A Competitive Theory of Mismatch

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July 10, 2024

Abstract

I study the distributions of unemployment, vacancies, and wages across local labor markets in an economy where workers and jobs are matched and *mis*matched based on more explicit assumptions and aggregation principles than in the reduced-form aggregate matching-function approach. The endogenous matching process formulated here is flexible and has practical value for applied work. Local and aggregate labor market adjustments to local productivity and aggregate demand shocks reproduce empirical Beveridge and wage curve patterns, offer an alternative perspective on empirical indices of mismatch unemployment, and deliver an endogenous and commonly used reduced-form aggregate matching function.

Keywords: unemployment, vacancies, mismatch, continuous-time search
JEL classification: E20; E24; D52.
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*I would like to thank the editor and two anonymous referees for their detailed comments and suggestions. I would also like to thank seminar participants at Northwestern University, UC Santa Cruz, Rice University, USC, UC Riverside, the conference "Business Cycles: Theoretical and Empirical Advances," the Midwest Macroeconomics meetings, the 2010 NBER summer meeting, the summer meeting of the Econometric Society, LACEA-LAMES, and the SED meetings for their comments and suggestions. Charlie Nusbaum and Yao Ma were invaluable research assistants.

1 Introduction

This paper studies how unemployment, vacancies, and wages vary across disaggregated labor markets through the lens of a model of *mismatch*. Unemployment and vacancies have been the purview of the matching-function based approach (Pissarides [52]). The matching-function approach assumes that there are some unemployed workers looking for work, and some vacant positions to be filled. This approach is not focused on the causes of a worker becoming unemployed or of an unfilled position arising in the first place, but rather on frictions that prevent workers, once unemployed, from finding a job. The matching-function based approach is tractable, but informational and market imperfections intrinsic to the matching process are rarely made explicit. Instead, unspecified frictions influence equilibrium outcomes in reduced-form. The matching function, where these frictions are embedded, is an aggregate object that depends on the total numbers of unemployed workers and of vacant jobs.

I consider an economy where the ways workers and jobs are matched and *mis*matched rely on more explicit assumptions and aggregation principles that under the reduced-form matching approach. The spatial and informational structure is spelled out in detail, and the distributions of unemployment, vacancies, and real wages arise from an *endogenous matching process* that depends on individual decisions and their interplay with frictions and shocks embedded in the economic environment. The economy consists of spatially separated markets (i.e., "islands," as in Lucas and Prescott [42]) and the equilibrium is an assignment of workers and jobs to locations. Agents are aware of the frictions they face and their decisions are shaped by these frictions. Workers and jobs are able to move but, once in place, the short side of the market determines the number of workers and jobs used, as in Lagos [36]. In equilibrium, some locations feature a shortage of jobs (unemployment) while others feature a shortage of workers (vacancies). As in daily labor markets, workers know what wages to expect if they are hired, but do not know if local conditions will ensure employment. Local labor markets are perfectly competitive and clear, but only in the *ex ante* sense of incomplete markets. In the absence of complete insurance against labor demand uncertainty, surpluses become a feature of the competitive equilibrium.

The paper raises several points in relation to the traditional matching-function based approach. The first is a methodological point. The matching-function approach recognizes that "frictions derive from information imperfections about potential trading partners, heterogeneities, the absence of perfect insurance markets, slow mobility, congestion from large numbers, and other similar factors," but it is often argued that "modeling each one of these explicitly would introduce intractable complexities in macroeconomic models" (Petrongolo and Pissarides [50], p. 390). The main contribution of the paper is to show that an endogenous matching process may be formulated without making the model intractable or rigid for applied work. The model economy is able to explain the dynamics of local unemployment, vacancies, and wages over time, as well as their distribution across labor markets. Local and aggregate economies, for example, trace out perfectly negative *Beveridge curves* and exhibit stationary differences in unemployment, vacancies, and wages consistent with disaggregated labor market data.

The second is a measurement point at the base of the importance of mismatch unemployment. Several empirical mismatch indices have been proposed to measure mismatch.[1] In these indices, the matching process and the distributions of unemployment and vacancies are exogenous. Moreover, unemployed workers are required to be indifferent about where they search for work. In some cases, mismatch is measured relative to a benchmark with no dispersion in unemployment. The distributions of unemployment and vacancies in the present model are endogenous and reflect efficient adjustments to aggregate and local shocks, instead of inefficiencies. Not taking these distributions as given puts less weight on the dispersion of unemployment as a measure of mismatch, as there are no gains tied to the reallocation of unemployed workers. For instance, while outward shifts in the Beveridge curve have been viewed as evidence of an increase in mismatch (Petrongolo and Pissarides [50]), *shifts* in local and aggregate Beveridge curves driven by aggregate demand shocks suggest a decline rather than higher measured mismatch.

[1] Jackman and Roper [31], Jackman et al. [32], and Layard et al. [39] explored the idea that the high and persistent European unemployment rates in the 1980s were driven by mismatch. They proposed several mismatch indices that have led to puzzling findings; see, e.g., Entorf [16] and Padoa-Schioppa [49]. Şahin et al. [55], Herz and van Rens [27], Marinescu and Rathelot [43], and Barichon and Figura [6] refined the measurement of mismatch and matching efficiency. Şahin et al. [55] found a secondary role for mismatch unemployment during the Great Recession; see, however, Herz and van Rens [27] for a dissenting view.

Finally, the model economy incorporates optimal individual decisions and explicit aggregation. Since the formulation of the *Lucas critique*, economists have been wary of accepting reduced-form aggregate relationships as structurally-invariant, as such relationships may perform poorly in response to policy regime changes or shocks. When agents are aware of the frictions they face, policy changes affect their search strategies and the matching process, which can lead to potentially misleading results (Lagos [36]). In the matching-function based approach, for example, unmatched resources coexist even within finely disaggregated segments of the labor market. Adding unmatched resources to treated local markets must necessarily increase local employment and output (Kline and Moretti [34]). If resources are mismatched, adding unmatched workers to treated locations with job shortages would be wasteful. In the model economy, the local number of unemployment and vacancies is not sufficient to evaluate place-based policies because changes in the matching process influence even untreated locations.

To make mismatch models appealing for applied work, it is essential to address limitations related to price adjustments and distributional aspects in existing work. Lagos [36] considered an endogenous matching process based on exogenous prices, while wages in Shimer [56] and Mortensen [47] rest in a two-point distribution. The different degree of rigidity in *ex ante* and *ex post* outcomes allows local markets to adjust not only through quantities, but also through prices. Wages here depend continuously on the number of local workers and jobs, as in Hawkins [24], and unemployment and wages are jointly determined. For instance, under log-normal productivity shocks, the distribution of unemployment rates becomes Pareto with a tail behavior consistent with the empirical distribution of unemployment rates across various labor market segments in the US, while frictional wages settle at a stationary double Pareto distribution, also consistent with empirical wage distributions (Toda [58]). Finally, while Shimer [56] and Mortensen [47] characterize an endogenous aggregate matching function numerically, I derive an approximate Cobb-Douglas matching function, a popular functional form in applied work.[2]

[2] The derivation is related to Houthakker [30] and Lagos [37], but considers uncertain states of nature. Alternative micro-founded matching functions deal with stochastic pairwise trade arrangements based on urn-ball or a telephone-line queuing processes; see, e.g., Albrecht et al. [3], Petrongolo and Pissarides [50], and Stevens [57]. Coordination problems are central in the microeconomic approach proposed by Burdett

The main abstraction is the absence of aggregate uncertainty. In incomplete market models, equilibrium prices depend on the distribution of the local state, which becomes an infinite-dimensional endogenous state variable under aggregate uncertainty. It may be possible to bypass the central limit theorem through *granular shocks*, as in Gabaix [19]; to study *eternal boom* and *bust* episodes where aggregate shocks persist indefinitely, as in Gouge and King [21]; to study aggregate shocks using the *numerical approach* of Krusell and Smith [35]; or to switch off general equilibrium interactions, as in the *block recursive* approach in Carrillo-Tudela and Visschers [10]. Before addressing aggregate uncertainty, it is important to begin by accounting for stationary differences across local markets.

Some related literature. Recent work on mismatch has proceeded on two fronts best represented by Shimer [56] and Şahin et al. [55]. These fronts focus on the aggregate implications of mismatch, while I place greater emphasis on its disaggregate implications. I focus on differential adjustments across local labor markets and the dynamic behavior of local unemployment rates, vacancies rates, and wages.[3]

Shimer [56] studied an economy where workers and jobs were randomly assigned across locations, and employment was determined by the short side of the market, as in Lagos [36]. Shimer [56], as well as Mortensen [47], derived an aggregate matching function and reproduced cyclical movements along an aggregate Beveridge curve. Shimer [56], however, did not confront any disaggregated labor market data. In Shimer [56], all locations are identical and there is no wage dispersion, as labor markets clear in an *ex post* sense. *Ex post* market clearing is perhaps unsatisfactory for a number of reasons. First, it implies zero wages in locations with a surplus of workers. Second, an all-or-nothing wage is sensitive to small changes in local labor market conditions. A marginal change in the number of local workers or jobs leads to a discontinuous change in wages in marginal locations and to no changes in inframarginal locations. This high sensitivity played a central role in Shimer's [56] ability to match aggregate business cycle volatilities (Hawkins [24]).

et al. [8], while Heinesen [26] and Lambert [38] derived matching functions based on stochastic shortages.

