On the Anti-Competitive and Welfare Effects of Cross-Holdings with Product Differentiation

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Abstract

The anti-competitive effects of cross-holdings have been extensively analyzed in the literature. While the literature has generally focused on cases with homogenous products, this paper is the first attempt to analyze welfare as well as the anti-competitive effects of cross-holdings allowing for product differentiation. We show that the known results established with homogenous products continue to hold with product differentiation. However, both the strength of the anti-competitive effects and the conditions resulting in these effects critically depend on the degree of product differentiation. Furthermore, cross-holdings generally increase social welfare with complementary products in quantity competition but decrease it with substitutable products in price competition. Our analysis has useful empirical and policy applications.

Keywords: Cross-holdings, Oligopoly, Product differentiation, Cournot equilibrium

JEL Classification: D43, L13

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1 Introduction

Cross-holding takes place when a firm acquires *passive* ownership in another firm, which entitles the acquiring firm a share in the acquired firm's profit but not in decision making. This practice is commonly observed in the real world. Microsoft, for example, purchased 7% of Apple's nonvoting, convertible, preferred stock at the aggregate price of 150 million dollars.[1] Gillette, the international and U.S. leader in the wet shaving razor blade market, acquired 22.9% of the nonvoting stock of Wilkinson Sword, one of its largest rivals.[2] Other examples include cross-holdings in the automobile, airline, banking, and telecommunication industries.[3]

A substantial literature on the anti-competitive effects of cross-holdings has emerged. Reynolds and Snapp (1986) were the first to study the anti-competitive effects of cross-holdings. With homogenous products, they showed that the total industry Cournot equilibrium output becomes less competitive as cross-holdings increase. Intuitively, after a firm has entered into a long equity position in a rival firm, it is induced to take into consideration the effect of its own output decision on the rival's profit. This consideration makes the firm compete less aggressively, because in so doing the firm can increase the profit of the rival and hence its stake in the rival's profit. Farrell and Shapiro (1990) explored when a firm might rationally increase its initial ownership share in a rival. They concluded that joint profit falls if a big firm (in terms of market share) acquires partial ownership in a small firm, but it is profitable for a small firm to acquire partial ownership in a big firm. They also found that if a small firm increases its holdings of a rival in which it previously had no financial interest, welfare will rise. The

[1]See "Preferred Stock Purchase Agreement-Apple Computer Inc. and Microsoft Corp. (Aug 04, 1997)" at http://contracts.corporate.ndlaw.com/planning/purchase/ 971.html.

[2]United States v. Gillette Co., 55 Fed. Reg. 28,312 (July 10, 1990). Gillette's passive stake in Wilkinson Sword was approved by the Department of Justice Antitrust Division. See infra text accompanying notes 94-9

[3]We refer the reader to Airline Business (1998), Alley (1997), Brito et al (2014), and Dietzenbacher et al (2000) for further discussion.

reason is that the welfare effects associated with the output change depend upon firms' markups. If the acquiring firm is small, the welfare increase due to the expansion in production from all other firms is larger than the welfare decrease due to the reduction in the acquiring firm's output. In that case, such cross-holdings are welfare-enhancing.

Considering the effects of both direct and indirect cross-holdings, Flath (1991) adopted a different approach in a duopoly setting and demonstrated that no firm will acquire shares in a rival unless its own *operating earning* increases with cross holdings.[4] Dietzenbacher et al. (2000) extended the analysis in Flath (1991) to a general n-firm setting and conducted empirical studies using the data of the Dutch financial sector to analyze consequences of cross-holdings. The authors showed that competition is reduced for both Cournot and Bertrand oligopolistic firms.[5]

There have also been studies that look at the collusive effects in repeat settings. Malueg (1992) considered a collection of duopoly examples showing cross-holdings have an ambiguous effect on collusion in a repeated Cournot duopoly. The reason behind this finding is as follows. On the one hand, cross-holdings weaken the incentive to deviate from collusive agreements. On the other hand, they also soften the punishment that would follow after deviation has taken place. Thus, whether cross-holdings facilitate or hinder collusion depends on if they weaken the incentive to a greater extent than they soften the punishment.[6] In a recent paper, Qin et al (2017) identified a class of cross-holdings that soften the punishment to a greater extent than they weaken the incentive. They showed that cross-holdings hinder instead of facilitating tacit collusion

[4] Qin et al (2017) showed that depending on the existing structure of cross-holdings, two firms may be better off reducing cross-holdings in each other. The reason is because their less competitive behavior may induce the other firms to compete more aggressively, similar to the reason behind the *merger paradox* in Salant, Switzer, and Renolds (1983).

[5] For recent studies of possible effects of cross-holdings on entry deterrence, see, for example, Hansen and Lott (1995), Clayton and Jorgensen (2005), Mathews (2006) and Li et al. (2015).

[6] Gilo et al (2006) analyzed collusive effects of cross-holdings in repeated Bertrand competition. Due to the competitiveness of the Bertrand equilibrium, they showed that without product differentiation, cross-holdings generally facilitate collusion.

in repeated Cournot competition.

The analysis in the literature has largely focused on cross-holdings in oligopolies with homogenous products. Homogeneity is too extreme, as products in practice are mostly differentiated. It is therefore worth investigating the extent to which results established with product homogeneity continue to hold with product differentiation. This paper is the first attempt to analyze welfare as well as the anti-competitive effects of cross-holdings allowing for product differentiation. Two prevalent research streams for analyzing consumer and firm behavior with product differentiation have evolved: the class of spatial models in the spirit of Hotelling (1929) and Lancaster (1979), and the class of non-spatial models in the spirit of Chamberlin (1933), Spence (1976), and Dixit (1979). We conduct our analysis by following the later stream. Specifically, we adopt a common model of product differentiation in Singh and Vives (1984). This model generates a linear demand system from a representative consumer. The model has been subsequently advanced. For example, Vives (2000) extended the model to allow for n-firm ($n \geq 3$) differentiated goods. In a recent paper, Amir et al. (2017) provided a thorough exploration of the microeconomic foundations for the linear demand system. They found that strict concavity of the utility function is critical for the demand system to be well defined, and it imposes significant restrictions on the range of complementarity of products.

The results of this paper can be summarized as follows. For Cournot competition with substitutable products, the total industry equilibrium output decreases as cross-holdings increase. This is in line with the established results without product differentiation. However, both the strength of the anti-competitive effects and the conditions resulting in these effects critically depend on the degree of product differentiation. The less substitutable products become, the stronger the strength of the anti-competitive effect is. Furthermore, welfare effects of cross-holdings associated with the output change depend upon firms price-cost gaps. When a firm with the largest price-cost gap increases passive ownership in rivals, social welfare decreases.

With product complementarity, firms' choices have positive effects on other firms.

Such positive externalities are partially internalized with cross-holdings. Thus, as cross-holdings increase firms expand their production under conditions which become less restrictive the less complementary products are. Moreover, social welfare increases with cross-holdings, and less differentiation makes the welfare effects easier to be achieved.

We also analyze the anti-competitive and welfare effects of cross-holdings with product differentiation in Bertrand competition. With substitutable products, firms increase prices and social welfare falls with cross-holdings. With complementary products, a decrease in price has positive effects on the demand of other products. It follows that the total industry equilibrium price decreases with cross-holdings. Moreover, social welfare rises when a firm with the largest price-cost gap increases ownership share in any other firms.

Our findings have useful implications. For Cournot competition with substitutable products, industries with cross-holdings tend to be more concentrated. With complementary products, cross-holdings make firms more cooperative. From a social point of view, our analysis suggests that cross-holdings should be promoted with complementary products in Cournot competition but discouraged with substitutable products in Bertrand competition. Furthermore, a firm with the largest price-cost gap should be prohibited from raising their ownership shares in rivals with substitutable products in Cournot competition but encouraged from raising their ownership shares in rivals with substitutable complementary in Bertrand competition.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 and Section 4 present the results in Cournot competition with substitutable and complementary products, respectively. Section 5 presents the results in Bertrand competition. Section 6 concludes. Proofs of results are organized in an Appendix.

2 The Model

Consider an oligopolistic industry with n firms producing n differentiated products. We assume that each firm i's technology is of constant return to scale for simplicity.

Denote firm i's constant marginal cost is c_i. We assume that firms have zero fixed cost. Let us adopt the notational convention that vectors and matrices are denoted by bold letters and scalars by italic. Vives (2000) considers an n-firm ($n \geq 3$) differentiated goods oligopoly model that is a direct generalization of the duopoly model developed by Singh and Vives (1984). We will present the details of this quadratic and strictly concave utility function in the welfare analysis section. In this model, the inverse demand function for firm i's product is given by

$$p_i(q) = \alpha - \beta_i q_i - \sum_{j \neq i}^{n} \gamma_{ij} q_j, \tag{1}$$

where $\mathbf{q} = (q_1, \cdots, q_n)$ with q_i as the quantity produced by firm i, $\alpha, \beta_i, \gamma_{ij}$ are constant such that $\alpha > c_i \geq 0$, $\beta_i > 0$ and $\gamma_{ij} = \gamma_{ji}$, $i = 1, 2, \cdots, n$, $i \neq j$. The parameter γ_{ij} can be positive, negative or zero depending on whether the goods are substitutable, complementary or independent. All γ_{ij} have the same sign so that products are either all substitutable or all complementary. We define $Q = \sum_{i=1}^{n} q_i$ and $P = \sum_{i=1}^{n} p_i$ as the total industry output and price, respectively. The strict concavity of the quadratic utility function requires that $\gamma_{ij}^2 \leq \beta_i \beta_j$ for $i \neq j$. Notice that, when γ_{ij} in (1) is close to β_i, firm i's and j's products are homogenous. As γ_{ij} approaches 0, these two firms' products are independent. Consequently, γ_{ij} is an index of product differentiation. Singh and Vives (1984) take the ratio $\gamma_{ij}^2/(\beta_i \beta_j)$ as the degree of product differentiation between firm i's and firm j's products, ranging from zero when the goods are independent to one when the goods are perfect substitutes.[7] We follow their approach.

Firms can be linked through a web of cross-holdings. Let δ_{ij} be firm i's ownership share in firm j and let $\mathbf{\Delta}$ be the $n \times n$ matrix of cross-holdings

$$\mathbf{\Delta} = \begin{bmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \dots & \delta_{nn} \end{bmatrix}.$$

[7] When $\beta = -\gamma$ the demand system may not be well defined.

The ith row is the vector of ownership share held by firm i in firm j, $j = 1, 2, \cdots, n$, while the jth column is the vector of ownership share held by firm j in firm i. Normalize the total ownership share in each firm to one. Thus, the sum of each column in $\mathbf{\Delta}$ is strictly equal to one.

We consider cross-holdings that are passive. In the terminology of Bresnahan and Salop (1986), firms are assumed to have "silent financial interests" in rivals. This means that acquiring firms are not involved in decision-making of acquired firms. Most studies on cross-holdings make this assumption, such as Reynolds and Snapp (1986), Farrell and Shapiro (1990), and Flath (1991). With quantity competition each firm i decides on q_i independently. Let $p_i(q)q_i - c_iq_i$ denote firm i's operating earning, which is a measure of how much revenue will eventually become profit for firm i. Reynolds and Snapp (1986) assume that each firm keeps its operating earning net of those going to the acquiring firms. In addition, each firm receives financial interests in competitors' operating earnings. Following their approach, given cross-holding matrix $\mathbf{\Delta}$, firm i's objective is to solve the following profit maximization problem:[8]

$$\max_{q_i} \quad \delta_{ii}[p_i(q) - c_i)]q_i + \sum_{j \neq i}^{n} \delta_{ij}[p_j(q) - c_j]q_j, \quad i = 1, 2, \cdots, n, \tag{2}$$

taking other firms' quantities $q_{-i} = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_n)$ as given, where $\delta_{ii} = 1 - \sum_{j \neq i}^{n} \delta_{ji}$. The first-order condition for (2) is

$$2\beta_i\delta_{ii}q_i + \sum_{j \neq i}^{n}[\gamma_{ij}(\delta_{ii} + \delta_{ij})q_j] - \delta_{ii}(\alpha - c_i) = 0. \tag{3}$$

Solving q_i from (3), firm i's reaction function is

$$q_i = -\sum_{j \neq i}^{n}[\frac{\gamma_{ij}}{2\beta_i}(1 + \frac{\delta_{ij}}{\delta_{ii}})q_j] + \frac{\alpha - c_i}{2\beta_i}. \tag{4}$$

Observe that the slope of q_i's reaction function depends on the sign of γ_{ij}. When $\gamma_{ij} > 0$, products are substitutable and firm i's best response to an increase in another

[8]This profit formulation in the presence of cross-holdings was prosed in Reynolds and Snapp (1986) and was followed by Farrell and Shapiro (1990), and Alley (1997), among others.

firm's output is to decrease its own output. In contrast, when $\gamma_{ij} < 0$, products are complementary and firm i's best response to an increase in another firm's output is to expand its production.

