goods. It follows that, a similar theorem to Theorem 4 establishes the competitive effects of cross-holdings in Bertrand competition with substitutable products.

Theorem 7. *Suppose cross-holding matrix satisfies*

$$\delta_{ii} > -\frac{n-1}{2(\frac{\beta}{\gamma} + n - 2)}, \quad i = 1, 2, \cdots, n. \quad (32)$$

Then, all firms raise the prices for their products as cross-holdings increase.

The upper bound in (32) takes value from 0 to 1/2. In Bertrand competition with substitutable products, firms' reaction functions are upward sloping. Without cross-holdings, when a firm contemplates raising price, it does not care about the positive externality it would confer upon other firms. But with cross-holdings, this externality will be partially internalized, and all firms set higher prices. Our result is similar to the findings in Dietzenbacher et al. (2000). They found that competition is reduced due to shareholding interlocks in Bertrand competition. As an empirical example, the Dutch financial sector is used in their paper. Comparing the case of shareholding with the case of no-shareholding, the price-cost margins are found to be up to 2% higher in a Bertrand oligopoly.

The change in social welfare due to a small increasing of firm 1's ownership share in firm k, is now defined as

$$\frac{\partial W(\mathbf{\Delta})}{\partial \delta_{1k}} = \sum_{i=1}^{n} \{[-b(p_i - c_i) + g \sum_{j \neq i} (p_j - c_j)] \frac{\partial p_i}{\partial \delta_{1k}}\} \tag{33}$$

By Theorem 7, all firms raise prices as cross-holdings increase. Consequently, it may not be socially desirable. Theorem 8 below provides a confirmation.

Theorem 8. *Let cross-holding matrix satisfies*

$$\delta_{ii} > -\frac{n-1}{2(\frac{\beta}{\gamma} + n - 2)}, \quad i = 1, 2, \cdots, n.$$

Then, social welfare decreases as cross-holdings increase .

According to Singh and Vives (1984), $a + bc_i - g \sum_{j \neq i}^{n} c_j$ in (31) is positive. Consequently, the right side in (31) is positive. It concludes that when one firm raises ownership shares in its competitors, social welfare falls. Condition (32) is weakened as products become less substitutable. It follows that, when products are less substitutable, social welfare are more likely to decrease as cross-holdings increases.

5.2 The Case with Complementary Products

Bertrand competition with complementary products is the dual of Cournot competition with substitutable goods. It follows that, similar theorems to Theorem 1 and 2 establish the competitive effects of cross-holdings in Bertrand competition with complementary products.

Theorem 9. *Suppose cross-holding matrix satisfies*

$$\delta_{ii} > \frac{1}{2(\frac{\beta}{\gamma} + n - 2)}, \quad i = 1, 2, \cdots, n.$$

Then, the total price decreases as cross-holdings increase.

Theorem 10. *Let cross-holding matrix be given. Then,* $\frac{\partial p_i}{\partial \delta_{ik}} < 0$ *under any following two conditions*

$$(i) \quad \delta_{kh} = 0, \quad \forall h \neq k; \tag{34}$$

$$(ii) \quad \sum_{j \neq k} \delta_{ij} + (n-2)\delta_{ii} > (\frac{\beta}{\gamma} + n - 2)(\delta_{ii} + \delta_{ik}). \tag{35}$$

In Bertrand competition with complementary products, firms' reaction functions are downward sloping. When one or more firms raises its ownership share in rivals, the total price falls. Under condition (34) or (35), when a single firm raises ownership share in a complementary firm, the price of this acquiring firm's product goes down.

Theorem 11. *Let firm i be a firm such that*

$$p_i^\star - c_i = \max_{1 \leq j \leq n} (p_j^\star - c_j).$$

Suppose that either condition (34) or (35) is satisfied. Then,

$$\frac{\partial W(\mathbf{\Delta})}{\partial \delta_{ij}} > 0.$$

When a big firm acquires partial ownership in other firms, social welfare rises. A policy implication is that, big firms should be encouraged from raising their ownership shares in their competitors under price competition.

5.3 Discussion

Whether price or quantity competition is more efficient has been widely debated. Singh and Vives (1984) studied this conjecture in a differentiated duopoly and conclude that Cournot competition entails higher prices and profits than Bertrand competition, whereas both firms output and social welfare are higher under Bertrand competition. Hackner (2000) extended the work in Singh and Vives to allow for an arbitrary number of firms. He concludes that the results in Singh and Vives are sensitive to the duopoly assumption. Hence, it is not evident which type of competition is more efficient. Hsu and Wang (2005) showed that both consumer surplus and total surplus are higher under price competition than under quantity competition, regardless of whether goods are substitutable or complementary.

This paper has provided the competitive and welfare effects of cross-holdings with product differentiation for both quantity and price competition. It is not our intention to analyze the efficiency between Cournot competition and Bertrand competitions. However, in each type of competition, with each kind of product, one can find how changes in cross-holdings affect an industry performance.

6 Conclusion

In this paper we have analyzed the competitive and welfare effects of cross-holdings in oligopolies with product differentiation. Our results show that the standard anti-competitive effect, that cross-holdings in general make industries more concentrated, is robust with respect to product differentiation. In the case with product substitutability, we showed that this anti-competitive effect is smaller the less substitutable the products are. For the case with product complementarity, we showed that cross-holdings induce firms to cooperate, and this response is easier to be achieved the more differentiated the products are.

We also conducted welfare analysis. We showed that cross-holdings are more likely to be harmful the more differentiated the products are when the products are substitutable. In comparison, with complementary products, everyone is generally better off with cross-holdings. Additionally this welfare implication is more likely to be achieved when products are less complementary.

Moreover, we used the duality structure of Cournot and Bertrand competition to analyze the competitive and welfare effects of cross-holdings with product differentiation in Bertrand competition. For the case with complementary products, the total price decreases as cross-holdings increase. Big firm's increasing ownership in rivals makes social welfare fall. It was shown that all firms charge higher prices for their products and social welfare decreases when cross-holdings increase, in industries with substitutable products..