[3] Distributional aspects were central in the study of sectoral reallocation by Lilien [40] and Abraham and Katz [1]. Gallipoli and Pelloni [20], in a recent survey, noted that analyses gravitate around two views: Lucas and Prescott's [42] "island" approach (e.g., Rogerson [54]) and the reduced-form aggregate matching-function approach (e.g., Hosios [29]). Carrillo-Tudela and Visschers [10], Pilossoph [51], and Chang [11] are just a few recent papers that integrate both approaches.

Şahin et al. [55] measured the importance of mismatch in aggregate unemployment through indices based on a reduced-form aggregate matching function; see also Herz and van Rens [27], Marinescu and Rathelot [43], and Barichon and Figura [6]. Local unemployment rates are important in this strand of the literature, but they are treated as exogenous and, for unspecified reasons, as inefficient. Inefficiencies are tied to the concavity of the aggregate matching function and are assumed worse when the distribution of unemployment is more unequal. Market efficiency is associated with benchmarks that, through arbitrage, reduce labor market dispersion. As a consequence, "mismatch" could be solved if unemployed workers are costlessly reallocated across markets. In Herz and van Rens ([27], p. 1620), for example, job-finding probabilities should be equalized across labor markets: Regardless of their skills, unemployed workers should be indifferent to the choice of the local labor market of the economy in which they search for work.

There is good reason to believe that not all labor market heterogeneity is inefficient. The dispersion in disaggregated labor market data in the model reflects, at least in part, adjustments to changes in aggregate demand and local productivities. Because the dispersion of unemployment is efficient, differences between actual and ideal allocations provide no indication of mismatch. I illustrate the importance of mismatch unemployment in a version of the model calibrated to match frictional wage data. I find that mismatch plays a larger role in accounting for aggregate unemployment than worker search. The reason, as pointed out by Hornstein et al. [28], is that search and matching models generally have difficulties generating the large frictional wage dispersion seen in the data. As in Alvarez and Shimer [4], Carrillo-Tudela and Visschers [10], Herz and van Rens [27], Michaillat [44], Michaillat and Saez [45], frictions other than search contribute to unemployment here.

2 Mismatch in a Static Assignment Problem

This section introduces a simple model of mismatch. Its goal is to derive the distributions of unemployment and vacancies across local labor markets as functions of primitive frictions. The informational and trading frictions are explicit, but the model is static. Job capital accumulation, worker search, and persistent shocks will be added in the next section.

2.1 Production and Market Equilibrium

Environment. There is a continuum of "islands" or locations indexed by $x \in [0,1]$. Locations represent a potential market, not necessarily a geographic location (e.g., skill, occupation, industry, firm, or a combination of such categories). There are L workers in the economy and $l(x) \leq L$ workers in location x. The distribution of workers $\{l(x)\}_{x\in[0,1]}$ is exogenous for now, but will be part of the equilibrium in the dynamic model.

Jobs are divisible assignments of homogeneous capital. Let $K > 0$ denote aggregate capital, which is given for now. The jobs assigned to location x are denoted by $k(x)$, and the resource constraint is

$$\int_0^1 k(x)dx = K. \tag{1}$$

In each location, production is determined by the short side of the market under diminishing returns to labor and uncertain factor requirements. *If* there is enough demand for output in location x, output $y(x,\omega)$ will be produced according to: $y(x,\omega) = z(x)\min\{k(x), \epsilon(\omega)l(x)^{\phi}\}$, for $0 < \phi < 1$, where $z(x)$ is an observable *local productivity* shock, and $\epsilon(\omega) \in \mathbb{R}_+$, is an *uncertain factor requirement.*

As the local match outcome will be limited by the short side of the market, the jobs assigned to location x will be filled probabilistically, according to $\min\{1, \epsilon(\omega)l(x)^{\phi}/k(x)\}$. This local matching probability depends on how many jobs are available, and on the number and efficiency of local workers. Jobs available in a given location, cannot be reshaped instantaneously to accommodate any number of workers. As in putty-clay models, the main margin of adjustment will be the *ex ante* assignment of jobs. Diminishing returns are not essential in the static model because labor is immobile, but they avoid indeterminacies in the dynamic model where both jobs and workers are mobile. The term $\epsilon(\omega)$ makes local labor market conditions uncertain. Its distribution and importance are discussed later on.

Preferences for consumption are linear, and workers do not value leisure. Aggregate demand is uncertain. The random variable $\zeta(\omega) \in [0,1]$ represents *aggregate demand*, as $\zeta(\omega)$ is the number of active locations or the number of locations visited by consumers. Consumption in location x will be positive if and only if location x is active, i.e., if $x \leq \zeta(\omega)$. While it is not possible to know for certain which locations will be active *ex post*, there is no output wasted. In active locations, workers and jobs are hired locally, and output is produced to satisfy demand. Local consumption and output are interchangeable. Remarks about demand uncertainty are also provided below.

Given $K > 0$ and $\{l(x)\}_{x\in[0,1]}$, the *static assignment problem* allocates job capital across locations in order to maximize (mean) aggregate consumption,

$$Y \equiv \max \mathbb{E}_\omega \left\{ \int_0^{\zeta(\omega)} z(x) \min\{k(x), \epsilon(\omega) l(x)^\phi\} dx \right\} \text{ s.t. (1).} \tag{2}$$

In problem (2), it is key that jobs are assigned before the resolution of uncertainty and cannot be reassigned once $\epsilon(\omega)$ and $\zeta(\omega)$ are realized, or conditional on their realizations.

Some remarks on the matching frictions. The assignment problem (2) is fairly straightforward, but it is helpful to preface the role of the market frictions. First, *local productivity shocks* $z(x)$ are observable and serve to regularize static shortages. For now, I treat $z(x)$ as bounded from below and spatially independent. The other two shocks, $\epsilon(\omega)$ and $\zeta(\omega)$, are both static and unobserved. They make it impossible to know for certain how many workers and jobs are needed at any time in a particular location.

Consider first the *factor requirement* $\epsilon(\omega)$. For a value of $\epsilon_+ > 0$ specified in the Appendix, a convenient assumption to obtain reasonable closed-form aggregate production and matching functions is the following:

Assumption 1. *Factor requirements $\epsilon(\omega)$ are uncertain, with a stationary flat probability density function over $[0, \epsilon_+)$ and a Pareto density with index $(1-\alpha) > 0$ over $[\epsilon_+, \infty)$.*

Uncertainty in $\epsilon(\omega)$ makes it impossible to know how many local jobs are needed. In an active location with $l(x)$ workers and $k(x)$ jobs, there will be an *ex post* surplus of workers if $\epsilon(\omega)l(x)^\phi/k(x) > 1$, or an *ex post* shortage of workers if $\epsilon(\omega)l(x)^\phi/k(x) < 1$.[4] While the notion that shortages exist only when one side of the market is in short supply may seem

[4] If $\epsilon(\omega)$ was observable, each location will receive the "correct" number of jobs, and there will be no local imbalances. Early on, Walters [59] discussed the role of uncertain labor services associated with unpredictable "off days" due to sickness, weather, or other accidental influences. Lucas [41] and Akerlof [2] also made use of fixed factor requirements to study labor utilization. Lambert [38] and Michaillat [44] examined rationing from different and complementary perspectives.

restrictive, jobs often involve a variety of tasks that are not easily matched to the skills of available workers, even when workers and jobs are in the same physical space.[5]

Assumption 1 is different, and perhaps simpler, than in Shimer [56] and Mortensen [47] where both sides of the market are randomly assigned and there is no local margin of adjustment. Assumption 1 focuses on one side of the market, and allows for job reallocations. Locations will still feature *ex post* rationing of jobs (or workers) because uncertainty on the worker (or firm) side is sufficient to produce local imbalances in active locations.

The particular distribution for $\epsilon(\omega)$ will deliver reasonable (i.e., Cobb-Douglas) local and aggregate production functions. More general distribution functions would still deliver local imbalances as a surplus of workers, for example, requires only that $\Pr(\{\omega : \epsilon(\omega) > k(x)/l(x)^{\phi}\}) > 0$. I treat $\epsilon(\omega)$ as an aggregate variable for simplicity, but location-specific uncertainty, as in $\epsilon(x, \omega)$, will mostly modify the way output is aggregated without altering the nature of local imbalances.