3 The Case with Substitutable Products

We begin our analysis with a simplified case, in which $\beta_1, \beta_2, \cdots, \beta_n$ are identical and $\gamma_1, \gamma_2, \cdots, \gamma_n$ are identical, and then extend our analysis to a more general case allowing for β_i be different and γ_{ij} be different, $i = 1, 2, \cdots, n$. With symmetry, we use β and γ to denote the common values, respectively. This symmetric case is commonly used, such as Hackner (2000), Amir (2001, 2017) and Zanchettin (2008). In these studies, β is taken as one and γ is regarded as the degree of product differentiation. To be consistent with our analysis of the asymmetric case in later subsection, we do not normalize the value of β to one. With symmetry, (1) becomes

$$p_i(q) = \alpha - \beta q_i - \gamma \sum_{j \neq i}^{n} q_j, \tag{5}$$

and correspondingly the ratio γ^2/β^2 is regarded as the degree of product substitutability. Recall that the concavity of the utility requires $\gamma^2 \leq \beta^2$. The effect of a change in a firm's own quantity on its price is stronger than the effect of the same amount of change in any competitor's quantity when $\gamma < \beta$. The more substitutable (less differentiated) products are, the closer γ^2/β^2 is to 1. Firm i's reaction function in (4) now becomes

$$q_i = -\sum_{j \neq i}^{n} [\frac{\gamma}{2\beta}(1 + \frac{\delta_{ij}}{\delta_{ii}})q_j] + \frac{\alpha - c_i}{2\beta}.$$

Observe that the magnitude of the adjustment in firm i's output as a response to an increase in a competitor's output depends on the degree of product substitutability and the ratio of firm i's ownership share in this competitor to firm i's retained share. Firm i reduces its output more to an increase in a competitor's output when products are more substitutable.

As usual, the existence of a Cournot equilibrium is not automatically guaranteed. The presence of cross-holdings may complicate the existence problem. Since most cross-holdings in practice do not stop firms from producing, it is necessary to establish the existence of a positive Cournot equilibrium with cross-holdings and product differentiation. We then conduct comparative static analysis of total and individual firm's output with respect to changes of cross-holdings.

3.1 Cournot equilibrium

We can write firms' reaction functions together in vector notation as

$$\mathbf{q} = \mathbf{S}\mathbf{q} + \mathbf{m},$$

which is equivalent to

$$(\mathbf{I} - \mathbf{S})\mathbf{q} = \mathbf{m}, \tag{6}$$

where $\mathbf{m}$ denotes a column vector, with its transpose $\mathbf{m}' = ((\alpha - c_1)/2\beta, \cdots, (\alpha - c_n)/2\beta)$, and $\mathbf{S} = (s_{ij})$ denotes a $n \times n$ matrix with $s_{ii} = 0$ and $s_{ij} = -(\gamma/2\beta)(1+\delta_{ij}/\delta_{ii})$ for $j \neq i$. Notice that $\mathbf{I} - \mathbf{S}$ is a positive matrix because $\gamma > 0$. If $\mathbf{I} - \mathbf{S}$ is invertible, then a positive Cournot equilibrium can be solved from (6) as

$$\mathbf{q} = (\mathbf{I} - \mathbf{S})^{-\mathbf{1}}\mathbf{m}. \tag{7}$$

We will find conditions that can guarantee the equilibrium quantities in (7) to be positive in the next subsection.

The following lemma provides a sufficient condition for when $\mathbf{I} - \mathbf{S}$ is invertible.

Lemma 1. *Given $\mathbf{\Delta}$, β and γ, $\mathbf{I} - \mathbf{S}$ is invertible if the following condition, to be referred to as **A1**, holds:*

$$\delta_{ii} > \frac{\gamma}{2\beta}, \quad i = 1, 2, \cdots, n.$$

A1 puts a lower bound on how much share each firm i needs to retain. Intuitively, in order to have incentives to produce, each firm must retain certain amount of ownership. Observe that **A1** is less restrictive as products become less substitutable. Recall that $\gamma \leq \beta$ and $\gamma > 0$ when products are substitutable. Consequently, this lower bound in **A1** takes value from 0 to 1/2. In addition, **A1** implies $\delta_{ij}/\delta_{ii} < 2\beta/\gamma - 1$ for all i. In that case, each firm's ownership share in rivals has an upper bound, which also depends on the degree of product substitutability. Specifically, the less substitutable products are, the larger share a firm is allowed to hold in each rival. For example, if $2\beta/\gamma = 3$, then firm i's ownership share in one of competitors could be twice the share that firm i retains. It is worth noticing that in the case of an oligopoly without cross-holdings, **A1** is automatically satisfied since $\delta_{ii} = 1$ for all i.

3.1.1 Technologies are Symmetric

When firms' technologies are symmetric, their marginal cost are the same. We use c to express this common value. Denote $(\mathbf{I} - \mathbf{S})^{-\mathbf{1}} = (t_{ij})$. When firms have the same constant marginal cost, (7) reduces to

$$q_i = (\frac{\alpha - c}{2\beta}) \sum_{j=1}^{n} t_{ij}, \quad i = 1, 2, \cdots, n.$$

Consequently, if a given cross-holding matrix satisfies $\sum_{j=1}^{n} t_{ij} > 0$ for all i, a strictly positive Cournot equilibrium exists. The following proposition provides a condition for when $\sum_{j=1}^{n} t_{ij} > 0$.

Proposition 1. *For $\mathbf{\Delta}$, β and γ satisfying **A1** and*

$$\frac{\sum_{j \neq i} \delta_{ij}}{\delta_{ii}} < \frac{2\beta}{\gamma} - 1, \quad i = 1, 2, \cdots, n, \tag{8}$$

a positive Cournot equilibrium exists.

Condition (8) requires that the ratio of the sum of a firm's ownership shares in rivals to its retained share has an upper bound. This upper bound rules out the case when a large enough fraction of a firm's revenue comes from other firms' profits, this firm may produce nothing. This condition is not as strong as it appears. For example, in the case of a duopoly, condition (8) is automatically satisfied because (8) reduces to $\delta_{ij}/\delta_{ii} < 2\beta/\gamma - 1$, $i = 1, 2$. In addition, in the case of oligopoly with a symmetric cross-holding structure condition (8) is automatically satisfied since $\sum_{j\neq i} \delta_{ij} = \sum_{j\neq i} \delta_{ji} = 1 - \delta_{ii}$. In a usual case of oligopolies without cross-holdings, condition (8) is also automatically satisfied because $\delta_{ij} = 0$ for $j \neq i$. In this case, $(\mathbf{I} - \mathbf{S})^{-\mathbf{1}} = (t_{ij})$ is a $n \times n$ symmetric matrix with $t_{ii} = 1$ and $t_{ij} = \gamma/(2\beta)$ for $i \neq j$. It concludes that, without cross-holdings, a linear Cournot oligopoly with symmetric cost has a unique Cournot equilibrium, which is symmetric.

A direct application of proposition 1 establishes

Corollary 1. *For $\mathbf{\Delta}$, β and γ satisfying **A1** and*

$$\sum_{j\neq i} \delta_{ij} < 1 - \frac{\gamma}{2\beta}, \quad i = 1, 2, \cdots, n, \tag{9}$$

(8) is satisfied.

The upper bound in (9) on the sum of each firm's ownership shares in rivals takes value between 1/2 and 1, and is less restrictive as products become less substitutable. The following example illustrates that, if condition (9) does not hold, a positive Cournot equilibrium may not exist.

Example 1. Consider a Cournot oligopoly with 3 firms. Suppose $\alpha = 30$, $c = 2$, $\beta = 1.5$ and $\gamma = 1$. Next, let cross-holding matrix be

$$\mathbf{\Delta} = \begin{bmatrix} 0.3 & 0.6 & 0.2 \\ 0.4 & 0.4 & 0 \\ 0.3 & 0 & 0.8 \end{bmatrix}.$$

Notice that the sum of firm 1's ownership share in firm 2 and 3 is 0.8, which is greater than $1 - \gamma/2\beta = 2/3$. It follows that condition (9) is not satisfied. The inverse of $\mathbf{I} - \mathbf{S}$

is given by

$$(\mathbf{I}-\mathbf{S})^{-1} = \begin{bmatrix} -3.62 & 4.65 & 0.63 \\ 2.85 & -2.4 & -1.07 \\ 0.97 & -1.81 & 1.09 \end{bmatrix}.$$

Observe that the sum of the second row in $(\mathbf{I}-\mathbf{S})^{-1}$ is $2.85 - 2.4 - 1.07 < 0$. In that case, the first-order condition yields a negative quantity for firm 2 in equilibrium. $\square$

As we mentioned before, without cross-holdings a linear Cournot oligopoly with symmetric cost has a unique Cournot equilibrium which is symmetric. With cross-holdings, however, firms' equilibrium outputs are not necessarily symmetric. For instance, with the cross-holdings matrix in Example 1, all three firms' outputs are different because the sums of each row are different. This example illustrates that cross-holdings can have a large effect on firm performance.

3.1.2 Asymmetric Technologies

Firms' technologies are asymmetric when their marginal costs are different. Firms with better technologies have lower marginal costs. Recall that the first-order conditions in (3) are independent of costs. Thus, a positive Cournot equilibrium that allows firms to have different technologies can also be solved from (6). Assume the condition in lemma 1 is satisfied, then $(\mathbf{I}-\mathbf{S})$ is invertible. In that case, a positive Cournot equilibrium exists if and only if

$$q_i = \sum_{j=1}^{n} \frac{(\alpha - c_i)t_{ij}}{2\beta} > 0, \quad i = 1, 2, \cdots, n.$$

Unlike the symmetric case, the proceeding sufficient conditions for a strictly positive Cournot equilibrium depends on firms's cost structure. As such, even if condition (9) is satisfied, solutions of (6) can still be negative, as the following example illustrates.

Example 2. Consider a Cournot oligopoly with 3 firms. Suppose that $\alpha = 30$, $c_1 = 5$,

$c_2 = 10$, $c_3 = 20$, $\beta = 0.15$ and $\gamma = 0.1$. Let cross-holding matrix be

$$\mathbf{\Delta} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.8 & 0 \\ 0.1 & 0 & 0.8 \end{bmatrix}.$$

Denote q_i^* be each firm i's equilibrium quantity, $i = 1, 2, 3$. Observe that condition (9) is satisfied. Simple calculation shows in equilibrium, $q_1^* = 208$, $q_2^* = 110$. However, the first-order condition yields a negative quantity $q_3^* = -15$ for firm 3. It shows that, a strictly positive Cournot equilibrium does not exist in this example. $\square$

Finding conditions that guarantee positive solutions of (6) for a duopoly is possible, as Example 3 below illustrates. However, due to the asymmetric cost structures, it is not clear how to extend the result to a more general case.