Our analysis has both empirical and policy implications, which we will pursue in future research.

Appendix: Proof

Proof of Lemma 1

Proof. Assume $\mathbf{I}-\mathbf{S}$ is not invertible. Then, it must be true that the column vectors in $\mathbf{I}-\mathbf{S}$ are linearly dependent, which means that there exists a non-zero column vector, $\boldsymbol{\lambda}' = (\lambda_1, \lambda_1, \cdots, \lambda_n)$, such that $(\mathbf{I}-\mathbf{S}) \cdot \boldsymbol{\lambda} = 0$. Notice that $(\mathbf{I}-\mathbf{S})$ is positive, so λ_i can not be all negative. Assume there exist at least one element in $\boldsymbol{\lambda}$ is negative. It follows that there exist $\lambda_{j_1}, \lambda_{j_2}, \cdots \lambda_{j_{n_1}}$ such that $\lambda_{j_k} < 0$ for $1 \leq k \leq n_1 < n$. Consequently, the rest $n - n_1$ elements are non-negative, which means that $\lambda_{i_h} \geq 0$ for $1 \leq h \leq n_2 = n - n_1$. First, take the

j_kth equations in $(\mathbf{I}-\mathbf{S})\cdot\boldsymbol{\lambda}=0$, $1\le k\le n_1<n$. Second, multiply each equation by $\frac{2\beta}{\gamma}\delta_{j_k}$, $1\le k\le n_1<n$. Next, add these n_1 equations yielding

$$\begin{aligned}
&(\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k=1}^{n_1}\delta_{j_ki_1})\lambda_{i_1}+\cdots+(\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k=1}^{n_1}\delta_{j_ki_{n_2}})\lambda_{i_{n_2}}+[\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k\neq 1}^{n_1}\delta_{j_kj_1}\\
&+(\frac{2\beta}{\gamma}-1)\delta_{j_1j_1}]\lambda_{j_1}+\cdots+[\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_1}}+(\frac{2\beta}{\gamma}-1)\delta_{j_{n_1}j_{n_1}}]\lambda_{j_{n_1}}\\
=&(\sum_{k=1}^{n_1}\delta_{j_kj_k})(\sum_{k=1}^{n_1}\lambda_{j_k}+\sum_{h=1}^{n_2}\lambda_{i_h})+\sum_{k=1}^{n_1}\delta_{j_ki_1}\lambda_{i_1}+\cdots+\sum_{k=1}^{n_1}\delta_{j_ki_{n_2}}\lambda_{i_{n_2}}\\
&[\sum_{k\neq 1}^{n_1}\delta_{j_kj_1}+(\frac{2\beta}{\gamma}-1)\delta_{j_1j_1}]\lambda_{j_1}+\cdots+[\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_1}}+(\frac{2\beta}{\gamma}-1)\delta_{j_{n_1}j_{n_1}}]\lambda_{j_{n_1}}=0.
\end{aligned}$$

If $\sum_{k=1}^{n_1}\lambda_{j_k}+\sum_{h=1}^{n_2}\lambda_{i_h}\le 0$, the following must be true

$$\begin{aligned}
&\sum_{k=1}^{n_1}\delta_{j_ki_1}\lambda_{i_1}+\cdots+\sum_{k=1}^{n_1}\delta_{j_ki_{n_2}}\lambda_{i_{n_2}}+[\sum_{k\neq 1}^{n_1}\delta_{j_kj_1}+(\frac{2\beta}{\gamma}-1)\delta_{j_1j_1}]\lambda_{j_1}\\
&+\cdots+[\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_1}}+(\frac{2\beta}{\gamma}-1)\delta_{j_{n_1}j_{n_1}}]\lambda_{j_{n_1}}\ge 0.
\end{aligned}$$

However, under condition (6)

$$\begin{aligned}
&\sum_{k=1}^{n_1}\delta_{j_ki_1}\lambda_{i_1}+\cdots+\sum_{k=1}^{n_1}\delta_{j_ki_{n_2}}\lambda_{i_{n_2}}+[\sum_{k\neq 1}^{n_1}\delta_{j_kj_1}+(\frac{2\beta}{\gamma}-1)\delta_{j_1j_1}]\lambda_{j_1}\\
&+\cdots+[\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_1}}+(\frac{2\beta}{\gamma}-1)\delta_{j_{n_1}j_{n_1}}]\lambda_{j_{n_1}}\\
\le&(1-\delta_{i_1i_1})\lambda_{i_1}+\cdots+(1-\delta_{i_{n_1}i_{n1}})\lambda_{i_{n_1}}+(1-\frac{\gamma}{2\beta})\lambda_{j_1}+\cdots+(1-\frac{\gamma}{2\beta})\lambda_{j_{n_1}}\\
&+(\sum_{k\neq 1}^{n_1}\delta_{j_kj_1})\lambda_{j_1}+\cdots+(\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_1}}\lambda_{j_{n_1}})\\
<&(1-\frac{\gamma}{2\beta})(\sum_{k=1}^{n_1}\lambda_{j_k}+\sum_{h=1}^{n_2}\lambda_{i_h})+(\sum_{k\neq 1}^{n_1}\delta_{j_kj_1})\lambda_{j_1}+\cdots+(\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_1}}\lambda_{j_{n_1}})<0,
\end{aligned}$$

which is a contradiction. Then, it must be true that

$$\sum_{k=1}^{n_1}\lambda_{j_k}+\sum_{h=1}^{n_2}\lambda_{i_h}>0.$$

Now first, take the i_hth equations in $(\mathbf{I}-\mathbf{S})\cdot\boldsymbol{\lambda}=0$, $1\le h\le n_2$. Second, multiply each equation by $\frac{2\beta}{\gamma}\delta_{i_h}$, $1\le h\le n_2$. Next, add these n_2 equations yielding