Consider next $\zeta(\omega)$. As a consequence of an *uncertain aggregate demand* $\zeta(\omega) \in [0, 1]$, not all locations will be active. The probability that location x is active is: $q(x) \equiv \Pr(\{\omega : x \leq \zeta(\omega)\})$, so $q(x)$ represents the degree of demand (un)certainty in location x.

Assumption 2. *Aggregate demand $\zeta(\omega)$ is uncertain, and drawn from a stationary and continuous distribution function.*

Uncertainty in the extensive margin of demand is common in industrial organization, as factor demands and prices are often set before demand is known (Butters [9], Eden [15], Deneckere and Peck [14]).[6] Aggregate demand matters because it determines local factor demands. Active locations will feature either a surplus of workers *or* a surplus of jobs, but not both. Inactive locations will feature a surplus of workers *and* jobs. Unused factors (i.e., idle capacity) will coexist in inactive locations because a match has no value, as there is no

[5] When analyzing over- or under-qualified workers, the literature typically infers a worker's qualifications for a job from indirect data, such as the technical requirements of a job (Guvenen et al. [22]), or the qualifications of peers in the same occupation and establishment (Fredriksson et al. [18]).

[6] Prescott [53], for example, studied unused factors and monopoly pricing under a stochastic demand for hotel rooms. In Prescott [53], cheaper rooms fill first, but there may still be some vacant rooms (i.e., unused factors) at the end of the day. In Michaillat and Saez [45], due to matching frictions, aggregate demand is also probabilistic and some capacity remains idle; see also Ottonello [48] for a study of capital markets.

demand to be satisfied. There is no *ex post* consumption in locations where $x > \zeta(\omega)$, but linear preferences yield no insurance value against $\zeta(\omega)$.

Aggregate demand propagates differentially across locations. By definition, $q(x') \leq q(x)$ so location $x' \geq x$ faces a more uncertain demand than location x.[7] The function $q(x)$ is generally unrestricted. Because the demand for local output acts as an all-or-nothing Bernoulli random variable, I assume that $q(1) > 1/2$. This means that a decline in $q(x)$ leads to higher uncertainty, even in the location with the most uncertain demand.

The static model is very parsimonious. The frictions captured by $\epsilon(\omega)$ and $\zeta(\omega)$ induce a probabilistic matching between workers and jobs, and consumers and producers. The local matching between workers and jobs depends on the jobs assigned *ex ante*, whereas the matching between consumers and producers is exogenous. Demand uncertainty is simplistic, but it can be micro-founded under strong consumer complementarities. Even in reduced-form, demand uncertainty and market incompleteness imply that local unemployment rates will depend on the likelihood of trading in output markets.

Production and aggregation. Consumption and output are interchangeable, so expression (2) represents the *aggregate production function*. Given $z(x)$, let $y(x) \equiv \mathbb{E}_\omega[y(x,\omega)|z(x)]$ be the *mean* of the distribution of output $y(x,\omega)$ or simply (mean) local output. Let $\varphi(x) \equiv q(x)z(x)$ be an *augmented local shock*, and $k^*(x)$ the assigned jobs to location x.

Proposition 1 *Given $\varphi(x)$, local output $y(x) \equiv \mathbb{E}_\omega[z(x)\min\{k(x), \epsilon(\omega)l(x)^\phi\}|z(x)]$ satisfies*

$$y(x) = \varphi(x)k(x)^\alpha l(x)^{\phi(1-\alpha)}; \tag{3}$$

the optimally assigned jobs $k^*(x)$ are increasing in $\varphi(x)$; and the aggregate production function satisfies

$$Y = K^\alpha \left(\int_0^1 \varphi(x)^{1/(1-\alpha)} l(x)^\phi\right)^{1-\alpha}. \tag{4}$$

Although *ex post* matches are limited by the short side of the market, the *ex ante* local production function is Cobb-Douglas. The *ex ante* assignment of jobs is given by

[7] Differential sensitivity to aggregate demand was central in Abraham and Katz [1] to counter Lillien's [40] work on the effect of sectoral shocks on unemployment. As advocated by Abraham and Katz [1], aggregate demand in the model acts differentially on unemployment rates.

$\alpha\varphi(x)[k^*(x)/l(x)^\phi]^{\alpha-1} = r$, where r is the Lagrange multiplier on (1), i.e., the opportunity cost of a job. This first-order condition implies that $k^*(x)$ and $y(x)$ are both increasing in $\varphi(x)$. Therefore, better locations (i.e., more productive and more certain), receive more jobs and produce more output.[8] The aggregate production function is also Cobb-Douglas. For example, if $l(x) = L$, then $Y = ZK^\alpha L^{\phi(1-\alpha)}$, with

$$Z \equiv \left(\int_0^1 \varphi(x)^{1/(1-\alpha)} dx \right)^{1-\alpha}. \tag{5}$$

In (4) and (5), a proportional increase in $\varphi(x)$ increases Y and Z proportionally. By convexity, a mean-preserving spread in $\varphi(x)$ also increases Y and Z.

An incomplete-market equilibrium. The assignment problem can be decentralized in an incomplete market economy where state-contingent contracts are ruled out. Let $\{w(x), r\}_{x\in[0,1]}$ represent local wages and the rental price of capital, respectively.

Definition 1 *In an incomplete competitive market equilibrium: (i) labor is inelastically supplied, i.e., $l^s(x) = l(x)$ whenever $w(x) > 0$; (ii) firms maximize expected profits, i.e., $\varphi(x)k^d(x)^\alpha l^d(x)^{\phi(1-\alpha)} - w(x)l^d(x) - rk^d(x)$; and (iii) factor markets clear in an ex ante sense: $l^d(x) = l^s(x)$ for all $x \in [0,1]$ and $\int_0^1 k^d(x)dx = K$.*

The only relevant decision for a worker is whether or not to participate in the local labor market. Profit maximization equalizes factor prices to the expected marginal products for job capital and labor. (There are local profits, but their distribution is not specified, as they do not affect the workers' or the firms' incentives.) As decisions are made before $\epsilon(\omega)$ and $\zeta(\omega)$ are known, labor markets clear from an *ex ante* perspective, before the identity of the matched and unmatched workers and jobs is determined. Wages, for instance, cannot condition on the realization of $\epsilon(\omega)$ and $\zeta(\omega)$. In equilibrium, the rental price of capital is constant across locations, and all workers expect to receive the same wage within a location.

Local wages are a function of the state of a local market $(\varphi(x), l(x))$ and the rental price of capital r, as in

$$w(x) = \phi(1-\alpha)(\alpha/r)^{\alpha/(\alpha-1)}\varphi(x)^{1/(1-\alpha)}l(x)^{\phi-1}. \tag{6}$$

[8] Demand uncertainty and local productivity differences are indistinguishable in $\varphi(x)$, but higher moments of the distribution of $y(x,\omega)$ help identify differences in $q(x)$ and $z(x)$, as the Appendix shows.

Wages are increasing in $q(x)$ and $z(x)$, with $q(x)$ acting as a *compensating differential* due to demand uncertainty. All else equal, firms in locations with a more uncertain demand are less likely to trade in the output market, making local factors less desirable. The reduced demand for jobs and workers imply lower wages in more uncertain markets.[9]

Markets operate as casual daily labor markets. Workers know what wages to expect if hired, but they do not know if local labor demand will be sufficiently high to ensure that they will be hired. If contingent decisions on $\epsilon(\omega)$ and $\zeta(\omega)$ are allowed, or if jobs can be moved *ex post*, no jobs will be assigned to inactive locations, and the "correct" number of jobs will be assigned to active locations. In these cases, there will be a surplus on one and only one side of the market. In an *ex ante* perspective, workers participate in the market expecting positive wages but "incorrectly" face adverse *ex post* local conditions. If local labor markets clear *ex post*, workers in the short side of the market would capture the entire surplus (Shimer [56]) and wages would be too sensitive to local conditions (Hawkins [24]).

2.2 *Ex Post* Labor Market Imbalances

The previous characterizations dealt with *ex ante* allocations and prices. If workers turn out to be very efficient, i.e., $\epsilon(\omega)l(x)^{\phi}/k^{*}(x) > 1$, there will be a shortage of jobs relative to the local needs. Conversely, if $\epsilon(\omega)l(x)^{\phi}/k^{*}(x) < 1$, there will be a shortage of workers.