Example 3. Consider a Cournot duopoly. Let cross-holdings matrix be

$$\mathbf{\Delta} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}.$$

Under **A1**,

$$det(\mathbf{I} - \mathbf{S}) = 1 - \frac{\gamma^2}{4\beta^2}(1 + \frac{\delta_{12}}{\delta_{11}})(1 + \frac{\delta_{21}}{\delta_{22}}) > 0.$$

Consequently,

$$(\mathbf{I} - \mathbf{S})^{-\mathbf{1}} = \frac{1}{1 - \frac{\gamma^2}{4\beta^2}(1 + \frac{\delta_{12}}{\delta_{11}})(1 + \frac{\delta_{21}}{\delta_{22}})} \begin{bmatrix} 1 & -\frac{\gamma}{2\beta}(1 + \frac{\delta_{12}}{\delta_{11}}) \\ -\frac{\gamma}{2\beta}(1 + \frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix}.$$

It follows that, a strictly positive Cournot equilibrium for a duopoly exists if and only if

$$\frac{\gamma}{2\beta}\frac{\delta_{11} + 1 - \delta_{22}}{\delta_{11}} < \frac{\alpha - c_1}{\alpha - c_2} < \frac{2\beta}{\gamma}\frac{\delta_{22}}{\delta_{22} + 1 - \delta_{11}}. \tag{10}$$

In a duopoly without cross-holdings, (10) reduces to

$$\frac{\gamma}{2\beta} < \frac{\alpha - c_1}{\alpha - c_2} < \frac{2\beta}{\gamma}, \tag{11}$$

which only depends on the cost structure and degree of product substitutability. Comparing (10) with (11), it is clear that cross-holdings complicate the existence of a strictly positive Cournot equilibrium. $\square$

3.1.3 Cross-Holdings and Exit

In this model, when **A1** fails to be satisfied, the ratio of the sum of a firm's ownership shares in all other firms to its retained share is large enough, this firm will produce nothing. In other words, if a large enough fraction of a firm's revenue comes from other firms' profits, its profit shares from acquired firms are significant so that the total revenue of this acquiring firm diminishes as it increases its own output. In that case, there will be an equilibrium in which one firm produces nothing and other firms are producing. It follows that industries will tend to be more concentrated. Policy makers use the Herfindahl-Hirschman Index (HHI) to measure market concentration. A horizontal merger of a given size, as measured by the HHI delta, is of greater concern the greater the market concentration. Besides a merger, additional concentration resulting from exit with cross-holdings is a matter of significant anti-competitive concern as well. One fold from a policy perspective is that in industries with differentiated products, the antitrust authorities should be concerned when firms acquire massive ownerships in competitors to exit. However, this type of cross-holdings is typically ignored by antitrust regulations, for example, the acquisition of partial cross ownership in Chinese ride-hailing market.

Didi Chuxing, which is a major ride-sharing company in China, supplied about 76% of the country's app-requested rides while Uber ranked second with nearly 17% market share in 2016. The competition between ride-sharing companies was among the world's fiercest in China, where Uber was estimated to be losing $1 billion a year (and Didi Chuxing even more) because of subsidies and discounts to gain drivers and users. This competition between Uber and Didi ended after Uber sold its China operation to rival Didi Chuxing in late 2016.[9] Under the terms of the deal, Uber got 17.5% passive ownership of Didi Chuxing while Didi Chuxing invested $1 billion into Uber. In this

[9] We refer the reader to "Uber Sells China Operations to Didi Chuxing" at https://www.wsj.com/articles/china-s-didi-chuxing-to-acquire-rival-uber-s-chinese-operations-1470024403.

environment, both companies had reason to be happy with their deal: Uber stopped the bloodletting in China, and Didi Chuxing not only eliminated its top competitor but gained a veritable monopoly power over the world's biggest market for ride-sharing services. Chinese antitrust authorities initially investigated whether Didi Chuxing's proposed takeover of Uber China business violates the country's antitrust regulations. The investigation last several months but the deal eventually went through by Chinese government. The only losers would be consumers, who would likely be facing higher fares, less choice and worse service. A survey, which was conducted by web portal Sina a year after Uber exited Chinese markets, revealed 81.7% percent of respondents believe that hailing a ride in China is more difficult than it was a year ago, and 86.6% say it is more expensive than ever before.

In reality, most cross-holdings do not stop firms from producing. To be consistent with this empirical observation, we will assume a strictly positive Cournot equilibrium exists in the next subsection.

3.2 Anti-Competitive Effects

We now analyze the anti-competitive effects of cross-holdings, beginning with the case of a single firm raises ownership share in one of competitors. We establish conditions under which the total equilibrium output decreases as one firm acquires ownership in rivals. By decomposing the total change into a chain of individual changes, we can show that total output decreases as more firms simultaneously increase ownership shares in their competitors .

To determine the effect of a small increase of firm 1's ownership share in firm k on total output, we totally differentiate firms' individual quantities characterized by the first-order conditions in (3) to get [10]

$$(\mathbf{I} - \mathbf{S}) \cdot \mathbf{dq} = \mathbf{k}, \tag{12}$$

[10] To determine the effect of a small increase from one acquiring firm's ownership share in a acquired firm on total output, without loss of generality we assume firm 1 increases its share in firm k, $k \neq 1$.

where $\mathbf{k}$ is a $n \times 1$ matrix with the first element as $-q_k \gamma d\delta_{1k}/(2\beta\delta_{11})$, kth element as $-\sum_{j\neq k} \delta_{kj} q_j \gamma \cdot d\delta_{1k}/(2\beta\delta_{kk}^2)$, and zero otherwise. By lemma 1, $\mathbf{I}-\mathbf{S}$ is invertible under **A1**. In that case, (12) can be solved by

$$\mathbf{dq} = (\mathbf{I}-\mathbf{S})^{-\mathbf{1}}\mathbf{k}. \tag{13}$$

Lemma 2 below summarizes some properties that are useful to determine the total output change due to a small increase of firm 1's ownership share in firm k.

Lemma 2. *For $\mathbf{\Delta}$, β and γ satisfying **A1**,*

$$t_{ii} > 0 \quad \textit{and} \quad \sum_{j=1}^{n} t_{ji} > 0,$$

where t_{ij} represents the element from the ith row and jth column in $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$.

Under **A1**, the sum of each column in $\mathbf{I}-\mathbf{S}^{-1}$ is positive. (13) characterizes each firm's output reaction to a small increase of firm 1's ownership share in firm k. Adding each firm's output reaction together yields the total industry output reaction below.

$$\frac{\partial Q^*}{\partial \delta_{1k}} = -\frac{\gamma}{2\beta}\left[\frac{q_k^*}{\delta_{11}}\left(\sum_{i=1} t_{i1}\right) + \frac{\sum_{j\neq k}\delta_{kj}q_j^*}{\delta_{kk}^2}\left(\sum_{j=1} t_{jk}\right)\right].$$

By lemma 2, the sum of each column in $\mathbf{I}-\mathbf{S}^{-1}$ is positive. It follows that

$$\frac{\partial Q^*}{\partial \delta_{1k}} < 0. \tag{14}$$

(14) indicates that the total industry output falls when a single firm increases ownership share in one of competitors. The following example provides an illustration.

Example 4. Consider a Cournot duopoly. Assume cross-holding matrix satisfies **A1**. (13) reduces to

$$\begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \frac{1}{1-\frac{\gamma^2}{4\beta^2}(1+\frac{\delta_{12}}{\delta_{11}})(1+\frac{\delta_{21}}{\delta_{22}})} \begin{bmatrix} 1 & -\frac{\gamma}{2\beta}(1+\frac{\delta_{12}}{\delta_{11}}) \\ -\frac{\gamma}{2\beta}(1+\frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix} \begin{bmatrix} -\frac{q_2}{\delta_{11}}\frac{\gamma}{2\beta}d\delta_{12} \\ -\frac{\delta_{21}q_1}{\delta_{22}^2}\frac{\gamma}{2\beta}d\delta_{12} \end{bmatrix}.$$

Consequently, the total output change, due to firm 1's increasing ownership share in firm 2, is given by

$$\frac{\partial Q^*}{\partial \delta_{12}} = -\frac{(\frac{2\beta}{\gamma}\delta_{22} - \delta_{22} - \delta_{21})q_2^* + (\frac{2\beta}{\gamma}\delta_{11} - \delta_{22} - \delta_{21})\frac{\delta_{21}q_1^*}{\delta_{22}}}{\frac{4\beta^2}{\gamma^2}\delta_{11}\delta_{22} - (\delta_{11} + \delta_{12})(\delta_{22} + \delta_{21})}.$$

Recall that **A1** implies $\delta_{12}/\delta_{11} < 2\beta/\gamma - 1$ and $\delta_{21}/\delta_{22} < 2\beta/\gamma - 1$. Consequently, $\partial Q^*/\partial \delta_{12} < 0$. In other words, total equilibrium output decreases when firm 1 acquires passive ownership in firm 2. □

Now we are ready to analyze what effect increasing cross-holdings would have on the total industry equilibrium output.

Theorem 1. *For* $\mathbf{\Delta}$, β *and* γ *satisfying* ***A1****, total industry equilibrium output decreases as cross-holdings increase.*

Simultaneous changes in cross-holdings can be achieved through a sequence of unilateral changes, with only one firm changing ownership in one of rivals. According to lemma 2, when a single firm raises ownership share in a rival, the total industry output decreases. Applying lemma 2 to each step in this sequence can establish Theorem 1. By Theorem 1, industry becomes less competitive as cross-holdings increase. This effect arises because cross-holdings link the fortunes of competitors, producing a positive correlation among their profits. The linking of profits gives each firm an incentive to compete less aggressively and adopt behavior more conducive to joint profit maximization than otherwise. A well known implication of this finding is that industries with cross-holdings tend to be more concentrated.

This is in line with the established result without product differentiation. Reynolds and Snapp (1986) showed that cross-holding arrangements could result in less aggregate output and higher prices than when there are no cross-holdings, even if the ownership shares are relatively small. Farrell and Shapiro (1990) found that when firm raises its shares of a rival in which it previously had no financial interest, total output falls. We have extended this result to the inclusion of product differentiation. Our result critically depends on the degree of product substitutability. In other words, the condition resulting in this result is weakened as products become less substitutable. Another implication is that the anti-competitive effect of cross-holdings is more likely to be achieved, as products are less substitutable.

It has been shown in literature that when a firm raises ownership share in rivals, this acquiring firm competes less aggressively while all other firms expand production with homogenous products. We now examine how this result may depend on the degree of product substitutability.

Firm 1's output change due to a small change of ownership share in firm k is given by [11]

$$\frac{\partial q_1^*}{\partial \delta_{1k}} = -[q_k t_{11} + \frac{p_k - \beta q_k - c_k}{\gamma} t_{1k}]. \tag{15}$$

As shown in (15), the sign of $\partial q_1 / \partial \delta_{1k}$ depends not only on the cross-holding structure of the acquired firm, but also relies on each firm's output. This shows that the acquiring firm might not alway reduce its output when it acquires passive ownership in competitors. An illustration is provided in Example 5 below.

Example 5. Consider a Cournot duopoly. Suppose that $\delta_{11} = 0.7$, $\delta_{22} = 0.8$, $\beta = 2$ and $\gamma = 1$. Under **A1**, (15) becomes

$$\frac{\partial q_1^*}{\partial \delta_{1k}} = \frac{-\frac{2\beta}{\gamma}\delta_{22} q_2 + (\delta_{11} + \delta_{12})\frac{\delta_{21} q_1}{\delta_{22}}}{\frac{4\beta^2}{\gamma^2}\delta_{11}\delta_{22} - (\delta_{11} + \delta_{12})(\delta_{22} + \delta_{21})}.$$

An increase in firm 1's share in firm 2 causes its output to decrease if and only if

$$\frac{\alpha - c_1}{\alpha - c_2} < \frac{\frac{2\beta}{\gamma}\delta_{22}^2 + \frac{\gamma}{2\beta}\frac{1}{\delta_{11}}(\delta_{11} + 1 - \delta_{22})^2(1 - \delta_{11})}{1 + \delta_{22}^2 - \delta_{11}^2} = 2.3.$$

$\square$

As Example 5 illustrates, firm 1's output reaction to its increasing ownership in a rival is not definite. The theorem below characterizes the conditions that are sufficient for the acquiring firm to produce less when it acquires passive ownership in competitors.