$$\begin{aligned}
&[\sum_{h=1}^{n_2}\delta_{i_hi_h}+\sum_{h\neq 1}^{n_2}\delta_{i_hi_1}+(\frac{2\beta}{\gamma}-1)\delta_{i_1i_1}]\lambda_{i_1}+\cdots+[\sum_{h=1}^{n_2}\delta_{i_hi_h}+\sum_{h=1}^{n_2-1}\delta_{i_hi_{n_2}}+\\
&(\frac{2\beta}{\gamma}-1)\delta_{i_{n_2}i_{n_2}}]\lambda_{i_{n_2}}+(\sum_{h=1}^{n_2}\delta_{i_hi_h}+\sum_{h=1}^{n_2}\delta_{i_hj_1})\lambda_{j_1}+\cdots+(\sum_{h=1}^{n_2}\delta_{i_hi_h}+\sum_{h=1}^{n_2}\delta_{i_hj_{n_1}})\lambda_{j_{n_1}}\\
&=(\sum_{h=1}^{n_2}\delta_{i_hi_h})(\sum_{k=1}^{n_1}\lambda_{j_k}+\sum_{h=1}^{n_2}\lambda_{i_h})+[\sum_{h\neq 1}^{n_2}\delta_{i_hi_1}+(\frac{2\beta}{\gamma}-1)\delta_{i_1i_1}]\lambda_{i_1}+\cdots+\\
&[\sum_{h=1}^{n_2-1}\delta_{i_hi_{n_2}}+(\frac{2\beta}{\gamma}-1)\delta_{i_{n_2}i_{n_2}}]\lambda_{i_{n_2}}+(\sum_{h=1}^{n_2}\delta_{i_hj_1})\lambda_{j_1}+\cdots+(\sum_{h=1}^{n_2}\delta_{i_hj_{n_1}})\lambda_{j_{n_1}}=0.
\end{aligned}$$

Sicne $\sum_{k=1}^{n_1}\lambda_{j_k}+\sum_{h=1}^{n_2}\lambda_{i_h}>0$, It must be true that

$$\begin{aligned}
&[\sum_{h\neq 1}^{n_2}\delta_{i_hi_1}+(\frac{2\beta}{\gamma}-1)\delta_{i_1i_1}]\lambda_{i_1}+\cdots+[\sum_{h=1}^{n_2-1}\delta_{i_hi_{n_2}}+(\frac{2\beta}{\gamma}-1)\delta_{i_{n_2}i_{n_2}}]\lambda_{i_{n_2}}\\
&+(\sum_{h=1}^{n_2}\delta_{i_hj_1})\lambda_{j_1}+\cdots+(\sum_{h=1}^{n_2}\delta_{i_hj_{n_1}})\lambda_{j_{n_1}}\le 0.
\end{aligned}$$

However,

$$\begin{aligned}
&[\sum_{h\neq 1}^{n_2}\delta_{i_hi_1}+(\frac{2\beta}{\gamma}-1)\delta_{i_1i_1}]\lambda_{i_1}+\cdots+[\sum_{h=1}^{n_2-1}\delta_{i_hi_{n_2}}+(\frac{2\beta}{\gamma}-1)\delta_{i_{n_2}i_{n_2}}]\lambda_{i_{n_2}}\\
&+(\sum_{h=1}^{n_2}\delta_{i_hj_1})\lambda_{j_1}+\cdots+(\sum_{h=1}^{n_2}\delta_{i_hj_{n_2}})\lambda_{j_{n_2}}\\
&\ge[(\frac{2\beta}{\gamma}-1)\delta_{i_1i_1}]\lambda_{i_1}+\cdots+[(\frac{2\beta}{\gamma}-1)\delta_{i_{n_2}i_{n_2}}]\lambda_{i_{n_2}}+\sum_{k=1}^{n_1}[(1-\delta_{j_kj_k})\lambda_{j_k}]\\
&\ge(1-\frac{\gamma}{2\beta})\lambda_{i_1}+\cdots+(1-\frac{\gamma}{2\beta})\lambda_{i_{n_2}}+(1-\frac{\gamma}{2\beta})\lambda_{j_1}+\cdots+(1-\frac{\gamma}{2\beta})\lambda_{j_{n_1}}\\
&=(1-\frac{\gamma}{2\beta})(\sum_{k=1}^{n_1}\lambda_{j_k}+\sum_{h=1}^{n_2}\lambda_{i_h})>0,
\end{aligned}$$

which is a contradiction, thus all elements in λ are non-negative. Since $\mathbf{I}-\mathbf{S}$ is positive, all λ_i has to be zero. Therefore, the column vectors in $\mathbf{I}-\mathbf{S}$ are linearly independent. It follows that, $\mathbf{I}-\mathbf{S}$ is invertible.

$\square$

Proof of Proposition 1

Proof. According to Lemma 1, $\mathbf{I}-\mathbf{S}$ is invertible under condition (8). It follows that the transpose of $\mathbf{I}-\mathbf{S}$ is also invertible under (8). Denote $(\mathbf{A^T})^{-\mathbf{1}} = (z_{ij})$. To prove $\sum_{j=1}^{n} t_{ij} > 0$ is equivalent to prove $\sum_{j=1}^{n} z_{ji} > 0$, $i = 1, 2\cdots, n$, since $(\mathbf{A^T})^{-\mathbf{1}} = (\mathbf{A^{-1}})^{\mathbf{T}}$. Given $\mathbf{A^T}$ is a positive matrix, there exists at least one element is positive from ith column in $(\mathbf{A^T})^{-\mathbf{1}}$. Assume there exist $z_{j_1 i}, z_{j_2 i}, \cdots, z_{j_{n_2} i}$ such that $z_{j_k i} > 0$ for $1 \leq k \leq n_2 < n$. Then the rest $n - n_2$ elements are non-positive such that $z_{i_h i} \leq 0$ for $0 \leq h \leq n_1 < n$. Use the i_hth row in $\mathbf{A^T}$ times the ith column in $(\mathbf{A^T})^{-\mathbf{1}}$, $0 \leq h \leq n_1 < n$. Then, add these n_1 equations yielding