Unemployment. Workers will be unemployed either because no consumer visited location x, i.e., $x > \zeta(\omega)$, or because location x is active (i.e., visited) but there is a shortage of jobs, i.e., $x \leq \zeta(\omega)$ but only $[k^{*}(x)/\epsilon(\omega)]^{1/\phi}$ workers are needed *ex post*:

$$u(x,\omega) = \begin{cases} l(x) & \text{if } x > \zeta(\omega) \\ l(x) - [k^{*}(x)/\epsilon(\omega)]^{1/\phi} & \text{if } x \leq \zeta(\omega) \text{ and } k^{*}(x) \leq \epsilon(\omega)l(x)^{\phi} \\ 0 & \text{if } x \leq \zeta(\omega) \text{ and } k^{*}(x) > \epsilon(\omega)l(x)^{\phi}. \end{cases} \tag{7}$$

Let $u(x) \equiv \mathbb{E}_{\omega}[u(x,\omega)|z(x)]$ denote the *mean* of the distribution of total unemployment in

[9] Workers and firms in location $x' > x$ may condition their decisions on the demand situation of location x. The equilibrium remains unchanged unless $\zeta(\omega)$ is fully revealed. Uncertain and sequential trade models interpret this property as delivering *rigid* prices (Butters [9], Eden [15], and Burdett et al. [8]). Price posting, originally studied by Prescott [53], has been recently generalized by Deneckere and Peck [14].

location x conditional on the local productivity shock or simply (mean) local unemployment.

Proposition 2 *Given $\varphi(x)$, local unemployment rates $\tilde{u}(x) \equiv u(x)/l(x)$ satisfy*

$$\tilde{u}(x) = [1 - q(x)] + q(x) \left(\frac{1}{1 + \phi(1-\alpha)} \frac{r}{\varphi(x)} \right). \tag{8}$$

Unemployment rates in Proposition 2 differ across locations by differences in aggregate demand sensitivities $q(x)$ and local productivities $z(x)$. The first term is associated with demand-driven unemployment and the second with local job shortages.

In expression (8), better locations have lower unemployment rates as a result of less prevalent shortages. First, more output will be produced and more labor will be used in locations with more certain demands, i.e., locations with higher values of $q(x)$. If $q(x) = 1$, there will be no demand-driven unemployment. The influence of aggregate demand on unemployment differs from Lucas and Prescott [42], where negative demand shocks increase unemployment through changes in the supply of labor, as more workers search for alternative locations; and from Michaillat and Saez [45], where negative demand shocks increase unemployment through changes in the demand for labor, but only when wages are fixed. Under flexible pricing, demand shocks are absorbed by the wage, not by unemployment.

Second, more productive locations receive more *ex ante* assignments of job capital $k^*(x)$ and experience fewer job shortages. Job shortages in (8) are proportional to $r/\varphi(x)$. As $\varphi(x) \equiv q(x)z(x)$, more productive locations will have fewer job shortages. Local job shortages tend to zero when $r \to 0$, as aggregate job capital would be in infinite supply, or when $\varphi(x) \to \infty$, as local jobs would be in infinite supply. In these cases, local jobs are so abundant that all workers in active locations will be employed *ex post*. Unemployment rates are also increasing in the rental price of capital r. This rental price is endogenous and, through general equilibrium effects, it will change in response to aggregate shocks.

Unemployment is consistent with actual labor market measurement. Unemployed workers should not be employed, should be available for work, and should have made minimal effort to find a job during a reference period. Unemployed workers are idle, but available for work. If different realizations of $\epsilon(\omega)$ and $\zeta(\omega)$ had prevailed, some of the unemployed workers would be hired. Although $q(x) = 0$ has been ruled out, if a location is known to be inactive, there will be no unemployment. With $\varphi(x) = 0$, these locations will receive no jobs, $k^*(x) = 0$, and workers would not participate in the market. If workers anticipate *ex ante* that no trading will take place, there will be no unemployment.

Vacancies. Mirroring unemployment, assigned jobs will remain vacant either because location x is not visited, i.e., $x > \zeta(\omega)$, or because location x is active (i.e., visited) but workers are in short supply, i.e., $x \leq \zeta(\omega)$ but only $\epsilon(\omega)l(x)^\phi$ jobs are needed:

$$v(x,\omega) = \begin{cases} k^*(x) & \text{if } x > \zeta(\omega) \\ k^*(x) - \epsilon(\omega)l(x)^\phi & \text{if } x \leq \zeta(\omega) \text{ and } k^*(x) > \epsilon(\omega)l(x)^\phi \\ 0 & \text{if } x \leq \zeta(\omega) \text{ and } k^*(x) \leq \epsilon(\omega)l(x)^\phi. \end{cases} \tag{9}$$

Let $v(x) \equiv \mathbb{E}_\omega[v(x,\omega)|z(x)]$ denote the mean of the distribution of total vacancies in x conditional on the local productivity shock or simply (mean) local vacancies. Vacancy rates can be defined in several ways. Following Shimer ([56], Eq. (4)), I measure vacancy rates in terms of the assigned jobs.

Proposition 3 *Given $\varphi(x)$, local vacancy rates $\tilde{v}(x) \equiv v(x)/k^*(x)$ satisfy*

$$\tilde{v}(x) = 1 - \frac{r}{\alpha}\frac{q(x)}{\varphi(x)}. \tag{10}$$

Proposition 3 considers demand-driven vacancies and vacancies due to worker shortages. However, as $\varphi(x) \equiv q(x)z(x)$, the model explains differences in local vacancy rates only by differences in local productivities $z(x)$. Local vacancies are not directly influenced by demand uncertainty because differences in $q(x)$ are offset by the job assignment. Demand conditions influence local vacancies indirectly, as a general equilibrium effect, through r. More productive locations have higher vacancy rates as they have more assigned jobs $k^*(x)$. Local vacancy rates would equal one if aggregate capital is in infinite supply, i.e., $r \to 0$, or if local jobs are in infinite supply, i.e., $z(x) \to \infty$.

Vacancies are unmatched assignments of homogeneous job capital, as in existing models of mismatch (Lagos [36], Shimer [56], Mortensen [47]), classical models of labor underutilization (Lucas [41], Akerlof [2]), and recent models of capital unemployment (Ottonello [48]).

Vacancies in job capital, for instance, are complementary to capital unemployment because not all forms of physical capital matter for a job, i.e., residential housing, and because the job side of a match does not have to rely on durable goods.[10] Moreover, vacancies here are costly due to an opportunity cost and not because of a posting or hiring cost. Even if posting and recruiting costs are zero, firms would have no incentive to pursue unlimited vacancies, as job capital always has alternative uses.

Beveridge and wage curves. The state of a local market is $(\varphi(x), l(x))$. As the augmented shock is $\varphi(x) \equiv q(x)z(x)$, unemployment rates $\tilde{u}(x)$ in (8) will be different across locations, even if all locations experience the same productivity shock $z(x)$. Vacancy rates $\tilde{v}(x)$, however, will differ across locations due only to differences in local productivities. I have not yet specified the distribution of local productivity shocks, but for any well-behaved distribution of local shocks, the *ex ante* assignment of local jobs will fluctuate randomly due to local productivities. An implication of Propositions 2 and 3 is that:

Proposition 4 *Local productivity shocks induce a perfect negative correlation between local unemployment rates $\tilde{u}(x)$ and local vacancy rates $\tilde{v}(x)$, and a negative correlation between local wages $w(x)$ and local unemployment rates $\tilde{u}(x)$.*

The logic behind the perfect negative correlation between $\tilde{u}(x)$ and $\tilde{v}(x)$ is simple. In active locations, unemployment and vacancies are *mutually exclusive*, and thus negatively dependent events. Conditional on a local shock $z(x)$, if labor efficiency $\epsilon(\omega)$ is low, an active location will experience vacancies but not unemployment, as labor will be in short supply. If $\epsilon(\omega)$ is high, an active location will experience unemployment but not vacancies, as jobs will be in short supply. As $z(x)$ fluctuates over time, these (discrete) events are regularized in (8) and (10). As $z(x)$ fluctuates over time, local unemployment and vacancy rates will trace out a perfectly negative relationship, i.e., a local *Beveridge curve.*[11] The negative

[10] Ottonello [48] showed that substantial amounts of physical capital remain unmatched at any given time and approached this issue using non-Walrasian capital markets characterized by reduced-form matching frictions. In Ottonello [48], capital trading depends on the tightness of the capital market. He showed that frictions and financial shocks are important to account for the slow recovery in investment following the Great Recession, as the demand for capital first made use of existing unemployed capital before new capital goods (i.e., investment) make it into the market.

[11] The correlation between local unemployment and vacancies relies on mean conditional values. As $\epsilon(\omega)$ and $\zeta(\omega)$ are unobservable, it is not interesting to measure correlations in *ex post* outcomes. It is possible

correlation between unemployment rates and wages follows the same logic. These reduced-form relationships, driven by local productivity shocks being a common causal factor, are part of a natural self-correction process in competitive markets with shortages.

2.3 Measuring Mismatch Unemployment

The distributions of unemployment and vacancy rates depend on the stochastic properties of the augmented shock $\varphi(x)$ in ways that will only be fully specified in the dynamic problem below (Assumption 3). The aim of the following discussion is to examine the economy's response to simple aggregate "shocks." Let U and V denote aggregate unemployment and vacancies, as in

$$U \equiv \int_0^1 u(x)dx \text{ and } V \equiv \int_0^1 v(x)dx,$$

and let $\Delta_\lambda X$ denote the change in X as a function of a proportional shifter λ, as in $\Delta_\lambda X \equiv X(\lambda < 1) - X(\lambda = 1)$, for a generic variable X.