[11] From firm k's first-order condition, $(p_k - \beta q_k - c_k)/\gamma = (\sum_{j \neq k} \delta_{kj} q_j)/\delta_{kk}$.

Theorem 2. *Let cross-holding matrix be given and k be the acquired firm. Then, $\partial q_i^*/\partial \delta_{ik} < 0$ under any following conditions*

$$(i) \quad \delta_{kh} = 0, \quad \forall h \neq k; \tag{16}$$

$$(ii) \quad \sum_{j \neq k} \delta_{ij} + (n-2)\delta_{ii} > \frac{2\beta}{\gamma}(\delta_{ii} + \delta_{ik}). \tag{17}$$

Under condition (16), firm k does not hold any passive ownership in rivals. Its objective is just to maximize its own operating earning. The first-order condition that characterizes its optimal output becomes $p_k - \beta q_k - c_k = 0$. In that case, equation (15) reduces to

$$\frac{\partial q_1 *}{\partial \delta_{1k}} = -\frac{q_k}{\delta_{11}} t_{11}.$$

By lemma 2, t_{11} is positive. Under condition (16), this result is similar to the findings in Qin et al. (2016) for cases with a single acquiring firm cross-holdings. They showed that each time when the acquiring firm raises its share in one of competitors, it will be more conservative in production with homogenous products.

Unlike condition (16), (17) extends this result by allowing for the acquired firm to hold partial ownership in competitors. Under condition (17), t_{1k} from (16) is positive. The left side of (17) depends on the sum of firm i's ownership shares in competitors beside firm k, the number of firms and the share firm i retains. The right side relies on the degree of product substitutability, the ownership share firm i retains and the share firm i holds in firm k. Notice that, (17) is less restrictive as γ increases. In this sense, the more substitutable products are, the more likely the acquiring firm compete less aggressively. When the sum of this acquiring firm's shares in competitors beside this acquired firm, is much larger than this acquiring firm's share in this acquired firm, (17) is easier to be achieved.

Under conditions in Theorem 2, when a firm increases ownership in rivals, this acquiring firm reduces its output. Intuitively, when a single firm increases ownership in one of rivals, this acquiring firm is induced to take into consideration the effect of its output decision on the acquired firm's profit. It realizes that increasing its output reduces the profit it earns on ownership share of the acquired firm. This consideration makes the acquiring firm compete less aggressively, because in doing so it can augment the profit at the acquired firm and hence its share in the acquired firm's profit.

In the literature it is well established that, with homogenous product and homogenous technology, when a firm increases ownership share in rivals, all its rivals produce more. Reynolds and Snapp (1986, p. 145) found that when the acquiring firm decreases its output all other firms only expand production until marginal revenue equals marginal cost, so they will never fully replace the output contraction. But with heterogeneous technology and product differentiation, this result is not guaranteed. The following example provides an illustration.

Example 6. Consider a Cournot duopoly. Suppose that $\alpha = 4000$, $c_1 = 1000$, $c_2 = 2000$, $\delta_{11} = 0.9$, $\delta_{22} = 0.7$, $\beta = 2$ and $\gamma = 0.99$. In Cournot equilibrium, $q_1* = 671$ and $q_2* = 276$. When firm 1 raises ownership in firm 2 by 0.1, in the new Cournot equilibrium firm 1 produces 663 units while firm 2 produces 272 units. It is clear that both firms reduce production when firm 1 acquires ownership in firm 2. The intuition is that when firm 1 increases ownership in firm 2, firm 1 reduces its output to augment firm 2's operating earning. It follows that, firm 1's reduction in output decreases its own operating earning and hence the financial return that goes to firm 2. However, if firm 2 reacts to expand its output, the increasing amount of its operating earning due to its expansion in output is less than the reduction from firm 1's financial return because firm 2 has higher marginal cost. It turns out that firm 2 produces less in order to compensate the loss due to the reduction in firm 1's output. □

Our proceeding result shows how other firms' reactions may depend on the existing cross-holding matrix and specific cost structure. The following example illustrates that for cross-holdings with a single acquiring firm, all other non-acquiring firms react by expanding their production when the acquiring firm raises ownership in rivals.

Example 7. Consider an oligopoly with n firms. Suppose cross-holding matrix is

$$\mathbf{\Delta} = \begin{bmatrix} 1 & \delta_{12} & \dots & \delta_{1n} \\ 0 & 1-\delta_{12} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\delta_{1n} \end{bmatrix}.$$

Notice that condition (16) is satisfied since only firm 1 holds ownership in rivals. By Theorem 2, firm 1 reduces its output when it raises ownership share in of these non-acquiring firms. By Theorem 1, total industry equilibrium output decreases as cross-holdings increase. Other firms' output responses to firm 1's small increasing ownership shares firm k are determined by

$$\frac{\partial q_j^*}{\partial \delta_{1k}} = -\frac{1}{\frac{2\beta}{\gamma}-1}\frac{\partial Q^*}{\partial \delta_{1k}}, \quad j \neq 1.$$

According to lemma 2, $\partial Q^*/\partial \delta_{1k} < 0$. It follows that

$$\frac{\partial q_j}{\partial \delta_{1k}} > 0.$$

Given the stabilized Cournot equilibrium and downward sloping reaction functions, all other firms react to output contraction by expanding their own production. At the new equilibrium, however, the total output decreases. With such cross-holdings, our result is consistent with the findings in Reynolds and Snapp (1986), and Farrell and Shapiro (1990). Interestingly, such cross-holdings often occur in many industries throughout the world. The typical example involves Apple, Microsoft, and RealNetwork. In 1997, Microsoft acquired a 10% stake in RealNetworks Inc, and in the same year, Microsoft also acquired 7% of the nonvoting stock of Apple. □

We have showed the anti-competitive effects of cross-holdings, holding the degree of product substitutability constant. The following proposition establishes comparative statics with respect to changes in the degree of product substitutability, holding cross-holdings constant.

Proposition 2. *For $\mathbf{\Delta}$, β and γ satisfying **A1**, total industry equilibrium output increases as products become less substitutable.*

For $\gamma = 0$, products are independent. Products become increasingly homogenous when γ approaches 0. In other words, as products become less substitutable, competition gets less intensified. On the other hand, the less substitutable products are, the steeper of the slopes of firms' demand curves become. Thus, the less substitutable products become, the stronger the strength of the anti-competitive effect of cross-holdings is. From a policy perspective, an implication of this finding is that industries with firms producing similar products are more concentrated than industries with firms producing more differentiated products.

3.3 Welfare Analysis

Any change in cross-holdings in a Cournot oligopoly has ramifications for the outputs chosen by all of the oligopolists. As we have seen repeatedly, these induced shifts in output will affect industry performance. In this subsection we explore the changes in social welfare due to changes in cross-holdings.

We present the details of the representative consumer's utility function here as we promised at the beginning of this paper. Dixit (1979) introduced the quadratic and strictly concave utility function in a duopoly. In this model the representative consumer's utility function is given by

$$U(q, I) = \alpha_1 q_1 + \alpha_2 q_2 - \frac{1}{2}(\beta_1 q_1^2 + \beta_2 q_2^2 + 2\gamma q_1 q_2) + I,$$

where I is the total expenditure on other goods that are outside this duopoly.

Vives (2000) generalized the Dixit (1979) model to allow for an arbitrary number of firms. In that case, the representative consumer's utility function is given by [12]

$$U(q, I) = \boldsymbol{\alpha}\mathbf{q} - \frac{1}{2}\mathbf{q}'\mathbf{B}\mathbf{q} + I, \tag{18}$$

[12]Oligopoly pricing: old ideas and new tools by Xaviver Vives (2000), page 180.

where $\mathbf{B} = (l_{ij})$ is a symmetric $n \times n$ positive definite matrix with $l_{ii} = \beta_i$ and $l_{ij} = \gamma_{ij}$ for $j \neq i$. Most studies interpret $\boldsymbol{\alpha}$ as quality vector for vertical product differentiation, such as Hackner (2000, 2005), Hsu and Wang (2005), and Zanchettin (2008). Since we only consider horizontally differentiated products in this paper, we let each firm have the same α. The inverse demand function for firm i's product (1) is derived from (18). The representative consumer maximizes the utility function (18) subject to a linear budget constraint of the form

$$\sum_{i=1}^{n} p_i(q)q_i + I \leq M.$$

As usual, social welfare, denoted as $W(\boldsymbol{\Delta})$, is the sum of consumer surplus,

$$U(q, I) - (\sum_{i=1}^{n} p_i(q)q_i + I),$$

and producer surplus,

$$\sum_{i=1}^{n} [p_i(q)q_i - c_i q_i].$$

To determine the social welfare change due to a small change of firm 1's ownership share in firm k, we totally differentiate social welfare, yielding

$$\frac{\partial W(\boldsymbol{\Delta})}{\partial \delta_{1k}} = \sum_{i=1}^{n} [(p_i(q) - c_i) \frac{\partial q_i}{\partial \delta_{1k}}]. \tag{19}$$

As we mentioned at the beginning of this section, a symmetric version of (18) is commonly used (e.g., Hackner, 2000, Amir, 2001, 2017). We shall continue to analyze the cross-holding effect on social welfare with the symmetric case of (18) here. The analysis with asymmetric utility will be provided in the next subsection.

With symmetry, the representative consumer's utility function becomes

$$U(q, I) = \sum_{j=1}^{n} \alpha q_j - \frac{1}{2}[\beta \sum_{j=1}^{n} q_j^2 + \gamma \sum_{j=1}^{n} q_j (\sum_{s \neq j}^{n} q_s)] + I. \tag{20}$$

The inverse demand functions in (5) are derived from (20).

Notice that the social welfare reaction in (19), due to a small change of δ_{1k}, depends on the price-cost gaps and the output reactions to a small change of δ_{1k}. With these terms in place, we are ready to present our characterization of conditions for when social welfare increases or decreases with cross-holdings. Denote $p_i^\star(q)$ as the equilibrium price for firm i, $i = 1, 2, \cdots, n$.

Theorem 3. *Let firm i be a firm such that*

$$p_i^\star(q) - c_i = \max_{1 \le j \le n} (p_j^\star(q) - c_j).$$

Suppose that either condition (16) or (17) is satisfied. Then,

$$\frac{\partial W(\boldsymbol{\Delta})}{\partial \delta_{ij}} < 0.$$

We rank firms by their price-cost gaps in equilibrium. In other words, the larger firms have bigger price-cost gaps. By Theorem 1, the total output decreases as one firm increases ownership in a rival. When the firm with largest price-cost gap acquires partial ownership in rivals, the negative effect on social welfare from this firm's reduction in output dominates the total change of social welfare. Thus, overall social welfare decreases.

Recall that for cross-holdings with a single acquiring firm in Example 7, all non-acquiring firms react to produce more when the acquiring firm increases ownership in competitors. At the new equilibrium, however, the total output decreases. In this case, the social welfare effect depends on the interaction of two basic forces. First, social welfare decreases because of the reduction in output by the acquiring firm. The social welfare impact associated with this reduction in output depends upon the acquiring firm's price-cost gap. Second, social welfare increases due to the expansion in production from all non-acquiring firms because these firms move along their reaction curves in response to the reduction in the acquiring firm's output. It concludes that, if the acquiring firm is a larger firm, the negative social welfare effect is larger than the positive social welfare effect. As a result, overall welfare decreases when the acquiring firm raises ownership share in competitors. However, welfare may well rise if the acquiring firm is smaller. In that case the negative social welfare effect from the acquiring firm is smaller than the positive social welfare effect from all non-acquiring firms. This result is consistent to the finding in Farrell and Shapiro (1990) with a homogenous product. They found that if a small firm acquires partial ownership in a rival in which it previously had no financial interest, welfare may well rise. In their paper, firms' ranking is determined by their market shares. With a homogenous product and a linear demand function, firms with lower marginal cost yield a larger market share. Therefore, Farrell and Shapiro (1990)'s finding can be interpreted as when a firm, which has small price-cost gap, increases ownership in a rival in which it previously had no financial interest, welfare may well rise. A policy implication is that, larger firms should be prohibited from raising their ownership shares in their competitors. Nonetheless, antitrust concerns might not be raised when smaller firms acquire ownership in their rivals.