$$
\begin{aligned}
&\frac{\gamma}{2\beta}(n_1 + \frac{\sum_{h\neq 1}^{n_1} \delta_{i_1 i_h}}{\delta_{i_1 i_1}} + \frac{2\beta}{\gamma} - 1)z_{i_1 i} + \cdots + \frac{\gamma}{2\beta}(n_1 + \frac{\sum_{h=1}^{n_1-1} \delta_{i_{n_1} i_h}}{\delta_{i_{n_1} i_{n_1}}} + \frac{2\beta}{\gamma} - 1)z_{i_{n_1} i} \\
&+ \frac{\gamma}{2\beta}[n_1 + \frac{\sum_{h=1}^{n_1} \delta_{j_1 i_h}}{\delta_{j_1 j_1}}]z_{j_1 i} + \cdots + \frac{\gamma}{2\beta}[n_1 + \frac{\sum_{h=1}^{n_1} \delta_{j_{n_2} i_h}}{\delta_{j_{n_2} j_{n_2}}}]z_{j_{n_2} i} \\
=&n_1 \frac{\gamma}{2\beta}(z_{i_1 i} + \cdots + z_{i_{n_1} i} + z_{j_1 i} + \cdots + z_{j_{n_2} i}) + \frac{\gamma}{2\beta}(\frac{\sum_{h\neq 1}^{n_1} \delta_{i_1 i_h}}{\delta_{i_1 i_1}} + \frac{2\beta}{\gamma} - 1)z_{i_1 i} + \cdots + \\
&\frac{\gamma}{2\beta}(\frac{\sum_{h=1}^{n_1-1} \delta_{i_{n_1} i_h}}{\delta_{i_{n_1} i_{n_1}}} + \frac{2\beta}{\gamma} - 1)z_{i_{n_1} i} + \frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1} \delta_{j_1 i_h}}{\delta_{j_1 j_1}}]z_{j_1 i} + \cdots + \frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1} \delta_{j_{n_2} i_h}}{\delta_{j_{n_2} j_{n_2}}} z_{j_{n_2} i}] \\
=&0 \quad \text{or} \quad 1.
\end{aligned}
$$

If $\sum_{j=1}^{n} z_{ji} \leq 0$, it must be true that

$$
\begin{aligned}
&\frac{\gamma}{2\beta}(\frac{\sum_{h\neq 1}^{n_1} \delta_{i_1 i_h}}{\delta_{i_1 i_1}} + \frac{2\beta}{\gamma} - 1)z_{i_1 i} + \cdots + \frac{\gamma}{2\beta}(\frac{\sum_{h=1}^{n_1-1} \delta_{i_{n_1} i_h}}{\delta_{i_{n_1} i_{n_1}}} + \frac{2\beta}{\gamma} - 1)z_{i_{n_1} i} \\
&+ \frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1} \delta_{j_1 i_h}}{\delta_{j_1 j_1}}]z_{j_1 i} + \cdots + \frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1} \delta_{j_{n_2} i_h}}{\delta_{j_{n_2} j_{n_2}}}]z_{j_{n_2} i} \geq 0.
\end{aligned}
$$

However,

$$
\begin{aligned}
&\frac{\gamma}{2\beta}(\frac{\sum_{h\neq 1}^{n_1}\delta_{i_1 i_h}}{\delta_{i_1 i_1}}+\frac{2\beta}{\gamma}-1)z_{i_1 i}+\cdots+\frac{\gamma}{2\beta}(\frac{\sum_{h=1}^{n_1-1}\delta_{i_{n_1} i_h}}{\delta_{i_{n_1} i_{n_1}}}+\frac{2\beta}{\gamma}-1)z_{i_{n_1} i}\\
&+\frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1}\delta_{j_1 i_h}}{\delta_{j_1 j_1}}]z_{j_1 i}+\cdots+\frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1}\delta_{j_{n_2} i_h}}{\delta_{j_{n_2} j_{n_2}}}]z_{j_{n_2} i}\\
&\leq \frac{\gamma}{2\beta}(\frac{2\beta}{\gamma}-1)(\sum_{h=1}^{n_1}z_{i_h i})+\frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1}\delta_{j_1 i_h}}{\delta_{j_1 j_1}}]z_{j_1 i}+\cdots+\frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1}\delta_{j_{n_2} i_h}}{\delta_{j_{n_2} j_{n_2}}}]z_{j_{n_2} i}\\
&< \frac{\gamma}{2\beta}[(\frac{2\beta}{\gamma}-1)z_{i_1 i}+\cdots+(\frac{2\beta}{\gamma}-1)z_{i_{n_1} i}+(\frac{2\beta}{\gamma}-1)z_{j_1 i}+\cdots+(\frac{2\beta}{\gamma}-1)z_{j_{n_2} i}]\\
&= \frac{\gamma}{2\beta}[(\frac{2\beta}{\gamma}-1)(z_{i_1 i}+\cdots+z_{i_{n_1} i}+z_{j_1 i}+\cdots+z_{j_{n_2} i})]\leq 0
\end{aligned}
$$

which is a contradiction. Therefore,

$$\sum_{j=1}^{n} z_{ji} > 0.$$

It follows that,

$$\sum_{j=1}^{n} t_{ij} > 0.$$

□

Proof of Lemma 2

Proof. According to Lemma 1, $\mathbf{I}-\mathbf{S}$ is invertible under condition (8).

Prove $t_{ii} > 0$ using contradiction.

Suppose that $t_{ii} \leq 0$. It must be true that at least one element in the ith column is positive. Assume there exist $t_{j_1 i}, t_{j_2 i}, \cdots, t_{j_{n_1} i}$ such that $t_{j_k i} > 0$, $1 \leq k \leq n_1 < n-1$. It follows that the rest $n-n_1-1$ elements(other than t_{ii}) from the ith column in $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ is non-positive, such as $t_{i_h i} \leq 0$, $0 \leq h \leq n_2 \leq n-1-n_1$. First, use the j_kth row in $(\mathbf{I}-\mathbf{S})$ times the ith column in $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ and multiply each equation by $\frac{2\beta}{\gamma}\delta_{j_k}$, $1 \leq k \leq n_1 < n-1$.