Suppose first that local productivity decreases to $\lambda z(x)$ for some $0 < \lambda < 1$ and for all locations $x \in [0, 1]$. All locations experience a uniform and adverse aggregate productivity "shock." Since $\Delta_\lambda z(x) = -(1 - \lambda)z(x)$, and $\Delta_\lambda Z = -(1 - \lambda)Z$ in (5), aggregate factor productivity and aggregate output in (4) decline. Labor market conditions, however, are unchanged because the opportunity cost of capital absorbs the entire shock. As a consequence of the Envelope Theorem applied to (2), $\Delta_\lambda r = -(1-\lambda)r$, thus local unemployment and vacancies remain unchanged, because the ratio $r/z(x)$ is itself unchanged.

The neutrality of the distributions of unemployment and vacancies to proportional changes in productivity is specific to the type of change in $z(x)$, and to the fact that the total number of jobs is constant. In the dynamic model, r will be constant and K will be endogenous. Hence the adjustment to adverse aggregate productivity shocks will take place along a quantity margin by movements along local and aggregate Beveridge curves.

Next consider an adverse aggregate demand "shock."

to think of Proposition 4 as if measurement happens at discrete sampling frequencies and over a finite set of categories, as in reality. Hence, *measured* unemployment and vacancies coexist in local markets.

Proposition 5 *Suppose that demand uncertainty increases to $\lambda q(x)$ for some $0 < \lambda < 1$ and all locations $x \in [0, 1]$. Local unemployment and vacancies shift by*

$$\Delta_\lambda u(x) = (1 - \lambda)\,[l(x) - u(x)] > 0, \text{ and } \Delta_\lambda v(x) = (1 - \lambda)\,[k^*(x) - v(x)] > 0. \tag{11}$$

As locations experience differential demand sensitivities, aggregate demand shocks are not neutral. Proposition 5 says that a shift in demand uncertainty increases unemployment and vacancies everywhere. In fact, local and aggregate Beveridge curves shift by the same orders of magnitude, $\Delta_\lambda U = (1 - \lambda)[L - U] > 0$, and $\Delta_\lambda V = (1 - \lambda)[K - V] > 0$. Because vacancies in (10) do not vary with $q(x)$, the shifts in Proposition 5 are a general equilibrium response associated with a shift in the opportunity cost of a job r.

Shifts in local and aggregate Beveridge curves are typically associated with changes in matching efficiency and mismatch. I next ask if the previous shifts in local and aggregate Beveridge curves would be recognized as higher mismatch in existing indices. This is equivalent to asking if mismatch indices reveal the cause of the changes in the distributions of unemployment and vacancies in the model.

Mismatch indices. Empirical indices of mismatch consider cross-sectional distributions of unemployment and vacancies, such as (8) and (10), and assume invariant reduced-form wage and aggregate matching functions.

Indices based on the convexity of reduced-form wage functions vary with the dispersion of local unemployment,

$$\mathsf{MM}^L = \frac{1}{2}\int_0^1 \left(\frac{u(x) - U}{U}\right)^2 dx, \tag{12}$$

so that, to solve the "mismatch problem," unemployment should be equalized across locations; see Layard et al. ([39], pp. 309-313) for a derivation of MM^L and several variants. Additional indices consider pairwise differences between local unemployment and vacancies, as in

$$\mathsf{MM}^J = \frac{1}{2}\int_0^1 \left|\frac{u(x)}{U} - \frac{v(x)}{V}\right| dx, \tag{13}$$

so that unemployed workers must be reallocated to access the given vacant jobs; see, e.g., Jackman and Roper ([31], p. 12). In (13), $\mathsf{MM}^J = 0$ occurs when $u(x)/v(x) = U/V$ for all

x, while $\mathsf{MM}^J = 1$ occurs when all unemployed workers and all vacancies are in a single and separate location. In Şahin et al. ([55], Eqs. (8)-(10)),

$$\mathsf{MM}^S = 1 - \int_0^1 \Phi(x) \left(\frac{v(x)}{V}\right)^a \left(\frac{u(x)}{U}\right)^{1-a} dx, \tag{14}$$

with $a \in (0,1)$ as the elasticity of a reduced-form Cobb-Douglas aggregate matching function and with $\Phi(x)$ capturing, for example, heterogeneity in matching efficiency. In (14), $\mathsf{MM}^S = 1$ if unemployed workers and vacant jobs are in separate locations and $\mathsf{MM}^S = 0$ if unemployed workers satisfy $u(x)/U = [\Phi(x)(v(x)/V)^a]^{1/(a-1)}$ with $v(x)/V$ given.

The MM-indices seek to measure imbalances in the distributions of unemployment and vacancies across local labor markets. None of the previous indices, however, view the outward shifts in the Beveridge curves as part of an increase in mismatch:

Proposition 6 *As the distributions of unemployment and vacancy rates shift outward in Proposition 5, the previous mismatch indices satisfy:* $\Delta_\lambda \mathsf{MM}^L = (\lambda_U^2 - 1)\mathsf{MM}^L < 0$*;* $\Delta_\lambda \mathsf{MM}^J \leq (\lambda_U - 1)\mathsf{MM}^J + |\lambda_V - \lambda_U|/2$*; and* $\Delta_\lambda \mathsf{MM}^S < 0$*, where* $\lambda_U \equiv \lambda U/(\lambda U + (1-\lambda)L) < 1$ *and* $\lambda_V \equiv \lambda V/(\lambda V + (1-\lambda)K) < 1$.

Proposition 6 says that the MM-indices decline even though local and aggregate unemployment and vacancies increase everywhere. The index MM^S is convex in local unemployment shares $u(x)/U$. Therefore, MM^S increases if the distribution of local unemployment experiences a *mean-preserving spread.* Episodes of high aggregate unemployment rates are inconsistent with mean-preserving spreads in the distribution of unemployment. The index MM^S, all else equal, also associates higher mismatch with a compression in the distribution of local unemployment shares across labor markets.[12] In Proposition 5, unemployment shares actually shift proportionally across locations, as $\Delta_\lambda[u(x)/U] = (1 - \lambda_U)[1 - u(x)/U] > 0$. A decline in the MM-indices, however, is not exclusive to (11). If all local unemployment increases by a constant amount, as in $u(x) + \tilde{\lambda}$, the mismatch indices also decline, as local unemployment shares increase, i.e., $[u(x) + \tilde{\lambda}]/[U + \tilde{\lambda}] > u(x)/U$ for any $\tilde{\lambda} > 0$.

[12] The mismatch MM indices are not informative about changes within local markets as convexity makes them sensitive to aggregation. As noted by Barichon and Figura ([6], p. 243) in the context of matching efficiency, "a higher level of disaggregation (i.e., a smaller definition of a segment) will mechanically generate a higher level of dispersion." Proposition 6 applies to differences between, and not within, labor markets.

Unemployment shares increase more in locations with low unemployment shares, so the disparity in unemployment shares across locations decline. This decline masks a change in the causes of unemployment. Demand-driven unemployment increases while the prevalence of job shortages declines by the decline in the cost of capital. As jobs become more abundant in active locations, vacancies increase. In Proposition 5, job and worker shortages covary negatively, along local and aggregate Beveridge curves, but with aggregate demand changes increasing unemployment in all locations. While the MM-indices are easy to interpret, they are unable to identify the causes behind the changes in local and aggregate imbalances.

The previous proposition rationalizes a puzzling finding in the early literature on mismatch. Jackman and Roper [31] and Jackman et al. [32] found that empirical indices of mismatch fell while unemployment rates increased and Beveridge curves shifted in Europe during the 1980s; see also Entorf [16] and Padoa-Schioppa [49].[13] One potential reason for the lack of explanatory power of mismatch indices is that unemployment shares typically decline during times when the aggregate Beveridge curve shifts out. In the US, unemployment shares across different segments of the labor market generally increase during times of high aggregate unemployment (see Appendix). Echoing Abraham and Katz [1], Proposition 6 suggests that aggregate changes, rather than sectoral causes, can account for the shifts in unemployment and vacancies during the Great Recession.

Mismatch indices treat the causes of unemployment in reduced-form and view the distribution of unemployment as inefficient without specifying the source of inefficiencies. Local labor market outcomes here are endogenous so the present exercise offers a polar extreme where all the dispersion in unemployment, vacancies, and real wages is efficient. As there is no incentive for reallocations relative to the (constrained) efficient benchmark, counterfactual outcomes provide no indication of mismatch.[14] Even in models of competitive search, efficiency requires a non-degenerate distribution of unemployed workers across locations.

[13] Mismatch was suggested as an explanation for outward shifts in Beveridge curves. That Beveridge curves shift everywhere, and in similar proportions, however, was taken as evidence against mismatch. For example, Petrongolo and Pissarides ([50], p. 409) state that "in Britain the shifts in the regional Beveridge curves were of the same order of magnitude as the aggregate curve, casting doubt on the power of regional mismatch to explain the shift in the aggregate curve." Layard et al. [39] made a similar point.