3.4 Asymmetric Utility Function

We have shown that the anti-competitive and welfare effects of cross-holdings in a symmetric case with product differentiation. It is therefore worth investigating the extent to which results established under symmetry are valid under asymmetry. We begin with the case allowing for β_i or γ_{ij} to be different. Then, we will extend the analysis to the case introduced in (18).

When γ remains identical, (18) becomes

$$U(q, I) = \sum_{j=1}^{n} \alpha q_j - \frac{1}{2}[\sum_{j=1}^{n}(\beta_j q_j^2) + \gamma \sum_{j=1}^{n} q_j(\sum_{s \neq j}^{n} q_s)] + I. \tag{21}$$

Firm i's inverse demand can be derived from (21) as

$$p_i = \alpha - \beta_i q_i - \gamma \sum_{j \neq i} q_j.$$

Notice that each firm has different effect on the price of its own product but has the same effect on the price of other firms' products. In this case, a natural extension of **A1** is

$$\delta_{ii} > \frac{\gamma}{2} \max\{\frac{1}{\beta_1}, \frac{1}{\beta_2}, \cdots, \frac{1}{\beta_n}\}. \tag{22}$$

Similarly, when β remains identical, a natural extension of **A1** is

$$\delta_{ii} > \frac{1}{2\beta} \max\{\gamma_{21}, \gamma_{31}, \cdots, \gamma_{n1}\}. \tag{23}$$

Lemma 1, 2, and proposition 1, 3 can be shown to hold under condition (22) and (23). Recall that the ratio $\gamma_{ij}^2/(\beta_i\beta_j)$ expresses the degree of product substitutability between firm i's and firm j's products. If the degree of product substitutability between any two firms' products is changing, it does not necessarily affect condition (22) and (23) since the lower bound depends only on the smallest of β_i and the biggest γ_{ij}.

Recall that the inverse demand function for firm i's product can be derived from (18) as

$$p_i = \alpha - \beta_i q_i - \sum_{j \neq i} \gamma_{ij} q_j.$$

Notice that each firm has different effects on the price of one single product. lemma 3 below provides the condition that is sufficient to ensure the results from lemma 1 and 2 still hold.

Lemma 3. *Assume that*

$$\sum_{j \neq i} [\frac{\gamma_{ji}}{2\beta_j}(1 + \frac{\delta_{ji}}{\delta_{jj}})] < 1, \quad i = 1, 2, \cdots, n. \tag{24}$$

Then, $(\mathbf{I} - \mathbf{S})$ *is invertible,* $t_{ii} > 0$ *and* $\sum_{j=1}^{n} t_{ji} > 0$.

Observe that, $(\mathbf{I} - \mathbf{S})$ is a column diagonal matrix under condition (24). Under condition (24), the upper bound of firm i's ownership shares held by competitors depend on the degrees of product differentiation between firm i's product and those of other firms. In that case, firm j is allowed to acquire more shares in firm i when these two firms' products are less substitutable. Furthermore, when firm j's retained share is not large, the ownership share that firm j holds in firm i has to be small.

Under condition (24), Theorem 1 and 3 can be shown to hold. Under condition (16) and (24), the acquiring firm reduces its production as it raises ownership shares in a firm which does not have ownership shares in others. Recall that the anti-competitive effect of cross-holdings on the total industry output is larger when products are more substitutable. The following example illustrates that the anti-competitive effect of cross-holdings on the total industry output is larger when a firm raises ownership share in a rival firm whose product is more substitutable to this acquiring firm's product, than the case when the acquiring firm acquires ownership in a rival firm whose product is less substitutable to this acquiring firm's product.

Example 8. Consider a Cournot oligopoly with 3 firms. Assume that $\alpha = 4000, c_1 = 1000, c_2 = c_3 = 3000, \beta_1 = 4, \beta_2 = 1.5, \beta_3 = 1.3, \gamma_{12} = 1.1, \gamma_{13} = 1.1, \gamma_{23} = 1.2$, and cross-holding matrix is given by

$$\mathbf{\Delta} = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}.$$

Simple calculation shows that Q decreases by 29 units when firm 1 raises its share in firm 2 by 0.1 and Q decreases by 23 units when firm 1 raises its share in firm 3 by 0.1. $\square$

4 The Case with Complementary Products

With complementary products, firms might view other firms as more of partners than competitors, as in the case with substitutable products. Given this fundamental difference, one would expect that the effects of cross-holdings with complementary products will also be different from the previous section. We follow a similar approach to the case with substitutable products, beginning our analysis with a simplified case using utility function (20). Then, we will extend our analysis to a more general case as introduced in (18) in the later subsection.

4.1 Cournot Equilibrium

By (20), the ratio γ^2/β^2 is regarded as the degree of product complementarity when $\gamma < 0$. Recall that if $(\mathbf{I}-\mathbf{S})$ is invertible, ignoring the positivity requirement, the Cournot equilibrium can be solved as (6). All off-diagonal elements in $\mathbf{S}$, $s_{ij} = -(1+\delta_{ij}/\delta_{ii})\gamma/(2\beta)$ for $j \neq i$, are now all positive because $\gamma < 0$ when products are complementary. Unlike the case with substitutable products, the complementary firms' products indicate that $\mathbf{S}$ is a non-negative matrix. Consequently, $\mathbf{I}-\mathbf{S}$ is an open *Leontief Matrix*. Takayama characterizes conditions to ensure the inverse of a Leontief Matrix to be non-negative in the theorem below. [13]

Theorem 4. *Given a Leontief Matrix* $(\mathbf{I}-\mathbf{S})$*, the following conditions are equivalent.*
(i) There exists an $\mathbf{x} \geq 0$ *such that* $(\mathbf{I}-\mathbf{S}) \cdot \mathbf{x} > 0$.
(ii) For any $\mathbf{c} \geq 0$*, there exists an* $\mathbf{x} \geq 0$ *such that* $(\mathbf{I}-\mathbf{S}) \cdot \mathbf{x} = c$.
(iii) The matrix $(\mathbf{I}-\mathbf{S})$ *is nonsingular and* $(\mathbf{I}-\mathbf{S})^{\mathbf{-1}} \geq 0$.

Recall that $\mathbf{m} > 0$ in (6). In that case, a positive solution to (6) depends on an $\mathbf{x}$ that satisfies (i) in Theorem 4. A candidate for such an $\mathbf{x}$ is given by $[1, 1, \cdots, 1]'$. Consequently, a direct application of Theorem 4 establishes

Proposition 3. *Assume that a given cross-holding matrix* $\mathbf{\Delta}$ *satisfies*

$$\frac{\sum_{j\neq i} \delta_{ij}}{\delta_{ii}} < -\frac{2\beta}{\gamma} - (n-1), \quad i = 1, 2, \cdots, n. \tag{25}$$

Then, $(\mathbf{I}-\mathbf{S})^{\mathbf{-1}}$ *is non-negative.*

Observe that under condition (25), $(\mathbf{I}-\mathbf{S})$ becomes a row diagonal dominant matrix. Condition (25) requires that the ratio of the sum of a firm's ownership shares in others to its retained share, has an upper bound, which depends on the degree of product complementarity and the number of firms. In a recent paper, Amir et al. (2017) found that the strict concavity of the utility function imposes significant restrictions on the range of complementarity of products. The valid parameter range for

[13] Takayama, Mathematical Economics, Theorem 4.C.4, page 383

the complementarity cross-slope is $\gamma \in (\frac{\beta}{1-n}, 0)$. In that case, the upper bound in (25) is greater than $n-1$. We refer the reader to Amir et al. (2017) for further discussions regarding this significant restrictions on γ.

Condition (25) is not as strong as it appears. For example, in the case of oligopolies without cross-holdings, condition (25) is automatically satisfied since $\delta_{ij} = 0$ for $j \neq i$. In this case, a positive Cournot equilibrium always exists when γ satisfies the concavity requirement. Under condition (25), firms whose retained shares are larger, are allowed to acquire more shares in others than firms whose retained shares are smaller. A simple corollary to proposition 3 is the following.

Corollary 2. *Given $\mathbf{\Delta}$, β and γ, (25) is satisfied if the following condition, to be referred to as **A2**, holds:*

$$\delta_{ii} > -\frac{\gamma(n-1)}{2\beta}, \quad i = 1, 2, \cdots, n.$$

A2 puts a lower bound on how much ownership share each firm i needs to retain. The same intuition to the case with substitutable products can be applied here. Each firm must have ownership share in its own firm to have incentives to produce. Recall that the strict concavity of the utility function requires that $\gamma \in (\frac{\beta}{1-n}, 0)$. Correspondingly, this lower bound in (28) takes value from 0 to 1/2. **A2** is weakened as products are less complementary. Therefore, a positive Cournot equilibrium is more likely to exist as products become less complementary. Observe that $\mathbf{m}$ in (6) does not require firms' technologies to be the same. Consequently, under **A2**, a positive Cournot equilibrium allows for firms to have different marginal costs. The following example provides an illustration.

Example 9. Consider a Cournot duopoly. $(\mathbf{I} - \mathbf{S})$ is given by

$$\begin{bmatrix} 1 & \frac{\gamma}{2\beta}(1 + \frac{\delta_{12}}{\delta_{11}}) \\ \frac{\gamma}{2\beta}(1 + \frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix}.$$

Under **A2**,

$$det(\mathbf{I} - \mathbf{S}) = 1 - \frac{\gamma^2}{4\beta^2}(1 + \frac{\delta_{12}}{\delta_{11}})(1 + \frac{\delta_{21}}{\delta_{22}}) > 0.$$

Consequently,

$$(\mathbf{I}-\mathbf{S})^{-\mathbf{1}} = \frac{1}{1-\frac{\gamma^2}{4\beta^2}(1+\frac{\delta_{12}}{\delta_{11}})(1+\frac{\delta_{21}}{\delta_{22}})} \begin{bmatrix} 1 & -\frac{\gamma}{2\beta}(1+\frac{\delta_{12}}{\delta_{11}}) \\ -\frac{\gamma}{2\beta}(1+\frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix} > 0.$$

It follows that,

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = (\mathbf{I}-\mathbf{S})^{-\mathbf{1}} \begin{bmatrix} \frac{\alpha-c_1}{2\beta} \\ \frac{\alpha-c_2}{2\beta} \end{bmatrix} > 0.$$

□

4.2 Anti-Competitive Effects

A firm's individual decision has positive effects on other firms with product complementarity. Such positive externalities are not taken into consideration without cross-holdings. However, the positive externalities are partially internalized with cross-holdings. Therefore, we should expect that cross-holdings increase production.

Recall that the marginal changes of firms' outputs due to a small change of firm 1's ownership share in firm k are given by (13)

$$(\mathbf{I}-\mathbf{S})\cdot\mathbf{dq} = \mathbf{k},$$

where $\mathbf{k}$ is a $\mathbf{n}\times\mathbf{1}$ matrix with the first element as $-\gamma q_k \cdot d\delta_{1k}/(2\beta\delta_{11})$ and kth elements as $-\gamma(\sum_{j\neq i}\delta_{ij}q_j)d\delta_{1k}/(2\beta\delta_{kk}^2)$, zero otherwise. Observe that $\mathbf{k}$ is non-negative since $\gamma < 0$. By Corollary 2, under **A2**

$$\mathbf{dq} = (\mathbf{I}-\mathbf{S})^{-\mathbf{1}}\mathbf{k} > 0.$$

It follows that when a single firm raises ownership share in one other firm, all firms produce more. Example 11 below provides an illustration.

