Name ______________________

Final Exam, Economics 210A, December 2011

Here are some remarks to help you with answering the questions.

Question 1. *A firm has a production function*

$$F(x_1, x_2) = (\sqrt{x_1} + \sqrt{x_2})^2 .$$

It is a price taker in the factor markets.

A) *Is this production function homogeneous? If so, of what degree?*

It is homogeneous of degree 1.

B) *Is this production function concave? Prove your answer.*

Yes. You could prove this by showing that the Hessian is negative semi-definite. Alternatively, you can show that it is quasi-concave (you need to actually show this) and homogeneous of degree 1 and using the result that such a function is concave.

Irving Fernandez offered an even neater proof. Note that

$$(\sqrt{x_1} + \sqrt{x_2})^2 = x_1 + 2\sqrt{x_1}x_2 + x_2.$$

Show that this is the sum of concave functions and hence a concave function.

C) *Define the elasticity of substitution between two factors and calculate the elasticity of substitution between x_1 and x_2 for the firm in this problem.*

Elasticity of substitution for this function is 2.

D) *Where the prices of the two inputs are p_1 and p_2, find the amount of each factor that would be used to produce one unit of output in the cheapest possible way.*

$$x_1(p, 1) = \left(\frac{p_2}{p_1 + p_2}\right)^2$$

$$x_2(p, 1) = \left(\frac{p_1}{p_1 + p_2}\right)^2$$

E) *Where the prices of the two in puts are p_1 and p_2, find the amount of each factor that would be used to produce y units of output in the cheapest possible way.*

$$x_1(p,y) = \left(\frac{p_2}{p_1+p_2}\right)^2 y$$

$$x_2(p,y) = \left(\frac{p_1}{p_1+p_2}\right)^2 y$$

F) *Find the cost function for producing this good.*

$$p_1x_1(p,y) + p_2x_2(p,y) = \left(\frac{p_1p_2}{p_1+p_2}\right) y$$

Question 2. A consumer has utility function

$$U(x_1,x_2) = \left(x_1^{-1} + x_2^{-1}\right)^{-1}$$

defined over the set $\{(x_1,x_2)|x_1 > 0, x_2 > 0\}$.

A)*Is this utility function strictly increasing in both goods at all points in its domain? (Prove your answer.)*

Yes, take derivatives and show that they are positive.

B) *Find this consumer's Marshallian demand functions for goods 1 and 2.*

$$x_1(p,m) = \frac{m}{p_1+\sqrt{p_1p_2}}$$

$$x_2(p,m) = \frac{m}{p_2+\sqrt{p_1p_2}}$$

C) *Find this consumer's indirect utility function.*

$$V(p,m) = \frac{m}{p_1+p_2+\sqrt{p_1p_2}} = \frac{m}{(\sqrt{p_1}+\sqrt{p_2})^2}$$

D) *Verify that Roy's identity applies.*

This is a matter of straightforward calculation.

Question 3. A consumer has utility function

$$V(x_1,x_2,x_3) = x_3^aU(x_1,x_2)^{1-a}$$

where $0 < a < 1$ and

$$U(x_1, x_2) = \left(x_1^{-1} + x_2^{-1}\right)^{-1}.$$

A) *Solve for the Marshallian demand functions for goods 1, 2, and 3, using a two step procedure in which you first find the highest utility that a consumer can achieve if he spends a total amount of money m on goods 1 and 2. (Hint: use the answer from Question 2 to help you.) Now if the consumer has total income M, find the best way to divide his expenditure, spending $M - m$ on good 3 and m on goods 1 and 2.*

If the consumer spends a total of m on goods 1 and 2 and $M - m$ on good 3, her utility will be

$$x_3^a V(p_1, p_2, m)^{1-a} = \left(\frac{M-m}{p_3}\right)^a \left(\frac{m}{p_1 + p_2 + \sqrt{p_1 p_2}}\right)^{1-a}$$

The consumer will choose m to maximize this function. This is maximized when $M - m = aM$ and $m = (1-a)M$. But then

$$x_3(p, M) = \frac{M-m}{p_3} = \frac{aM}{p_3}$$

and (using the answer to problem 2) we have

$$x_1(p, m) = \frac{(1-a)M}{p_1 + \sqrt{p_1 p_2}}$$

$$x_2(p, m) = \frac{(1-a)M}{p_2 + \sqrt{p_1 p_2}}$$

B *Write down the indirect utility function for this consumer.*

$$V(p, m) = Ma^a(1-a)^{1-a} p_3^{-a} (p_1 + p_2 + \sqrt{p_1 p_2})^{a-1}$$

Question 4.