[14] It is possible to consider reallocations relative to a first-best, but conventional practice measure efficiency gains relative to feasible (i.e., constrained) alternatives rather than relative to unconstrained alternatives.

2.4 An Aggregate Matching Function

The previous propositions explain the presence of unemployment and vacancies in perfectly competitive markets as outcomes of market incompleteness. They also account for their coexistence when workers and jobs are located in the "wrong" locations. I next derive an endogenous aggregate matching function based on aggregate unemployment and vacancies.

Assume that matched workers and jobs remain attached indefinitely. Unemployed workers and vacant jobs are reassigned across locations in an uncoordinated and random way. The number of unemployed workers in each location is given by $u'(x)$ and the number of vacant jobs is given by $v'(x)$. As in the basic assignment problem, assume that unemployed workers and vacant jobs match locally. Local output is produced according to $\varphi(x)\mathbb{E}_\omega[\min\{v'(x), \epsilon(\omega)u'(x)^\phi\}]$, where $\epsilon(\omega)$ satisfies Assumption 1.

As the assignment of unemployed workers and vacant jobs is random, $\varphi(x)$ plays no role in the matching process and $u'(x)$ and $v'(x)$ cannot be location-specific. For feasibility, $u'(x) = U$ and $v'(x) = V$ for all x. Hirings are also limited by the short side of the market, so, if $\epsilon(\omega)U^\phi < V$, $H(\omega) = U$ as all unemployed workers are hired, while if $\epsilon(\omega)U^\phi \geq V$, only $H(\omega) = U - [V/\epsilon(\omega)]^{1/\phi}$ of the available workers are hired *ex post*. The (mean) flow of new worker-job matches represents the output of the aggregate matching function.

Proposition 7 *The exit rate from unemployment, $h(U,V) \equiv \mathbb{E}_\omega[H(\omega)]/U$, has a log-linear relationship with labor market tightness V/U^ϕ,*

$$h(U,V) = 1 - \frac{\alpha\phi(1-\alpha)}{1+\phi(1-\alpha)}\left(\frac{V}{U^\phi}\right)^{\alpha-1}. \tag{15}$$

The exit rate from unemployment is increasing in market tightness. For example, $\lim_{V/U^\phi\to\infty} h(U,V) = 1$, so all workers exit unemployment as local markets becomes slack. A log-linear approximation of (15) around $V \simeq U^\phi$ yields

$$\ln h(U,V) \simeq \ln h_0 - \tilde{\alpha}\ln[U^\phi/V], \tag{16}$$

where h_0 depends on α and ϕ, and $\tilde{\alpha} \equiv \alpha\phi(1-\alpha)^2/[1+\phi(1-\alpha)^2] < 1$.

The relationship between hires and market tightness in (16) is consistent with a reduced-form aggregate matching function that is approximately Cobb-Douglas, i.e., $M(U,V) \simeq \exp\{h_0\}V^{\tilde{\alpha}}U^{1-\tilde{\alpha}\phi}$. The Cobb-Douglas is a common reduced form in empirical studies. According to Petrongolo and Pissarides ([50], p. 399), none of the existing microeconomic foundations for the aggregate matching function "convincingly says why the aggregate matching function should be of the Cobb-Douglas form." The reduced-form elasticity $\tilde{\alpha}$ in (15) is governed by the distribution of $\epsilon(\omega)$, which also delivers constant output elasticities in Proposition 1. For example, if $\phi = 1$, both the aggregate production function and the aggregate matching function will exhibit constant returns to scale. There is a tight relationship between the aggregate production and matching functions here, as the aggregation over *ex ante* and *ex post* outcomes are just two sides of the same coin.

In closing this section, it may be useful to contrast the present model of mismatch and the matching-function approach. In both frameworks, trade is probabilistic. In the matching-function approach, the probability of trade is a reduced-form function of aggregate market tightness. In this paper, labor demand uncertainty is a primitive, and local market conditions responds to informational frictions. The mismatch approach here thus reverses the direction of causality between the probability of trade and market tightness. This reversal makes it is possible to integrate mismatch with competitive equilibrium theory.

It may also be helpful to point out that, as a reduced-form relationship, the aggregate matching function might be potentially problematic for policy analyses (Lagos [36]). Many placed-based policies direct resources to areas with low labor demand often with the explicit purpose of lowering unemployment (Kline and Moretti [34]). In the matching-function based approach, unemployment and vacancies coexist even within finely disaggregated segments of the labor market, and the total number of unmatched resources in treated areas is a sufficient statistic for policy evaluations, as they are givens into the matching process.[15] With mismatch, the matching process is endogenous and unmatched resources are in separate locations, so place-based policies need to take into account the entire distributions of unemployment and vacancies. When evaluated via an aggregate matching-function, adding jobs

[15] For example, consider an exogenous inflow of jobs to some local labor markets with unemployed workers. In the treated areas, local unemployment will decline and vacancies will increase, though not one-to-one as jobs will move to untreated markets. An aggregate matching function could characterize the treated areas well. However, understanding the endogenous decline in the opportunity cost of a job, which increases vacancies and reduces unemployment in untreated labor markets, requires an *endogenous* matching protocol.

to a local market must necessarily increase local employment and output. In a mismatch perspective, adding jobs to locations with worker shortages would be wasteful.

3 Mismatch in a Dynamic Assignment Problem

I now consider job capital accumulation and a frictional worker mobility protocol where workers direct their search and locations adjust to the inflow and outflow of workers and jobs. Frictional labor reallocations take place as within equilibrium search models, but I consider the job side of a match, which is important for mismatch. The closed-form solution yields sensible stationary wage and unemployment distributions that will serve to empirically validate the theory. I formulate the dynamic assignment problem and present key highlights of its solution here, but leave all the derivations in the Appendix.

3.1 Worker Search and Job Capital Accumulation

Time is continuous, indexed by $t \in [0, \infty)$, and discounted at a rate $\rho > 0$. Factor requirements $\epsilon(\omega)$ and aggregate demand $\zeta(\omega)$ satisfy Assumptions 1 and 2 for all $t \geq 0$. Regarding $z_t(x)$, I now assume that

Assumption 3. *Local productivity shocks $z_t(x)$ evolve as a spatially independent geometric Brownian motion.*

Local productivity evolves as $dz_t(x) = \mu_z z_t(x)dt + \sigma_z z_t(x)dB_t$, for a Brownian motion B_t, with drift μ_z and diffusion σ_z^2. As aggregate demand shocks are stationary, demand (un)certainty differs across locations, but it is fixed over time, i.e., $q_t(x) = q_0(x)$ for all t. Local conditions are hence summarized by the augmented shock $\varphi_t(x) \equiv q_0(x)z_t(x)$ evolving over time as

$$d\varphi_t(x) = \mu_z \varphi_t(x)dt + \sigma_z \varphi_t(x)dB_t. \tag{17}$$

Assumption 3 matters for the frictional reallocation of labor, and for the dispersion of unemployment rates and wages. On its own, however, Assumption 3 yields an uninteresting (degenerate) outcome because the mean and variance of the *unregulated* stochastic process

(17) grow unbounded as $t \to \infty$. Worker search will *regulate* local shocks to ensure a stationary distribution of the local state.

The state of a location is given by $(\varphi_t(x), l_t(x))$, and its probability density at t is $\psi(\varphi_t(x), l_t(x))$ or simply $\psi_t(x)$. A non-standard issue in (17) is that differences in demand uncertainty are subsumed in the initial values $\varphi_0(x)$. Instead of a deterministic initial condition where local shocks $\varphi_0(x)$ equal a single value, the economy has probabilistic initial conditions associated with the differential sensitivity to aggregate demand shocks.

Worker search. The number of workers in location x at time t is $l_t(x)$. Workers can now search for work across locations knowing $\varphi_t(x)$, but with no information about $\epsilon(\omega)$ and $\zeta(\omega)$. The output cost of moving is given by $\theta > 0$. Searchers cannot work while in transit, so the opportunity cost of search is the foregone wage.

The fraction of workers who move out of location x is $0 \leq s_t(x) \leq 1$. Search is partially directed in the sense that a fraction $0 \leq \eta < 1$ of searchers arrives to *random* locations and the remaining fraction, $1 - \eta$, *direct* their arrival to chosen locations. The rate at which workers arrive in x is $\eta\bar{a} + (1 - \eta)a_t(x)$, with $\bar{a}$ representing random arrivals and $a_t(x)$ representing directed arrivals or simply arrivals. A fraction $\bar{s} > 0$ of workers is separated from each location at each date. The local labor force, $l_t(x)(1 - s_t(x))$, is endogenous and evolves as

$$dl_t(x) = [(1 - \eta)a_t(x) - s_t(x) - \bar{\eta}]l_t(x)dt, \tag{18}$$

with $\bar{\eta} \equiv \bar{s} - \eta\bar{a}$ as the exogenous *net* separation rate. Pure directed and random search are special cases, but they have difficulties delivering a sensible distribution of wages. For feasibility, the number of searchers must equal the total number of arrivals.