Second add these n_1 equations yielding

$$
\begin{aligned}
&(\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k=1}^{n_1}\delta_{j_ki})t_{ii}+(\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k=1}^{n_1}\delta_{j_ki_1})t_{i_1i}+\cdots+(\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k=1}^{n_1}\delta_{j_ki_{n_2}})t_{i_{n_2}i}\\
&+[\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k\neq 1}^{n_1}\delta_{j_kj_1}+(\frac{2\beta}{\gamma}-1)\delta_{j_1j_1}]t_{j_1i}+\cdots+[\sum_{k=1}^{n_1}\delta_{j_kj_k}+\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_2}}+(\frac{2\beta}{\gamma}-1)\delta_{j_{n_1}j_{n_1}}]t_{j_{n_1}i}\\
=&(\sum_{k=1}^{n_1}\delta_{j_kj_k})(t_{ii}+t_{i_1i}+\cdots+t_{i_{n_2}i}+t_{j_1i}+\cdots+t_{j_{n_1}i})+(\sum_{k=1}^{n_1}\delta_{j_ki})t_{ii}\\
&+(\sum_{k=1}^{n_1}\delta_{j_ki_1})t_{i_1i}+\cdots+(\sum_{k=1}^{n_1}\delta_{j_ki_{n_2}})t_{i_{n_2}i}+[\sum_{k\neq 1}^{n_1}\delta_{j_ki}+(\frac{2\beta}{\gamma}-1)\delta_{j_1j_1}]t_{j_1i}\\
&+\cdots+[\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_1}}+(\frac{2\beta}{\gamma}-1)\delta_{j_{n_1}j_{n_1}}]t_{j_{n_2}i}=0.
\end{aligned}
$$

If $t_{ii}+\sum_{k=1}^{n_1}t_{j_ki}+\sum_{h=1}^{n_2}t_{i_hi}\geq 0$, then the following must be true

$$
\begin{aligned}
&(\sum_{k=1}^{n_1}\delta_{j_ki})t_{ii}+(\sum_{k=1}^{n_1}\delta_{j_ki_1})t_{i_1i}+\cdots+(\sum_{k=1}^{n_1}\delta_{j_ki_{n_2}})t_{i_{n_2}i}+\\
&[\sum_{k\neq 1}^{n_1}\delta_{j_kj_1}+(\frac{2\beta}{\gamma}-1)\delta_{j_1j_1}]t_{j_1i}+\cdots+[\sum_{k=1}^{n_1-1}\delta_{j_kj_{n_1}}+(\frac{2\beta}{\gamma}-1)\delta_{j_{n_1}j_{n_1}}]t_{j_{n_1}i}\leq 0.
\end{aligned}
$$

However,

$$
\begin{aligned}
&(\sum_{k=1}^{n_1}\delta_{j_k i})t_{ii} + (\sum_{k=1}^{n_1}\delta_{j_k i_1})t_{i_1 i} + \cdots + (\sum_{k=1}^{n_1}\delta_{j_k i_{n_2}})t_{i_{n_2} i} \\
&\quad + [\sum_{k\neq 1}^{n_1}\delta_{j_k j_1} + (\frac{2\beta}{\gamma}-1)\delta_{j_1 j_1}]t_{j_1 i} + \cdots + [\sum_{k=1}^{n_1-1}\delta_{j_k j_{n_2}} + (\frac{2\beta}{\gamma}-1)\delta_{j_{n_1} j_{n_1}}]t_{j_{n_1} i} \\
>&(1-\delta_{ii})t_{ii} + (1-\delta_{i_1 i_1})t_{i_1 i} + \cdots + (1-\delta_{i_{n_1} i_1})t_{i_{n_2} i} \\
&\quad + [\sum_{k\neq 1}^{n_1}\delta_{j_k j_1} + (\frac{2\beta}{\gamma}-1)\delta_{j_1 j_1}]t_{j_1 i} + \cdots + [\sum_{k=1}^{n_1-1}\delta_{j_k j_{n_2}} + (\frac{2\beta}{\gamma}-1)\delta_{j_{n_1} j_{n_1}}]t_{j_{n_1} i} \\
>&(1-\frac{\gamma}{2\beta})t_{ii} + (1-\frac{\gamma}{2\beta})t_{i_1 i} + \cdots + (1-\frac{\gamma}{2\beta})t_{i_{n_2} i} + [\sum_{k\neq 1}^{n_1}\delta_{j_k j_1} + (1-\frac{\gamma}{2\beta})]t_{j_1 i} \\
&\quad + \cdots + [\sum_{k=1}^{n_1-1}\delta_{j_k j_{n_1}} + (1-\frac{\gamma}{2\beta})]t_{j_{n_1} i} \\
=&(1-\frac{\gamma}{2\beta})(t_{ii} + \sum_{k=1}^{n_1}t_{j_k i} + \sum_{h=1}^{n_2}t_{i_h i}) + (\sum_{k\neq 1}^{n_1}\delta_{j_k j_1})t_{j_1 i} + \cdots + (\sum_{k=1}^{n_1-1}\delta_{j_k j_{n_1}})t_{j_{n_1} i} > 0,
\end{aligned}
$$

which is a contradiction. Then, it is must be true that $t_{ii}+\sum_{k=1}^{n_1} t_{j_k i}+\sum_{h=1}^{n_2} t_{i_h i} < 0$. Now use the ith and i_hth row in $(\mathbf{I}-\mathbf{S})$ times the ith column in $(\mathbf{I}-\mathbf{S})^{\mathbf{-1}}$ and multiply each equation by $\frac{2\beta}{\gamma}\delta_{i_h}$, $0 \le h \le n_2 \le n-1-n_1$. Next, add these n_2 equations yielding