A) *State the weak axiom of revealed preference.*

Let x^0 be the bundle chosen at prices p^0 and x^1 the bundle chosen at prices p^1. If $p^0x^1 \leq p^0x^0$, then $p^1x^0 > p^1x^1$.

B) *Prove that a utility-maximizing consumer who has strictly monotonic and strictly convex preferences will necessarily satisfy the weak axiom of revealed preference.*

If preferences are strictly convex and if x^0 is chosen at prices p^0, it must be that $u(x^0) > u(x)$ for all x such that $px \leq p^0$. (You should show why this is true.) Suppose that $p^0x^1 \leq p^0x^0$. Then it must be that $u(x^0) > u(x^1)$. If x^1 is chosen at price vector p^1, it must be that $u(x_1) > u(x)$ for all x such that $p^1x \leq p^1x^1$. Since $u(x^0) > u(x^1)$, it therefore cannot be that $p^1x^0 \leq p^1x^1$. It follows that if $p^0x^1 \leq p^0x^0$, it must be that $p^1x^0 > p^1x^1$.

C) *Show that a utility-maximizing consumer with weakly convex preferences might violate the weak axiom of revealed preference.*

Consider the following example. This is weakly convex but not strictly convex. Let $p^0 = (1, 1)$ and also let $p^1 = (1, 1)$. Then the commodity bundle $(x_1^0, x_2^0) = (2, 0)$ could be be chosen by a utility maximizing consumer with utility function $u(x_1, x_2) = x_1 + x_2$ and income 2. The bundle $x^1 = (1, 1)$ could also be chosen by the same utility maximizing consumer with income 2. Now $p^0x^1 = p^0x^0 = p^1x^1 = p^1x^0 = 2$. Therefore we have $p^0x^1 \leq p^0x^0$ and also $p^1x^0 \leq p^1x^1$ in violation of the weak axiom of revealed preference.

Question 5. *An economy has two consumers and two goods. Consumer A has the utility function*

$$U(x_1^A, x_2^A) = \alpha \ln x_1^A + (1 - \alpha) \ln x_2^A$$

and an initial endowment of ω_1^A *units of good 1 and no good 2. Consumer B has the utility function*

$$U(x_1^B, x_2^B) = \beta \ln x_1^B + (1 - \beta) \ln x_2^B$$

and an initial endowment of ω_2^B *units of good 2 and no good 1.*

A) *Let good 2 be the* numeraire *with a price of 1, and denote the price of good 1 by p. Solve for the demand for good 1 by each consumer as a function of p.*

Consumer A's income is $p\omega_1^A$. Solving A's maximization problem, we find that A spends the fraction α of her income on good 1. Thus we have

$$D_1^A(p) = \alpha\omega_1^A.$$

Consumer B's income is ω_2^B and B spends the fraction β of his income on good 1. This implies that

$$D_1^B(p) = \frac{\omega_2^B}{p}.$$

B) *Write an equation for excess demand for good 1 as a function of price p of good 1 when good 2 is the* numeraire. *At what price or prices p is this excess demand for good 1 equal to zero.*

$$E_1(p) = D_1^A(p) + D_2^A(p) - \omega_1^A = \alpha\omega_1^A + \frac{\beta\omega_2^B}{p} - \omega_1^A.$$

Therefore $E_1(\bar{p}) = 0$ when

$$\frac{\beta\omega_2^B}{\bar{p}} = (1-\alpha)\omega$$

which is the case when

$$\bar{p} = \frac{\beta\omega_2^B}{(1-\alpha)\omega_1^A}.$$

C) *If the price of good 1 is the price you found in Part B, at what price will excess demand for good 2 be zero?*

The same price, by Walras' Law.

D) *Is there a competitive equilibrium in which the price of good 2 is 3? If so, what must the price of good 1 be for there to be a competitive equilibrium with the price of good 2 equal to 3?*

Yes, multiplying all prices by a positive constant preserves competitive equilibrium, so $p_2 = 3$ and

$$p_1 = 3\bar{p} = 3\frac{\beta\omega_2^B}{(1-\alpha)\omega_1^A}$$

is also a competitive equilibrium.