Job capital accumulation. Given K_t, the resource feasibility constraint for job capital is of the same form as in (1),

$$\int_0^1 k_t(x)\psi_t(x)dx = K_t, \tag{19}$$

for all $t \geq 0$, but job capital changes through investment I_t and depreciation δK_t,

$$dK_t = (I_t - \delta K_t)dt, \tag{20}$$

where δ is the depreciation rate. For future reference, the Lagrange multiplier on (19) is given by r_t, which represents the opportunity cost of capital at date $t \geq 0$.

3.2 Dynamic Allocations

Given a set of initial conditions, and (17)-(20), the goal of the *dynamic assignment problem* is to accumulate aggregate job capital and to spatially allocate workers and jobs to maximize the present value of aggregate consumption, i.e., output net of mobility costs and investment:

$$\max \mathbb{E}_{\varphi} \int_0^{\infty} \left[\int_0^1 \{y_t(x) - \theta d l_t(x)\} \psi_t(x) dx - I_t \right] \exp\{-\rho t\} dt. \tag{21}$$

A stationary solution to the dynamic assignment problem is a spatial allocation of workers and jobs, an invariant cross-sectional density of the local state $\psi^*(x)$, a constant opportunity cost of a job r^*, and a constant aggregate job capital stock K^*.

The spatial allocation of workers and jobs, and the accumulation of aggregate job capital can be treated as separate problems. By the absence of aggregate uncertainty, capital accumulation is deterministic and the stationary value of the opportunity cost of capital satisfies $r^* = \rho + \delta$. Moreover, the spatial allocation of workers and jobs can be done in two steps. The first step nets out capital choices through an *indirect production function* that places the spatial allocation of capital in the background. That is, $g(\varphi_t(x), l_t(x); r_t) \equiv \max_{k_t(x)}\{\varphi_t(x) k_t(x)^{\alpha} l_t(x)^{\phi(1-\alpha)} - r_t k_t(x)\}$, simply $g_t(x)$, satisfies $g_t(x) = (1-\alpha)\varphi_t(x)^{1/(1-\alpha)} \left(\alpha/r_t\right)^{\alpha/(\alpha-1)} l_t(x)^{\phi}$.

The function $g_t(x)$ depends on r_t. This dependence is problematic under aggregate uncertainty, but unproblematic in a stationary environment where $r_t \to r^*$. The function $g_t(x)$ also features diminishing returns to labor for $\phi < 1$ and its expected marginal product equals the competitive market wage, as in expression (6). As the expected marginal product of labor coincides with $w_t(x)$, a competitive (incomplete) market decentralization is implicit. The coefficient on the augmented shock $\varphi_t(x)$ exceeds one, as there is a *complementarity* between labor and capital in each local market: better locations receive more workers and more jobs, with the expected marginal product of labor also increasing with the assigned jobs. Amplification by $1/(1-\alpha)$ matters because it increases the incentives to search.

For example, wage dispersion σ_w^2 is given by $\sigma_z^2/(1-\alpha)^2$. Job assignments, however, help equalize wages even in the absence of worker search. Abstracting from the job side of a match can therefore distort the contribution of worker search to frictional wage inequality.

Frictional wage dispersion. The main feature of the worker assignment is that inaction is optimal over a large range of wages. As search is costly, it is optimal to follow an *(S,s)* rule where workers leave the local market only when the local wage reaches a *search threshold* w^-. There is also an *arrival threshold* given by $w^+ > w^-$, which compensates searchers for the lost time and missed wage appreciation during the search process. Both thresholds are specified in the Appendix.

Wages determine the spatial allocation of labor and serve as a *sufficient statistic* to describe a location's state $(\varphi(x), l(x))$, as noted by Alvarez and Shimer [4].[16] That is, the cross-sectional density of the local state can be written as $\psi_t(\varphi(x), l(x)) = \psi_t(\tilde{w}(x))$, where $\psi_t(\tilde{w}(x))$ is the density of log-wages at date $t \geq 0$, i.e., $\tilde{w}(x) \equiv \ln(w(x))$. The spatial distribution of jobs $k^*(\tilde{w}(x))$ and K^* can be subsequently derived using $\psi^*(\tilde{w}(x))$.

The stationary closed-form log-wage density can be approximated by a double exponential,

$$\psi^*(\tilde{w}(x)) \simeq C_\psi \min\{\exp\{\gamma_1[\tilde{w}(x) - \tilde{w}^+], \exp\{\gamma_2[\tilde{w}(x) - \tilde{w}^+]\}\}, \tag{22}$$

where the roots satisfy $\gamma_1 > 0 > \gamma_2$; C_ψ is a constant of integration; and $\tilde{w}^+ \equiv \log w^+$. This density is continuous and it exhibits a *tent-shaped* pattern consistent with a *double Pareto* wage distribution. Wage data strongly agrees with an asymmetric double Pareto distribution (Toda [58]). To obtain a sensible wage distribution, random and directed search are needed. Under random search, wages feature a single Pareto distribution over $[w^-, \infty)$, and under directed search the support of the distribution of wages is only $[w^-, w^+]$.

3.3 Labor Market Imbalances Again

In addition to the informational imperfections and market incompleteness characteristic of the static model, workers now face costly mobility and uncertainty about future wages.

[16] The solution method builds on Alvarez and Shimer [4], but I focus on the stationary distribution of wages instead of their persistence, which is central to their work. They emphasized *rest unemployment*, which is determined by a worker's preferences, independently of the job side of the match.

As in Lucas and Prescott [42], workers in depressed labor markets leave to search for better locations. Search unemployment represents the amount of worker reallocation needed to ensure that log-wages remain above $\tilde{w}^- \equiv \log w^-$ (Alvarez and Shimer [4]). How much pressure must searchers exercise to ensure that $\tilde{w}^-$ is not crossed? Log-wages satisfy $d\tilde{w}_t(x) = (\phi - 1) dl_t(x)/l_t(x)$, with $dl_t(x)/l_t(x) = -s_t^*(x)dt$ from expression (18). The density of locations with log-wages reaching $\tilde{w}^-$ in a short time interval is $(1/2)\sigma_w^2 \tilde{\psi}_t(\tilde{w}^-)dt$ where $\psi_t(\tilde{w}^-)$ is the density of log-wages at $\tilde{w}^-$. The search rate needed to regulate log-wages at the threshold is

$$s_t^*(\tilde{w}^-) = \frac{1}{2}\frac{\sigma_z^2}{(1-\alpha)^2}\frac{\psi_t(\tilde{w}^-)}{1-\phi}. \tag{23}$$

Worker search transforms $\varphi_t(x)$ in (17) into a *regulated Brownian motion* (Harrison [23]) with a reflected boundary tied to the search threshold w^-. The distribution of local shocks $\varphi_t(x)$ converges asymptotically to a Pareto distribution $\Pr\{\varphi_t(x) \leq \varphi\} \to 1 - [\varphi^-/\varphi]^{\gamma_\varphi}$ for $\varphi \geq \varphi^-$ and $\gamma_\varphi \equiv 1 - 2\mu_z/\sigma_z^2 > 0$. Job and worker shortages are functions of $\varphi(x)$ and also follow Pareto-like distributions.

Demand and job shortages behave as in the static model with the appropriate distribution of local shocks. As initial log-wages $\tilde{w}_0(x)$ depend on the deterministic values of $(z_0(x), l_0(x))$, but also on $q_0(x)$, it is possible to map local demand conditions to the initial log-wages, as in $q_0(x) = q(\tilde{w}_0(x))$. Initial conditions are probabilistic, so the stationary value of demand uncertainty in the typical or representative location integrates spatial differences in $q(\tilde{w}_0(x))$ using $\psi^*(\tilde{w}_0(x))$, as in $q^* \equiv \mathbb{E}_{\tilde{w}_0(x)}[q(\tilde{w}_0(x))]$, with

$$q^* = \int_{\tilde{w}^-}^{\infty} q(\tilde{w}_0(x))\psi^*(\tilde{w}_0(x))d\tilde{w}_0(x), \tag{24}$$

which depends on the distribution of the state of the local economy. The role of aggregate demand on unemployment is hence mediated through the differential sensitivity across locations $q(\tilde{w}_0(x))$, but also through structural differences in the economy, i.e., $\psi^*(\tilde{w}_0(x))$.