Example 10. Consider a Cournot duopoly. Under **A2**, firm 1's and firm 2's output changes, due to firm 1's increasing ownership share in firm 2, are given by

$$\begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\gamma}{2\beta}(1+\frac{\delta_{12}}{\delta_{11}}) \\ \frac{\gamma}{2\beta}(1+\frac{\delta_{21}}{\delta_{22}}) & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\gamma q_2 \cdot d\delta_{12}/(2\beta\delta_{11}) \\ -\gamma(\sum_{j\neq i}\delta_{ij}q_j)d\delta_{1k}/(2\beta\delta_{kk}^2) \end{bmatrix}$$
$$= (\mathbf{I}-\mathbf{S})^{-\mathbf{1}} \begin{bmatrix} -\gamma q_2 \cdot d\delta_{12}/(2\beta\delta_{11}) \\ -\gamma(\sum_{j\neq i}\delta_{ij}q_j)d\delta_{1k}/(2\beta\delta_{kk}^2) \end{bmatrix} > 0.$$

Under **A2**, both firms expand their production when firm 1 acquires partial ownership in firm 2. □

Intuitively, when a single firm increases ownership in one of other firms, this acquiring firm is induced to take into consideration the effect of its output decision on the acquired firm's profit. It realizes that increasing its output raises the profit at the acquired firm and hence its share in the acquired firm's profit. This consideration makes this acquiring firm to expand production. Recall that reaction functions are upward sloping when products are complementary. It implies that the initial output increase by the acquiring firm will be followed by output increase on all other firms. In the new Cournot equilibrium, after full adjustment has taken place, outputs in the industry will rise, and firms will be better off.

Now we are ready to analyze what effects cross-holdings would have on the total industry equilibrium output when more firms simultaneously increase ownership shares in competitors.

Theorem 5. *For $\mathbf{\Delta}$, β and γ satisfying **A2**, firms increase their outputs with cross-holdings.*

When more firms acquire partial ownership shares simultaneously, the changes in cross-holdings can be decomposed into a sequence of unilateral changes made by each single firm. In each step from this sequence, only one firm acquires ownership share in one of other firms. According to Corollary 2, when a single firm raises share in one other firm, firms expand their production. Applying Corollary 2 to each step from this sequence can establish Theorem 5. This effect arises because a firm's individual decision has positive effects on other firms with product complementarity. With cross-holdings, the positive externalities are partially internalized. An implication of this finding is that with complementary products, cross-holdings make firms more cooperative.

A2 also points out the best strategy for firms to engage in cross-holdings. Firms should acquire ownership in a firm, whose retained share is large, not in a firm whose retained share is small. Otherwise, more cross-holdings may violate **A2**, as the following example illustrates.

Example 11. Consider a Cournot oligopoly with 3 firms. Suppose $\beta = 2$, $\gamma = -1$ $\alpha = 3$. Let the original cross-holding matrix be $\mathbf{\Delta}_1$.

$$\mathbf{\Delta}_1 = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.5 \\ 0.1 & 0.5 & 0.3 \end{bmatrix} \quad \longrightarrow \quad \mathbf{\Delta_2} = \begin{bmatrix} 0.7 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.5 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}$$

Notice that **A2** is satisfied. Consequently, a positive Cournot equilibrium exists, as shown a positive $(\mathbf{I}-\mathbf{S})_{\mathbf{1}}^{\mathbf{-1}}$.

$$(\mathbf{I}-\mathbf{S})_{\mathbf{1}}^{\mathbf{-1}} = \begin{bmatrix} 4.55 & 4.8 & 4.66 \\ 5.81 & 8.12 & 7.29 \\ 5.88 & 7.7 & 8 \end{bmatrix} \quad \longrightarrow \quad (\mathbf{I}-\mathbf{S})_{\mathbf{2}}^{\mathbf{-1}} = \begin{bmatrix} -2.48 & -4 & -4.35 \\ -5.77 & -6.45 & -7.5 \\ -5.11 & -6.19 & -6 \end{bmatrix}$$

If both firm 1 and 3 raise their ownership shares in firm 2 by 0.1, holding everything else constant. The cross-holding matrix becomes $\mathbf{\Delta}_2$. It is worth noticing that **A2** is violated. As a result, a positive Cournot equilibrium does not exist, as shown a negative $(\mathbf{I}-\mathbf{S})_{\mathbf{2}}^{\mathbf{-1}}$. However, if γ increases from -1 to -0.5, under the same cross-holdings matrix $\mathbf{\Delta}_2$, a positive Cournot equilibrium still exists. It concludes that, **A2** is weakened as products become less complementary. □

When products are substitutable, the total output is negatively correlated with the degree of product substitutability. We now examine how this result may vary when products are complementary.

Proposition 4. *For $\mathbf{\Delta}$, β and γ satisfying* ***A2****, total industry equilibrium output increases as products become less complementary.*

Similar reasoning as in the case of substitutable products, the less complementary products are, the steeper of the slopes of firms' demand curves become. Thus, the less complementary products become, the stronger the strength of the anti-competitive effect of cross-holdings is.

4.3 Welfare Analysis

When products are substitutable, industries tend to be more concentrated with cross-holdings. As a result, when a big firm raises share in competitors, social welfare falls. However, when products are complementary, we have found that cross-holdings increase total industry equilibrium output. We would expect that the effect of cross-holdings on social welfare will also be different from the previous section. Recall that the marginal change of social welfare due to a small change of firm 1's ownership share in firm k is given by (19)

$$\frac{\partial W(\mathbf{\Delta})}{\partial \delta_{1k}} = \sum_{i=1}^{n} [(p_i(q) - c_i)\frac{\partial q_i}{\partial \delta_{1k}}].$$

By Theorem 5 each firm reacts to a small increasing of firm 1's ownership share in firm k by expanding production. With these terms in place, the following Theorem reveals the effect of cross-holdings on social welfare.

Theorem 6. *For $\mathbf{\Delta}$, β and γ satisfying* ***A2****, social welfare rises as cross-holdings increase.*

Economides and Salop (1992) found that joint ownership or integration by firms with complementary products raises welfare. Our analysis shows encouraging firms to engage in cross-holdings can also enhance total welfare. **A2** is less restrictive as products become less complementary. In this sense, the less complementary products are, the more likely cross-holdings are socially preferable. Policy makers should encourage firms to acquire passive ownership shares in firms whose retained shares are large, but discourage firms to raise ownership shares in firms whose retained shares are small.

4.4 Asymmetric Utility Function

As we can see in section 3.4, the asymmetry complicates the conditions resulting the anti-competitive and welfare effects of cross-holdings. When β_i and γ_{ij} are not identical for firms, the representative consumer maximizes utility given by (18). By (18), $(\mathbf{I}-\mathbf{S})$ is still a Leontief Matrix. In that case, $[1, 1, \cdots, 1]'$ is also a candidate for an $\mathbf{x}$ in Theorem 4. Consequently, a direct application of Theorem 4 establishes

Corollary 3. *Suppose that a given cross-holding matrix $\boldsymbol{\Delta}$ satisfies*

$$\delta_{ii} > -\frac{n-1}{2} \max\left(\frac{\gamma_{i1}}{\beta_1}, \frac{\gamma_{i2}}{\beta_2}, \cdots, \frac{\gamma_{in}}{\beta_n}\right), \quad i = 1, 2, \cdots, n. \tag{26}$$

Then, $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ *is non-negative.*

Under condition (26), the upper bound of firm i's retained share depends on the largest ratio of firm i's effect on a firm's price to the slope of own-quantity of this firm. Theorem 5 and 6 can be shown to hold under condition (26). It follows that, as cross-holdings increase firms expand production and social welfare rises.

4.5 Discussion

The performance of a non-cooperative oligopoly model with cross-holdings depends on the interaction of two effects. First, cross-holdings allow linked firms to absorb negative or positive externalities. Second, cross-holdings elicit a spiral of responses from rival firms. When products are complementary, this response tends to be beneficial to all firms because reaction functions are upward sloping. In such an environment, firms are likely to have incentives to engage in cross-holdings. When products are substitutable, on the other hand, the response of rivals tend to hurt the acquiring firm because in this environment, reaction functions are typically downward sloping.

The welfare effects in the two situations are also very different. It is shown that cross-holdings generally increase social welfare when products are complementary, but decrease social welfare when the acquiring firm has the largest price-cost gap with substitutable products. Thus, from a social point of view, our analysis suggests that cross-holdings should be promoted with complementary products, while they should be discouraged if the acquiring firm has the largest price-cost gap with substitutable products.

There is a fundamental asymmetry between substitutable products and complementary products. For substitutability, the valid range for γ is indeed $(0, \beta]$, and this range is independent of the number of firms. However, for complementarity, the valid range is $(\beta/(1-n), 0)$, which monotonically shrinks with the number of firms, and converges to the empty set as the number of firms goes to positive infinity. This fundamental difference between the two kinds of products, makes the anti-competitive and welfare effects of cross-holdings different.

5 Bertrand Competition

Kreps and Scheinkman (1983) argued that whether firms compete in quantities or prices is ultimately an empirical question. In the real world, both Cournot and Bertrand behaviors are observed. For example, farmers set quantities at local farmers' markets, while restaurants set prices. These empirical observations indicate that the study of cross-holdings in Bertrand model with product differentiation is also necessary.

To get the analysis under price competition, we take advantage of the duality structure of Cournot and Bertrand competition in our differentiated product setting. This duality was first pointed out by Sonnenschein (1968) in a non-differentiated framework, and extended by Singh and Vives (1984). We will use the symmetric utility provided in (18) to analyze the anti-competitive and welfare effects of cross-holdings under price competition. Under symmetry, firm i's demand function is

$$q_i = a - bp_i + r\sum_{j\neq i}^{n} p_j, \quad i = 1, 2, \cdots, n,$$

where

$$a = \frac{\alpha}{[\beta + (n-1)\gamma]},$$
$$b = \frac{\beta + (n-2)\gamma}{[\beta + (n-1)\gamma](\beta - \gamma)},$$
$$r = \frac{\gamma}{[\beta + (n-1)\gamma](\beta - \gamma)}.$$

a, b and r are always positive for substitutable products. Recall that the strictly concavity of utility requires that $\gamma \in (\frac{\beta}{1-n}, 0)$ when products are complementary. In that case, a and b are positive while r is negative. Under Bertrand competition, firm i's first-order is given by

$$b(p_i - c_i) - r\sum_{j\neq i}(p_j - c_j) = \frac{r}{2}\sum_{j\neq i}^{n}\frac{\delta_{ij}}{\delta_{ii}}(p_j - c_j) + \frac{a + bc_i - r\sum_{j\neq i}^{n} c_j}{2}. \tag{27}$$

In our setting, it turns out that Cournot (Bertrand) competition with substitutable products is the dual of Bertrand (Cournot) competition with complementary products. This means that they share similar strategic properties. It is a matter of interchanging prices and quantities. A useful corollary is that one only needs to make computations or prove theorems for one type of competition (Cournot or Bertrand) or for one type of product (substitutable or complementary); the other cases follow by duality. Therefore, each of the results in Section III and IV corresponds a dual theorem dealing with Bertrand competition.

5.1 The Case with Substitutable Products

Bertrand competition with substitutable products is the dual of Cournot competition with complementary goods. It follows that, a similar theorem to Theorem 4 establishes the anti-competitive effects of cross-holdings in Bertrand competition with substitutable products.

Theorem 7. *Suppose cross-holding matrix satisfies*

$$\delta_{ii} > \frac{n-1}{2(\frac{\beta}{\gamma}+n-2)}, \quad i = 1, 2, \cdots, n. \tag{28}$$

Then, prices increase with cross-holdings.

The lower bound in (28) takes value from 0 to 1/2. With substitutable products, a decrease in price has positive effects on the demand of other products. Without cross-holdings, when a firm contemplates raising price, it does not care about the positive externalities it would confer upon competitors. But with cross-holdings, such externalities are partially internalized. Intuitively, a firm with interests in competitors is induced to increase the price of its product. It follows that the outputs of competitors will also increase. The acquiring firm may benefit from the increased operating earnings that result from the higher levels of output of competitors through cross-holdings. In Bertrand competition with substitutable products, firms' reaction functions are upward sloping. The acquiring firm's less competitive behavior induces the other firms to compete less aggressively. Thus, firms charge higher prices as cross-holdings increase. In other words, industry becomes less competitive as cross-holdings increase. Our result is similar to the findings in Dietzenbacher et al. (2000). They found that competition is reduced due to shareholding interlocks in Bertrand competition. As an empirical example, the Dutch financial sector is used in their paper. Comparing the case of shareholding with the case of no-shareholding, the price-cost margins are found to be up to 2% higher in a Bertrand oligopoly.