Since as we saw in our solution, there is only one competitive equilibrium with $p_2 = 1$, there also is only one competitive equilibrium in which $p_2 = 3$.

E) *Find the quantities of good 1 and good 2 consumed by person A in competitive equilibrium. Find the quantities of good 1 and good 2 consumed by person B in competitive equilibrium.*

$$x_1^A = \alpha\omega_1^A$$
$$x_1^B = (1-\alpha\omega_1^A$$
$$x_2^A = \beta\omega_2^B$$
$$x_2^B = (1-\beta)\omega_2^B$$

F) *Suppose that person B's initial endowment of of good 2 is increased by one unit. What is the effect on the equilibrium consumption of good 1 by person B? What is the effect on the equilibrium consumption of good 2 by person B?*

Differentiate your answers to Part E to find:

$$\frac{\partial x_1^B}{\partial \omega_2^B} = 0$$

$$\frac{\partial x_2^B}{\partial \omega_2^B} = 1-\beta$$

Question 6. *A pure exchange economy has 1000 people and one consumption good. There are two possible states of the world. If State A occurs, everyone will have an initial endowment of 10 units of the consumption good. If State B occurs, each person will have an initial endowment of 20 units of the consumption good. Each person is an expected utility maximizer, with von Neumann-Morgenstern (Bernoullian) expected utility* $\pi^i \ln c_a^i + (1-\pi^i)\ln c_B^i$ *where* π^i *is person i's subjective probability that Event A happens and* $1-\pi^i$ *is i's subjective probability that B happens, and where* c_A^i *and* c_B^i *are i's consumption contingent on these events.*

Before it is known which state will occur, persons can buy or sell either of two kinds of securities. A type A security will pay one unit of the consumption good if State A occurs and nothing if state B occurs. A type B security will pay one unit of the consumption good if state B occurs and nothing if state A occurs. The price of type A securities is p_A *per unit and the price of type B securities is* p_B *per unit. Let* c_A^i *and* c_B^i *be i's consumption in events A and*

B, respectively. The difference between the value of i's consumption in event A and i's endowment in event A is equal to the number of type A securities that i purchases if this difference is positive and sells if this difference is negative. Similarly for type B securities. For each person, the total value of securities purchased must be equal to the total value of securities sold. Thus $p_A(c^i_A - 10) + p_B(c^i_B - 20) = 0$ for all i.

A) Let the type B security be the numeraire. Find the demand of person i for type A securities as a function of p_A and of i's subjective probability π^i.

$$c_i^A = \frac{\pi^i}{p_A}(10p_A + 20)$$

B) *If everybody has the same subjective probability for $\pi^i = \pi$ for event A, what is the competitive equilibrium price for type A securities?*

In this case, excess demand for type A securities is

$$1000(\frac{\pi}{p_A}(10p_A + 20) - 10).$$

Therefore excess demand is zero when

$$\frac{\pi}{p_A}(10p_A + 20) - 10) = 0.$$

Solving this equation we have

$$\bar{p}_A = \frac{2}{1-\pi}.$$

C) *Solve for the competitive price for type A securities where different people have different subjective probabilities for event A.*

In equilibrium, we must have total excess demand for consumption contingent on event A equal to zero. Thus we need

$$\sum_{i=1}^{1000} \frac{\pi^i}{p_A}(10p_A + 20) = 10,000. \tag{1}$$

Define

$$\bar{p}i = \frac{1}{1000} \sum_{i=1}^{1000} \pi^i.$$

Thus $\bar{p}i$ is just the mean of the individual probabilities of event A. Divide both sides of equation 1 by 1000 to get

$$\frac{\bar{p}i}{p_A}(10p_A + 20) = 10.$$

Solving this one finds the equilibrium price is

$$\bar{p}_A = \frac{2}{1 - \bar{\pi}}.$$

D) *Suppose that $\pi^i \neq \pi^j$, what is the ratio of person i's competitive equilibrium consumption in event A to that of person j in event A?*

$$\frac{c_A^i}{c_A^j} = \frac{\pi^i}{\pi^j}.$$