$$
\begin{aligned}
&[\sum_{h=1}^{n_2} \delta_{i_h i_h} + \delta_{ii} + \sum_{h=1}^{n_2} \delta_{i_h i} + (\frac{2\beta}{\gamma} - 1)\delta_{ii}]t_{ii} + [\sum_{h=1}^{n_2} \delta_{i_h i_h} + \delta_{ii} + \sum_{h \neq 1}^{n_2} \delta_{i_h i_1} + \delta_{i i_1} + \\
&(\frac{2\beta}{\gamma} - 1)\delta_{i_1 i_1}]t_{i_1 i} + \cdots + [\sum_{h=1}^{n_2} \delta_{i_h i_h} + \delta_{ii} + \sum_{h=1}^{n_2-1} \delta_{i_h i_{n_2}} + \delta_{i i_{n_2}} + (\frac{2\beta}{\gamma} - 1)\delta_{i_{n_2} i_{n_2}}]t_{i_{n_2} i} \\
&+ (\sum_{h=1}^{n_2} \delta_{i_h i_h} + \delta_{ii} + \sum_{h=1}^{n_2} \delta_{i_h j_1})t_{j_1 i} + \cdots + (\sum_{h=1}^{n_2} \delta_{i_h i_h} + \delta_{ii} + \sum_{h=1}^{n_2} \delta_{i_h j_{n_1}})t_{j_{n_1} i} \\
&= (\sum_{h=1}^{n_2} \delta_{i_h i_h} + \delta_{ii})(t_{ii} + \sum_{k=1}^{n_1} t_{j_k i} + \sum_{h=1}^{n_2} t_{i_h i}) + [\sum_{h=1}^{n_2} \delta_{i_h i} + (\frac{2\beta}{\gamma} - 1)\delta_{ii}]t_{ii} \\
&+ [\sum_{h \neq 1}^{n_2} \delta_{i_h i_1} + \delta_{i i_1} + (\frac{2\beta}{\gamma} - 1)\delta_{i_1 i_1}]t_{i_1 i} + \cdots + [\sum_{h=1}^{n_2-1} \delta_{i_h i_{n_2}} + \delta_{i i_{n_2}} + (\frac{2\beta}{\gamma} - 1)\delta_{i_{n_2}}]t_{i_{n_2} i} \\
&+ (\delta_{i j_1} + \sum_{h=1}^{n_2} \delta_{i_h j_1})t_{j_1 i} + \cdots + (\delta_{i j_{n_1}} + \sum_{h=1}^{n_2} \delta_{i_h j_{n_1}})t_{j_{n_1} i} = \frac{2\beta}{\gamma} \sum_{h=1}^{n_2} \delta_{i_h}.
\end{aligned}
$$

Since $t_{ii} + \sum_{k=1}^{n_1} t_{j_k i} + \sum_{h=1}^{n_2} t_{i_h i} < 0$, the following must be true

$$
\begin{aligned}
&[\sum_{h=1}^{n_2} \delta_{i_h i} + (\frac{2\beta}{\gamma} - 1)\delta_{ii}]t_{ii} + [\sum_{h \neq 1}^{n_2} \delta_{i_h i_1} + \delta_{i i_1} + (\frac{2\beta}{\gamma} - 1)\delta_{i_1 i_1}]t_{i_1 i} + \cdots + \\
&[\sum_{h=1}^{n_2-1} \delta_{i_h i_{n_2}} + \delta_{i i_{n_2}} + (\frac{2\beta}{\gamma} - 1)\delta_{i_{n_2}}]t_{i_{n_2} i} + (\delta_{i j_1} + \sum_{h=1}^{n_2} \delta_{i_h j_1})t_{j_1 i} + \cdots + \\
&(\delta_{i j_{n_1}} + \sum_{h=1}^{n_2} \delta_{i_h j_{n_1}})t_{j_{n_1} i} > 0.
\end{aligned}
$$

However,

$$
\begin{aligned}
&[\sum_{h=1}^{n_2} \delta_{i_h i} + (\frac{2\beta}{\gamma} - 1)\delta_{ii}]t_{ii} + [\sum_{h \neq 1}^{n_2} \delta_{i_h i_1} + \delta_{i i_1} + (\frac{2\beta}{\gamma} - 1)\delta_{i_1 i_1}]t_{i_1 i} + \cdots + \\
&[\sum_{h=1}^{n_2-1} \delta_{i_h i_{n_2}} + \delta_{i i_{n_2}} + (\frac{2\beta}{\gamma} - 1)\delta_{i_{n_2}}]t_{i_{n_2} i} + \sum_{k=1}^{n_1}[(\delta_{i j_k} + \sum_{h=1}^{n_2} \delta_{i_h j_k})t_{j_k i}] \\
<&(1 - \frac{\gamma}{2\beta})t_{ii} + (1 - \frac{\gamma}{2\beta})t_{i_1 i} + \cdots + (1 - \frac{\gamma}{2\beta})t_{i_{n_2} i} + \sum_{k=1}^{n_1}[(1 - \frac{\gamma}{2\beta})t_{j_k i}] \\
=&(1 - \frac{\gamma}{2\beta})(t_{ii} + \sum_{k=1}^{n_1} t_{j_k i} + \sum_{h=1}^{n_2} t_{i_h i}) < 0
\end{aligned}
$$

which is a contradiction. Hence if $t_{ii} \leq 0$ we have proved that both $t_{ii}+\sum_{k=1}^{n_1} t_{j_k i}+\sum_{h=1}^{n_2} t_{i_h i} \geq 0$ and $t_{ii} + \sum_{k=1}^{n_1} t_{j_k i} + \sum_{h=1}^{n_2} t_{i_h i} < 0$ are contradictory, which means that such matrix does not exist! Therefore,

$$t_{ii} > 0$$

Let's use contradiction to prove that $\sum_{j=1}^{n} t_{ji} > 0$. Assume that $\sum_{j=1}^{n} t_{ji} \leq 0$, then there exist at least one element such that $t_{j_k i} \leq 0$ for $1 \leq k \leq n_1 < n-1$ then the rest $n - n_1 - 1$ elements(other than t_{ii}) from the ith column of $(\mathbf{I} - \mathbf{S})^{-\mathbf{1}}$ is negative, such that $t_{i_h i} > 0$ for $1 \leq h \leq n_2 \leq n - 1 - n_1$. Now use the j_kth equation in ($\mathbf{I} - \mathbf{S}$ times the ith colomn in $(\mathbf{I} - \mathbf{S})^{-\mathbf{1}}$ and multiply each equation by $\frac{2\beta}{\gamma}\delta_{j_k}$, $1 \leq k \leq n_1 < n-1$. Next, add these n_1 equations yielding