Our result critically depends on the degree of product substitutability. The condition resulting in this result is less restrictive as products become less substitutable. Another implication is that the anti-competitive effect of cross-holdings is more likely to be achieved, as products are less substitutable.

The change in social welfare due to a small increasing of firm 1's ownership share in firm k, is now defined as

$$\frac{\partial W(\mathbf{\Delta})}{\partial \delta_{1k}} = \sum_{i=1}^{n}\{[-b(p_i - c_i) + r\sum_{j\neq i}(p_j - c_j)]\frac{\partial p_i}{\partial \delta_{1k}}\}. \tag{29}$$

By Theorem 7, firms raise prices as cross-holdings increase. Consequently, it may not be socially desirable. Theorem 8 below provides a confirmation.

Theorem 8. *For $\mathbf{\Delta}$, β and γ satisfying (28), social welfare decreases with cross-holdings.*

According to Singh and Vives (1984), $a + bc_i - r\sum_{j\neq i}^{n} c_j$ in (27) is positive. Consequently, the right side in (27) is positive. It concludes that when one firm raises ownership shares in competitors, social welfare falls. Condition (28) is weakened as products become less substitutable. It follows that, when products are less substitutable, social welfare are more likely to decrease as cross-holdings increase.

5.2 The Case with Complementary Products

Bertrand competition with complementary products is the dual of Cournot competition with substitutable goods. It follows that, a similar theorem to Theorem 1 establishes the anti-competitive effects of cross-holdings in Bertrand competition with complementary products.

Theorem 9. *Suppose cross-holding matrix satisfies*

$$\delta_{ii} > -\frac{1}{2(\frac{\beta}{\gamma} + n - 2)}, \quad i = 1, 2, \cdots, n.$$

Then, the total industry equilibrium price decreases as cross-holdings increase.

With complementary products, a decrease in price has positive effects on the demand of other products. Such positive externalities are partially internalized with cross-holdings. Theorem 9 suggests that cross-holdings make firms more cooperative. Furthermore, less differentiation makes the anti-competitive effect easier to be achieved.

Theorem 10. *Let cross-holding matrix be given. Then, $\frac{\partial p_i}{\partial \delta_{ik}} < 0$ under any following* *two conditions*

$$(i) \quad \delta_{kh} = 0, \quad \forall h \neq k; \tag{30}$$

$$(ii) \quad \sum_{j \neq k} \delta_{ij} + (n-2)\delta_{ii} > (\frac{\beta}{\gamma} + n - 2)(\delta_{ii} + \delta_{ik}). \tag{31}$$

Under condition (30) or (31), when a single firm raises ownership share in rivals, the price of this acquiring firm's product goes down. For cross-holdings with a single acquiring firm in Example 7, all non-acquiring firms react to charge higher prices when the acquiring firm increases ownership in non-acquiring firms. At the new equilibrium, however, the total industry equilibrium price decreases. This result is similar as with substitutable products in Cournot competition since firms' reaction functions are downward sloping with complementary products in Bertrand competition. Thus, they have similar interpretations.

Theorem 11. *Let firm i be a firm such that*

$$p_i^\star - c_i = \max_{1 \leq j \leq n} (p_j^\star - c_j).$$

Suppose that either condition (34) or (35) is satisfied. Then,

$$\frac{\partial W(\mathbf{\Delta})}{\partial \delta_{ij}} > 0.$$

When a firm with largest price-cost gap acquires partial ownership in others, social welfare rises. A policy implication is that, firm with largest price-cost gap should be encouraged from raising their ownership shares in other firms under price competition.

6 Conclusion

In this paper, we have analyzed the anti-competitive and welfare effects of cross-holdings in oligopolies with product differentiation. Our results showed that the known anti-competitive effect, that cross-holdings in general make industries more concentrated, is robust with respect to product differentiation in Cournot competition. We found that this anti-competitive effect is stronger the less substitutable products are. With product complementarity, cross-holdings induce firms to cooperate, and this response is easier to be achieved the less complementary products are. We also conducted welfare analysis. When a firm with largest price-cost gap acquires partial ownership in rivals, cross-holdings are more likely to be harmful with substitutable products. In comparison, with complementary products, everyone is generally better off with cross-holdings. Additionally, this welfare effect is more likely to be achieved when products are less complementary.

Moreover, we used the duality structure of Cournot and Bertrand competition to analyze the anti-competitive and welfare effects of cross-holdings with product differentiation in Bertrand competition. With complementary products, the total industry equilibrium price decreases with cross-holdings. It was shown that firms charge higher prices for their products and social welfare decreases as cross-holdings increase with substitutable products.

Our analysis has both empirical and policy implications, which we will pursue in future research.

Appendix: Proof

Proof of Lemma 1: We use contradiction to prove this lemma. First, assume that $\mathbf{I} - \mathbf{S}$ is not invertible. It follows that the column vectors in $\mathbf{I} - \mathbf{S}$ are linearly dependent. Consequently, there exists a non-zero column vector, $\boldsymbol{\lambda}' = (\lambda_1, \lambda_1, \cdots, \lambda_n)$, such that $(\mathbf{I} - \mathbf{S}) \cdot \boldsymbol{\lambda} = 0$. Notice that $(\mathbf{I} - \mathbf{S})$ is positive. It follows that λ_i can not be all positive nor all negative. Thus, at least one element in $\boldsymbol{\lambda}$ is negative and at least one is positive. Next, assume that there exist $\lambda_{j_1}, \lambda_{j_2}, \cdots \lambda_{j_{n_1}}$ such that $\lambda_{j_k} < 0$ for $1 \leq k \leq n_1 < n$. Consequently, the rest $n - n_1$ elements are non-negative. Denote these elements as $\lambda_{i_h} \geq 0$ for $1 \leq h \leq n_2 = n - n_1$.

Multiplying each j_kth equation in $(\mathbf{I} - \mathbf{S}) \cdot \boldsymbol{\lambda} = 0$ by $2\delta_{j_k}\beta/\gamma$ for $1 \leq k \leq n_1 < n$, and adding these n_1 equations yield

$$(\sum_{k=1}^{n_1}\delta_{j_kj_k})(\sum_{u=1}^{n}\lambda_u)+\sum_{l=1}^{n_2}(\sum_{k=1}^{n_1}\delta_{j_ki_l}\lambda_l)+\sum_{s=1}^{n_1}\{[\sum_{k\neq s}^{n_1}\delta_{j_kj_s}+(\frac{2\beta}{\gamma}-1)\delta_{j_sj_s}]\lambda_{j_s}\}=0.$$

If $\sum_{u=1}^{n}\lambda_u\leq 0$, the following condition must be satisfied

$$F(\delta,\beta,\gamma,\lambda)=\sum_{l=1}^{n_2}(\sum_{k=1}^{n_1}\delta_{j_ki_l}\lambda_{i_l})+\sum_{s=1}^{n_1}\{[\sum_{k\neq s}^{n_1}\delta_{j_kj_s}+(\frac{2\beta}{\gamma}-1)\delta_{j_sj_s}]\lambda_{j_s}\}\geq 0.$$

Under **A1**,

$$\sum_{k=1}^{n_1}\delta_{j_kl}\lambda_{i_l}\leq(1-\delta_{i_li_l})\lambda_{i_l}<(1-\frac{\gamma}{2\beta})\lambda_{i_l},\ l=i_1,\cdots,n_2,$$

and

$$[\sum_{k\neq s}^{n_1}\delta_{j_kj_s}+(\frac{2\beta}{\gamma}-1)\delta_{j_sj_s}]\lambda_{j_s}<(1-\frac{\gamma}{2\beta})\lambda_{j_s},\ s=1,\cdots,n_1.$$

It follows that $F(\delta,\beta,\gamma,\lambda)<0$, which is a contradiction. Then, $\sum_{u=1}^{n}\lambda_u>0$. Next, Multiplying each i_hth equations in $(\mathbf{I}-\mathbf{S})\cdot\boldsymbol{\lambda}=0$ by $2\delta_{i_h}\beta/\gamma$ for $1\leq h\leq n_2$, and adding these n_2 equations yield

$$(\sum_{h=1}^{n_2}\delta_{i_hi_h})(\sum_{u=1}^{n}\lambda_u)+\sum_{l=1}^{n_1}[(\sum_{h=1}^{n_2}\delta_{i_hj_l})\lambda_{j_l}]+\sum_{s=1}^{n_2}\{[\sum_{h\neq s}^{n_2}\delta_{i_si_1}+(\frac{2\beta}{\gamma}-1)\delta_{i_si_s}]\lambda_{i_s}\}=0.$$

Since $\sum_{u=1}^{n}\lambda_u>0$, the following condition must hold

$$G(\delta,\beta,\gamma,\lambda)=\sum_{l=1}^{n_1}[(\sum_{h=1}^{n_2}\delta_{i_hj_l})\lambda_{j_l}]+\sum_{s=1}^{n_2}\{[\sum_{h\neq s}^{n_2}\delta_{i_si_1}+(\frac{2\beta}{\gamma}-1)\delta_{i_si_s}]\lambda_{i_s}\}\leq 0.$$

Under **A1**,

$$(\sum_{h=1}^{n_2}\delta_{i_hj_l})\lambda_{j_l}>(1-\frac{\gamma}{2\beta})\lambda_{j_l},l=1,\cdots,n_1$$

and

$$[\sum_{h\neq s}^{n_2}\delta_{i_si_1}+(\frac{2\beta}{\gamma}-1)\delta_{i_si_s}]\lambda_{i_s}\geq[(\frac{2\beta}{\gamma}-1)\delta_{i_hi_h}]]\lambda_{i_s}>(1-\frac{\gamma}{2\beta})\lambda_{i_s},\ s=1,\cdots,n_2.$$

Consequently, $G(\delta,\beta,\gamma,\lambda)>0$, which is a contradiction as well. Thus, all λ_i have to be zero. Therefore, the column vectors in $\mathbf{I}-\mathbf{S}$ are linearly independent. It follows that, $\mathbf{I}-\mathbf{S}$ is invertible. □

Proof of Lemma 2 According to lemma 1, $\mathbf{I}-\mathbf{S}$ is invertible under **A1**. First, we prove $t_{ii}>0$ using contradiction. Suppose that $t_{ii}\leq 0$. Recall that $\mathbf{I}-\mathbf{S}$ is positive. Then, at least one element in the ith column is positive. It follows that there exist $t_{j_1 i}, t_{j_2 i}, \cdots, t_{j_{n_1} i}$ such that $t_{j_k i}>0$, $1\leq k\leq n_1<n-1$. Thus, the rest $n-n_1-1$ elements(other than t_{ii}) from the ith column in $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ is non-positive, such as $t_{i_h i}\leq 0$, $0\leq h\leq n_2\leq n-1-n_1$. Multiplying the j_kth row in $(\mathbf{I}-\mathbf{S})$ by the ith column in $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ yields n_1 equations. Next, multiplying each equation by $2\beta\delta_{j_k}/\gamma$ for $1\leq k\leq n_1<n-1$ and adding these equations get

$$(\sum_{k=1}^{n_1}\delta_{j_k j_k})(\sum_{u=1}^{n}\lambda_u)+(\sum_{k=1}^{n_1}\delta_{j_k i})t_{ii}+\sum_{l=1}^{n_2}\{[(\sum_{k=1}^{n_1}\delta_{j_k i_l})t_{i_l i}]+ \sum_{s=1}^{n_2}[\sum_{k\neq s}^{n_1}\delta_{j_k i}+(\frac{2\beta}{\gamma}-1)\delta_{j_s j_s}]t_{j_s i}\}=0.$$

If $\sum_{u=1}^{n}\lambda_u\geq 0$, the following condition must hold

$$H(\delta,\beta,\gamma,t)=(\sum_{k=1}^{n_1}\delta_{j_k i})t_{ii}+\sum_{l=1}^{n_2}\{[(\sum_{k=1}^{n_1}\delta_{j_k i_l})t_{i_l i}]+\sum_{s=1}^{n_2}[\sum_{k\neq s}^{n_1}\delta_{j_k i}+(\frac{2\beta}{\gamma}-1)\delta_{j_s j_s}]t_{j_s i}\}\leq 0.$$

Under **A1**, $(\sum_{k=1}^{n_1}\delta_{j_k i})t_{ii}\geq(1-\delta_{ii})t_{ii}>(1-\gamma/(2\beta))t_{ii}$, and similarly $(\sum_{k=1}^{n_1}\delta_{j_k i_l})t_{i_l i}>(1-\gamma/(2\beta))t_{i_l i}$ for $l=1,\cdots,n_2$. Moreover,

$$[\sum_{k\neq s}^{n_1}\delta_{j_k i}+(\frac{2\beta}{\gamma}-1)\delta_{j_s j_s}]t_{j_s i}>[\sum_{k\neq s}^{n_1}\delta_{j_k i}+(1-\frac{\gamma}{2\beta})]t_{j_s i},\ \ s=1,\cdots,n_1.$$

It follows that $H(\delta,\beta,\gamma,t)>0$, which is a contradiction. Then, $\sum_{u=1}^{n}\lambda_u<0$.