$$\begin{aligned}
&(\sum_{k=1}^{n_1} \delta_{j_k j_k} + \sum_{k=1}^{n_1} \delta_{j_k i})t_{ii} + (\sum_{k=1}^{n_1} \delta_{j_k j_k} + \sum_{k=1}^{n_1} \delta_{j_k i_1})t_{i_1 i} + \cdots + (\sum_{k=1}^{n_1} \delta_{j_k j_k} + \sum_{k=1}^{n_1} \delta_{j_k i_{n_2}})t_{i_{n_2} i} + \\
&[\sum_{k=1}^{n_1} \delta_{j_k j_k} + \sum_{k \neq 1}^{n_1} \delta_{j_k j_1} + (\frac{2\beta}{\gamma} - 1)\delta_{j_1 j_1}]t_{j_1 i} + \cdots + [\sum_{k=1}^{n_1} \delta_{j_k j_k} + \sum_{k=1}^{n_1 - 1} \delta_{j_k j_{n_2}} + (\frac{2\beta}{\gamma} - 1)\delta_{j_{n_1} j_{n_1}}]t_{j_{n_1} i} \\
=&(\sum_{k=1}^{n_1} \delta_{j_k j_k})(t_{ii} + \sum_{k=1}^{n_1} t_{j_k i} + \sum_{h=1}^{n_2} t_{i_h i}) + (\sum_{k=1}^{n_1} \delta_{j_k i})t_{ii} + (\sum_{k=1}^{n_1} \delta_{j_k i_1})t_{i_1 i} + \cdots + (\sum_{k=1}^{n_1} \delta_{j_k i_{n_2}})t_{i_{n_2} i} + \\
&[\sum_{k \neq 1}^{n_1} \delta_{j_k i} + (\frac{2\beta}{\gamma} - 1)\delta_{j_1 j_1}]t_{j_1 i} + \cdots + [\sum_{k=1}^{n_1 - 1} \delta_{j_k j_{n_1}} + (\frac{2\beta}{\gamma} - 1)\delta_{j_{n_1} j_{n_1}}]t_{j_{n_1} i} = 0.
\end{aligned}$$

If $t_{ii} + \sum_{k=1}^{n_1} t_{j_k i} + \sum_{h=1}^{n_2} t_{i_h i} \leq 0$, the following must be true

$$\begin{aligned}
&(\sum_{k=1}^{n_1} \delta_{j_k i})t_{ii} + (\sum_{k=1}^{n_1} \delta_{j_k i_1})t_{i_1 i} + \cdots + (\sum_{k=1}^{n_1} \delta_{j_k i_{n_2}})t_{i_{n_2} i} + [\sum_{k \neq 1}^{n_1} \delta_{j_k i} + (\frac{2\beta}{\gamma} - 1)\delta_{j_1 j_1}]t_{j_1 i} + \cdots + \\
&[\sum_{k=1}^{n_1 - 1} \delta_{j_k j_{n_1}} + (\frac{2\beta}{\gamma} - 1)\delta_{j_{n_1} j_{n_1}}]t_{j_{n_1} i} > 0.
\end{aligned}$$

However,

$$
\begin{aligned}
&(\sum_{k=1}^{n_1}\delta_{j_k i})t_{ii} + (\sum_{k=1}^{n_1}\delta_{j_k i_1})t_{i_1 i} + \cdots + (\sum_{k=1}^{n_1}\delta_{j_k i_{n_2}})t_{i_{n_2} i} \\
&+ [\sum_{k\neq 1}^{n_1}\delta_{j_k i} + (\frac{2\beta}{\gamma}-1)\delta_{j_1 j_1}]t_{j_1 i} + \cdots + [\sum_{k=1}^{n_1-1}\delta_{j_k j_{n_1}} + (\frac{2\beta}{\gamma}-1)\delta_{j_{n_1} j_{n_1}}]t_{j_{n_1} i} \\
<&(1-\delta_{ii})t_{ii} + (1-\delta_{i_1 i_1})t_{i_1 i} + \cdots + (1-\delta_{i_{n_2} i_{n_2}})t_{i_{n_2} i} \\
&+ [\sum_{k\neq 1}^{n_1}\delta_{j_k j_1} + (\frac{2\beta}{\gamma}-1)\delta_{j_1 j_1}]t_{j_1 i} + \cdots + [\sum_{k=1}^{n_1-1}\delta_{j_k j_{n_1}} + (\frac{2\beta}{\gamma}-1)\delta_{j_{n_1} j_{n_1}}]t_{j_{n_1} i} \\
<&(1-\frac{\gamma}{2\beta})t_{ii} + (1-\frac{\gamma}{2\beta})t_{i_1 i} + [\sum_{k\neq 1}^{n_2}\delta_{j_k j_1} + (1-\frac{\gamma}{2\beta})]t_{j_1 i} + \cdots + [\sum_{k=1}^{n_1-1}\delta_{j_k j_{n_1}} + (1-\frac{\gamma}{2\beta})]t_{j_{n_1} i} \\
=&(1-\frac{\gamma}{2\beta})(t_{ii} + \sum_{k=1}^{n_1}t_{j_k i} + \sum_{h=1}^{n_2}t_{i_h i}) + (\sum_{k\neq 1}^{n_1}\delta_{j_k j_1})t_{j_1 i} + \cdots + (\sum_{k=1}^{n_1-1}\delta_{j_k j_{n_1}})t_{j_{n_1} i} < 0,
\end{aligned}
$$

which is a contradiction. It concludes that

$$
t_{ii} + \sum_{k=1}^{n_1}t_{j_k i} + \sum_{h=1}^{n_2}t_{i_h i} > 0.
$$

□

Proof of Proposition 2

Proof. Differentiating firms first-order conditions with respect to γ, holding β constant, yields that $(\mathbf{I}-\mathbf{S})\mathbf{dq} = \mathbf{R}$, where $\mathbf{R} = (R_i)$ denotes a column vector, with $R_i = -\frac{1}{\gamma}\sum_{j\neq i}[(\delta_{ii} + \delta_{ij}q_j]d\gamma$. If condition (5) is satisfied, then, $\frac{\partial Q}{\partial\gamma} = -\frac{1}{\gamma}[\sum_{j\neq 1}(\delta_{11}+\delta_{1j})q_j(\sum_{l=1}^{n}e_{l1}) + \cdots + \sum_{j\neq n}(\delta_{nn}+\delta_{nj})q_j(\sum_{l=1}^{n}e_{ln})] < 0$ according to Lemma 2.