Next, multiply the ith and i_hth row in $(\mathbf{I}-\mathbf{S})$ by the ith column in $(\mathbf{I}-\mathbf{S})^{-\mathbf{1}}$ for $0\leq h\leq n_2\leq n-1-n_1$. Next, multiplying each equation by $2\delta_{i_h}\beta/\gamma$ for $0\leq h\leq n_2\leq n-1-n_1$, and adding these n_2 equations yield

$$(\sum_{h=1}^{n_2}\delta_{i_h i_h}+\delta_{ii})(\sum_{u=1}^{n}\lambda_u)+[\sum_{h=1}^{n_2}\delta_{i_h i}+(\frac{2\beta}{\gamma}-1)\delta_{ii}]t_{ii}+\sum_{s=1}^{n_1}[(\delta_{ij_s}+\sum_{h=1}^{n_2}\delta_{i_h j_s})t_{j_s i}]+ \sum_{l=1}^{n_2}\{[\sum_{h\neq l}^{n_2}\delta_{i_h i_l}+\delta_{ii_l}+(\frac{2\beta}{\gamma}-1)\delta_{i_l i_l}]t_{i_l i}\}=\frac{2\beta}{\gamma}\sum_{h=1}^{n_2}\delta_{i_h}.$$

Since $\sum_{u=1}^{n}\lambda_u < 0$, the following expression must be satisfied

$$\begin{aligned}Z(\delta,\beta,\gamma,\lambda) =&[\sum_{h=1}^{n_2}\delta_{i_h i} + (\frac{2\beta}{\gamma}-1)\delta_{ii}]t_{ii} + \sum_{s=1}^{n_1}[(\delta_{ij_s} + \sum_{h=1}^{n_2}\delta_{i_h j_s})t_{j_s i}]\\ &+\sum_{l=1}^{n_2}\{[\sum_{h\neq l}^{n_2}\delta_{i_h i_l} + \delta_{ii_l} + (\frac{2\beta}{\gamma}-1)\delta_{i_l i_l}]t_{i_l i}\} > 0.\end{aligned}$$

Under **A1**, $[\sum_{h=1}^{n_2}\delta_{i_h i} + (2\beta/\gamma-1)\delta_{ii}]t_{ii} < (1-\gamma/(2\beta))t_{ii}$, $(\delta_{ij_s} + \sum_{h=1}^{n_2}\delta_{i_h j_s})t_{j_s i} < (1-\gamma/(2\beta))t_{j_s i}$ for $s = 1,\cdots,n_1$, and $\sum_{h\neq l}^{n_2}\delta_{i_h i_l} + \delta_{ii_l} + (2\beta/\gamma-1)\delta_{i_l i_l}]t_{i_l i} < (1-\gamma/(2\beta))t_{i_l i}$ for $l = 1,\cdots,n_2$. Consequently, $Z(\delta,\beta,\gamma,\lambda) < 0$, which is a contradiction as well. Thus, if $t_{ii} \leq 0$, we have proved that both $\sum_{u=1}^{n}\lambda_u \geq 0$ and $\sum_{u=1}^{n}\lambda_u < 0$ are contradictory. It concludes that such matrix does not exist! Therefore,

$$t_{ii} > 0.$$

Now, we use contradiction to prove that $\sum_{j=1}^{n} t_{ji} > 0$. Assume that $\sum_{j=1}^{n} t_{ji} \leq 0$. It follows that there exist at least one element such that $t_{j_k i} \leq 0$ for $1 \leq k \leq n_1 < n-1$. Thus, the rest $n - n_1 - 1$ elements(other than t_{ii}) from the ith column of $(\mathbf{I}-\mathbf{S})^{-1}$ is positive, such that $t_{i_h i} > 0$ for $1 \leq h \leq n_2 \leq n-1-n_1$. Multiply the j_kth equation in $(\mathbf{I}-\mathbf{S}$ by the ith colomn in $(\mathbf{I}-\mathbf{S})^{-1}$ for $1 \leq k \leq n_1 < n-1$. Next, multiplying each equation by $2\beta\delta_{j_k}/\gamma$ for $1 \leq k \leq n_1 < n-1$, and adding these n_1 equations get

$$\begin{aligned}&(\sum_{k=1}^{n_1}\delta_{j_k j_k})(\sum_{u=1}^{n}\lambda_u) + (\sum_{k=1}^{n_1}\delta_{j_k i})t_{ii} + \sum_{l=1}^{n_1}[(\sum_{k=1}^{n_1}\delta_{j_k i_l})t_{i_l i}]+\\ &\sum_{s=1}^{n_1}\{[\sum_{k\neq s}^{n_1}\delta_{j_k i} + (\frac{2\beta}{\gamma}-1)\delta_{j_s j_s}]t_{j_s i}\} = 0.\end{aligned}$$

If $\sum_{u=1}^{n}\lambda_u \leq 0$, the following condition must be satisfied

$$V(\delta,\beta,\gamma,t) = (\sum_{k=1}^{n_1}\delta_{j_k i})t_{ii} + \sum_{l=1}^{n_1}[(\sum_{k=1}^{n_1}\delta_{j_k i_l})t_{i_l i}] + \sum_{s=1}^{n_1}\{[\sum_{k\neq s}^{n_1}\delta_{j_k i} + (\frac{2\beta}{\gamma}-1)\delta_{j_s j_s}]t_{j_s i}\} > 0.$$

Under **A1**, $(\sum_{k=1}^{n_1}\delta_{j_k i})t_{ii} \leq (1-\delta_{ii})t_{ii} < (1-\gamma/(2\beta))t_{ii}$, and similarly $(\sum_{k=1}^{n_1}\delta_{j_k i_l})t_{i_l i} < (1-\gamma/(2\beta))t_{i_l i}$ for $l = 1,\cdots,n_2$. Moreover,

$$[\sum_{k\neq s}^{n_1}\delta_{j_k i} + (\frac{2\beta}{\gamma}-1)\delta_{j_s j_s}]t_{j_s i} < [\sum_{k\neq s}^{n_1}\delta_{j_k i} + (1-\frac{\gamma}{2\beta})]t_{j_s i},\ \ s = 1,\cdots,n_1.$$

It follows that $V(\delta, \beta, \gamma, t) < 0$, which is a contradiction. Hence,

$$\sum_{u=1}^{n} \lambda_u > 0.$$

$\square$

Proof of Proposition 1 According to lemma 1, $\mathbf{I} - \mathbf{S}$ is invertible under **A1**. It follows that the transpose of $\mathbf{I} - \mathbf{S}$ is also invertible under (8). Denote $(\mathbf{A^T})^{-1} = (z_{ij})$. To prove $\sum_{j=1}^{n} t_{ij} > 0$ is equivalent to prove $\sum_{j=1}^{n} z_{ji} > 0$, $i = 1, 2 \cdots, n$, since $(\mathbf{A^T})^{-1} = (\mathbf{A}^{-1})^{\mathbf{T}}$. Given $\mathbf{A^T}$ is a positive matrix, at least one element is positive from ith column in $(\mathbf{A^T})^{-1}$. Assume that there exist $z_{j_1 i}, z_{j_2 i}, \cdots, z_{j_{n_2} i}$ such that $z_{j_k i} > 0$ for $1 \leq k \leq n_2 < n$. Thus, the rest $n - n_2$ elements are non-positive such that $z_{i_h i} \leq 0$ for $0 \leq h \leq n_1 < n$. First we multiply the i_hth row in $\mathbf{A^T}$ by the ith column in $(\mathbf{A^T})^{-1}$, $0 \leq h \leq n_1 < n$. Next, adding these n_1 equations yields

$$n_1 \frac{\gamma}{2\beta}(\sum_{j=1}^{n} z_{ji}) + \sum_{l=1}^{n_1}[\frac{\gamma}{2\beta}(\frac{\sum_{h\neq l}^{n_1} \delta_{i_l i_h}}{\delta_{i_l i_l}} + \frac{2\beta}{\gamma} - 1)z_{i_l i}] + \sum_{s=1}^{n_2}[\frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1} \delta_{j_s i_h}}{\delta_{j_s j_s}}]z_{j_s i}] = 0 \text{ or } 1.$$

If $\sum_{j=1}^{n} z_{ji} \leq 0$, it must be true that

$$D(\delta, \beta, \gamma, z) = \sum_{l=1}^{n_1}[\frac{\gamma}{2\beta}(\frac{\sum_{h\neq l}^{n_1} \delta_{i_l i_h}}{\delta_{i_l i_l}} + \frac{2\beta}{\gamma} - 1)z_{i_l i}] + \sum_{s=1}^{n_2}[\frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1} \delta_{j_s i_h}}{\delta_{j_s j_s}}]z_{j_s i}] \geq 0.$$

Under **A1**,

$$\frac{\gamma}{2\beta}[\frac{\sum_{h=1}^{n_1} \delta_{j_s i_h}}{\delta_{j_s j_s}}]z_{j_s i} < (\frac{2\beta}{\gamma} - 1)z_{j_s i}, \ s = 1, \cdots, n_2.$$

It follows that $D(\delta, \beta, \gamma, z) < 0$, which is a contradiction. Thus, $\sum_{j=1}^{n} z_{ji} > 0$. It concludes that,

$$\sum_{j=1}^{n} t_{ij} > 0.$$

$\square$

Proof of Proposition 2 Differentiating firms' first-order conditions with respect to γ, holding β constant, yields that $(\mathbf{I} - \mathbf{S})\mathbf{dq} = \mathbf{R}$, where $\mathbf{R} = (R_i)$ denotes a column vector, with $R_i = -1/\gamma \sum_{j \neq i}[(\delta_{ii} + \delta_{ij} q_j] d\gamma$. If condition (5) is satisfied, according to lemma 2

$$\frac{\partial Q^*}{\partial \gamma} = -\frac{1}{\gamma}[\sum_{j \neq 1}(\delta_{11} + \delta_{1j}) q_j (\sum_{l=1}^{n} e_{l1}) + \cdots + \sum_{j \neq n}(\delta_{nn} + \delta_{nj}) q_j (\sum_{l=1}^{n} e_{ln})] < 0$$

$\square$

Proof of Proposition 4 Differentiating firms' first-order conditions with respect to γ, holding β constant, yields that $(\mathbf{I} - \mathbf{S})\mathbf{dq} = \mathbf{R}$, where $\mathbf{R} = (R_i)$ denotes a column vector, with $R_i = -2\beta \sum_{j \neq i}[(\delta_{ii} + \delta_{ij} q_j] d\gamma$. If condition (16) is satisfied, then, $(\mathbf{I} - \mathbf{S})^{-1}$ is a non-negative matrix. Therefore, $\partial Q / \partial \gamma < 0$. $\square$

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