□

Proof of Proposition 4

Proof. Differentiating firms first-order conditions with respect to γ, holding β constant, yields that $(\mathbf{I}-\mathbf{S})\mathbf{dq}=\mathbf{R}$, where $\mathbf{R}=(R_i)$ denotes a column vector, with $R_i=-2\beta\sum_{j\neq i}[(\delta_{ii}+\delta_{ij}q_j]d\gamma$. If condition (16) is satisfied, then, $(\mathbf{I}-\mathbf{S})^{-1}$ is a non-negative matrix. Therefore $\frac{\partial Q}{\partial\gamma}<0$.

□

References

[1] Aleskerov, Fuad, Hasan Ersel, Dmitri Piontkovski (2011), *Linear Algebra for Economists*, 174.

[2] Alley, Wilson A.(1997), *Partial ownership arrangements and collusion in the automobile industry*, The Journal of Industrial Encomomics, 45, 191-205.

[3] Airline Business (1998), *Airlines with Ownership by Other Carriers*, June 1998, News, Vol. 48.

[4] Amir, Ericksonz and Jin(2015), *On the microeconomic foundations of linear demand for differentiated products*, School of Economics & Finance Online Discussion Paper Series, 2055-303X.

[5] Bergstrom, Ted (1978), *Cournot Equilibrium in Factor Markets.*

[6] Carlton, Dennis W. and Perloff, Jeffrey M. (2004), *Modern Industrial Organization (4th Edition).*

[7] Dietzenbacher, Erik, Bert Smid, and Bjrn Volkerink (2000), *Horizontal intergration in the Dutch financial sector*, International Journal of Industrial Organization, 18, 1223-1242.

[8] Deneckere, Raymond and Davidson, Carl (1985) *Incentives to Form Coalitions with Bertrand Competition*, RAND Journal of Economics, 16, 473-486.

[9] Dixit, Avinash K. (1986), *Comparative statistics for oligopoly*, International Encomomic Review, 27, 107-122.

[10] Economides, Nicholas and Steven C Salop (1992), *Competition and Integration among Complements, and Network Market Structure*, Journal of Industrial Economics, 40, 105-23.

[11] Farrell, Joseph and Carl Shapiro (1990), *Asset Ownership and Market Structure in Oligopoly*, The Rand Journal of Encomomics, 21, 275-292.

[12] Jonas Hckner (2000), *A Note on Price and Quantity Competition in Differentiated Oligopolies*, Journal of Economic Theory, 93, 233-239.

[13] Hawkins, D., and H. A. Simon (1949), *Note: Some Conditions of Macroeconomic Stability*, Econometrica XVII, 1949, pp. 245-248.

[14] Hansen, Robert G. and Jr. John R. Lott (1995), *Profiting from Induced Changes in Competitors Market Values: The Case of Entry and Entry Deterrence*, The Journal of Industrial Economics, 43(3), 291-276.

[15] Gilo, David, Yossi Moshe and Yossi Spiegel (2006), *Partial Cross Ownership and Tacit Collusion*, The Rand Journal of Economics, 37(1), 81-99.

[16] Ledvina, Andrew and Sircar, Ronnie (2012), *Oligopoly Games under Asymmetric Costs and an Application to Energy Production* , Mathematics and Financial Economics, Volume 6, Issue 4, pp 291293. more

[17] Li, Sanxi, Hongkun Ma, and Chenhang Zeng (2015), *Passive cross-holding as a Strategic Entry Deterrence*, Economics Letters, 134, 37-40.

[18] Qin, Cheng-Zhong, Hongkun Ma and Chenhang Zeng (2016), *Competitive and Welfare Implications of Cross-Holdings in Oligopoly.*

[19] Reitman, David (1994), *Partial Ownership Arrangements and the Potential for Collusion*, The Journal of Industrial Economics, 42, 313-322.

[20] Reynolds, Robert J. and Bruce R. Snapp (1986), *The competitive effects of partial ownership interests and joint ventures*, International Journal of Industrial Organization, 4, 141-153.

[21] Takayama, Akira (1985), *Mathematical Economics.*

[22] Salant, Stephan W., Sheldon Switzer, and Robert J. Reynoldsl (1983), *Losses from Horizontal Merger: The Effects of an Exogenous Change in Industry Structure on Cournot-Nash Equilibrium* The Quarterly Journal of Economics, 98,185-199.

[23] Shapiro, Carl (1989), *Theory of Oligopoly Behavior* Handbook of Industrial Organization, 1, 329-414.

[24] Singh, N. and Vives, X. (1984), *Price and Quantity Competition in a Differentiated Monopoly* RAND Journal of Economics, 15: 546-554.

[25] Spence, A. Michael (1976), *Product differentiation and Welfare* American Economic Review, 66, 407-14.

[26] Vives, X. (1999), *oligopoly pricing: old ideas and new tools*

[27] Wang, X. (2005), *On Welfare under Cournot and Bertrand Competition in Differentiated Oligopolies* Review of Industrial Organization, 27:185191.

[28] Piercarlo Zanchettin (2006), *Differentiated Duopoly with Asymmetric Costs* Journal of Economics and Management Strategy, 4: 999-1015.