Overall, the aggregate unemployment rate converges to

$$\tilde{u}^* = \underbrace{1 - q^*}_{\text{Demand shortages}} + \underbrace{q^*\left(\frac{(\rho+\delta)}{1+\phi(1-\alpha)}\frac{\gamma_\varphi}{\varphi^-(1+\gamma_\varphi)}\right)}_{\text{Job shortages}} + \underbrace{\frac{1}{2}\frac{\sigma_z^2}{(1-\alpha)^2}\frac{C_\psi}{1-\phi}\frac{(w^-)^{\gamma_1-1}}{(w^+)^{\gamma_1}}}_{\text{Worker search}}, \tag{25}$$

where the first two terms are averages of the local unemployment rates in (8) and the last term is just (23). The aggregate vacancy rate converges asymptotically to

$$\tilde{v}^* = \underbrace{1-q^*}_{\text{Demand shortages}} + \underbrace{q^*\left(1-\frac{\rho+\delta}{\alpha}\frac{\gamma_\varphi}{\varphi^-(1+\gamma_\varphi)}\right)}_{\text{Worker shortages}}. \tag{26}$$

Several features are important. First, worker search sustains frictional wage dispersion in (22), but inequality in unemployment and vacancy rates across locations is due to differential local shortages. Worker search also insures workers against demand and job shortages. Insurance is limited because searchers might arrive to inactive locations or to active locations with job capital shortages. Unemployment rates and durations will thus differ across markets. (In Lucas and Prescott [42], all unemployment spells last one period.) As local shocks are asymptotically Pareto, with shape parameter γ_φ, local shortages converge to inverse Pareto distributions. Consider $q^* = 1$ for illustration. The stationary distribution of local unemployment rates is $\Pr\{\tilde{u}^*(x) \leq u\} = [u/u^+]^{\gamma_\varphi}$, defined over $[0, u^+]$ with $u^+ \equiv (\rho+\delta)/\varphi^-(1+\phi(1-\alpha))$.

Second, the inverse Pareto $[u/u^+]^{\gamma_\varphi}$ characterizes the lower tail of the unemployment distribution. This tail is associated with frequent and short unemployment durations. As workers and jobs are movable, mismatch helps explain short unemployment spells, but not why some workers remain unemployed for long periods of time. Third, the distribution of local unemployment rates features a heavy tail. The shape parameter γ_φ determines the degree of unemployment inequality. Finally, the cross-sectional dispersion of local unemployment rates, $\sigma_u = \tilde{u}^*/\sqrt{\gamma_\varphi(2+\gamma_\varphi)}$, is positively related to $\tilde{u}^*$. A direct relationship between σ_u and $\tilde{u}^*$ implies that episodes of high mean unemployment rates are accompanied by a high cross-sectional dispersion in local unemployment rates.

4 An Illustration

This section considers a quantitative illustration of the model. I calibrate the model to match (frictional) wage inequality and use the calibrated model to measure the contribution of job and demand shortages to aggregate unemployment. I also confront the distribution of unemployment rates across different segments of the US labor market.

4.1 Calibration

Some aggregate parameters such as $\delta = 0.012$, $\rho = 0.012$, and $\phi = 0.64$ are standard at quarterly frequencies, i.e., Cooley [12]. Not all forms of physical capital can be reallocated in the short run. For the capital share, I assume $\alpha = 0.06$ based on the importance of nonresidential equipment in aggregate capital. I assume an exogenous net worker flow rate of $\bar{\eta} = 0.02$, taken directly from Alvarez and Shimer ([4], p. 101), and normalize the direct cost of moving to $\theta = 1$. (I provide details of the calibration in the Appendix.)

The main challenge of the calibration is identifying the data analog of an "island." I often consider industrial sectors due to data availability (Şahin et al. [55]), but sectors do not discriminate across geographic and occupational categories. I calibrate demand uncertainty using capacity utilization in manufacturing sectors, as in Michaillat and Saez [45]. I assume that demand shortages are only relevant in low wage locations and that demand uncertainty declines exponentially across locations at a constant rate χ. Matching the empirical association between unemployment rates and the cross-sectional dispersion of capacity utilization measures over time yields $\chi = 0.0023$.

The drift and diffusion coefficients, μ_z and σ_z, and the threshold φ^- that regulates local productivity shocks are key parameters. I calibrate σ_z and μ_z to match the distribution of frictional wage inequality. I use $\sigma_z = 0.10$ so that log-wage dispersion is the average between the dispersion of *sectoral* level log-wages at a 5-digit industry level in Alvarez and Shimer ([4], Table 1) and the dispersion of the permanent component of *individual* log-wages in Heathcote et al. ([25], Figure 18). I do so because industrial sectors already aggregate workers and jobs in different occupations and geographic areas. Given σ_z and $\bar{\eta}$, I use $\mu_z = -0.0115$ to match the estimated upper Pareto tail exponent $\gamma_2 = -2.34$ in Toda ([58], p. 368). The lower tail exponent in Toda ([58], p. 368) is 1.15, which is slightly lower than here, $\gamma_1 = 1.51$. The resulting drift for log-wages is -0.0047 and the Pareto exponent γ_φ is 3.3. I normalize L to ensure that the local shock threshold satisfies $\varphi^- = (\rho + \delta)q^*/\alpha$.

4.2 Frictional Wage Inequality and Unemployment

In Table 1, unemployment due to demand shortages is 2.45 percent, and unemployment due to job shortages is 2.88 percent. The control exercised by worker search is limited, as search unemployment is $s^*(\tilde{w}^-) = 0.41$ percent. Total unemployment is about 5.6 percent, which agrees with postwar US data. The aggregate vacancy rate, however, is high at $\tilde{v}^* = 23.2$ percent. Table 1 also reports the Pareto tails of the stationary wage distribution. The wage distribution features a heavy upper tail. The Gini coefficient in a Pareto distribution is $(2|\gamma_2| - 1)^{-1}$. Ignoring the lower tail, the Gini coefficient for frictional wages is 0.27.[17]

Table 1. Aggregate labor market outcomes in the calibrated model.

	Unemployment rate				Frictional wage inequality	
	Demand shortages	Job shortages	Worker search	Vacancy rate	γ_1	γ_2
Baseline calibration	2.45	2.88	0.41	23.2	1.51	−2.34
Alternate values						
$\mu_z = -0.10$	0.02	3.63	2.63	2.97	0.20	−17.6
$\sigma_z = 0.05$	0.88	3.44	0.25	8.21	2.39	−5.89

Note: The table reports the stationary values of key aggregate labor market variables. All rates are reported in percentages. The coefficients $\gamma_{1,2}$ represent the Pareto exponents of the stationary wage distribution. The parameters in the baseline calibration are discussed in the text and the Appendix. The sensitivity analyses rely on the same normalization in the main text, but consider different values for the drift and diffusion of the local shocks.

The model delivers a reasonable view of aggregate unemployment and frictional wage dispersion, but vacancies that are too high. Normalizing the local shock threshold as $\varphi^- = (\rho + \delta)q^*/\alpha$ means that vacancy rates are $\tilde{v}^* = 1/(1 + \gamma_\varphi)$, which only vary with γ_φ. The value of γ_φ must be small to fit a large frictional wage dispersion. A small aggregate vacancy rate is therefore inconsistent with large frictional wage dispersion.

Why is search unemployment low? The drift of log-wages μ_w determines the frequency of worker reallocations. More negative values of μ_z make μ_w more negative and wages

[17] Another way to see that the wage distribution is reasonable is to consider the *mean-min wage ratio*, a central measure of wage inequality in Hornstein et al. [28]. Ignoring the lower tail of the wage distribution, the mean-min wage ratio in the baseline calibration is $\gamma_2/(\gamma_2 - 1) = 1.7$. The measured ratio of mean wages to the lowest wage in Hornstein et al.'s [28] is 1.8.

more likely to reach the search threshold $\tilde{w}^-$. For example, Table 1 considers an alternate value of $\mu_z = -0.10$, which is more negative than $\mu_z = -0.0115$ and implies $\gamma_\varphi = 21$. (The baseline value is $\gamma_\varphi = 3.3$. I recalibrate L so that $\varphi^- = (\rho+\delta)q^*/\alpha$, but leave all other parameters unchanged.) A more negative drift μ_z yields reasonable vacancy rates of $\tilde{v}^* = 2.97$ percent and higher search unemployment, $s^*(\tilde{w}^-) = 2.63$ percent. Unfortunately, $\mu_z = -0.10$ implies counterfactual wages. The upper tail of the wage distribution becomes $\gamma_2 = -17.6$, which yields a Gini coefficient of 0.03. A lower dispersion parameter $\sigma_z = 0.05$ also implies lower vacancy rates, but, as suggested by expression (23), an even smaller search unemployment. Under both $\mu_z = -0.10$ and $\sigma_z = 0.05$, the mass of workers in depressed locations is smaller, hence there is a less demand-driven unemployment.

As large frictional wage dispersion can only be sustained by infrequent worker reallocations, the main message of Table 1 is that frictions associated with aggregate demand uncertainty and job shortages might be more important than worker search in accounting for unemployment rates and the observed high dispersion in wages. In other words, "price" frictions that limit workers from knowing what wages they will earn if hired might not be as significant as "quantity" frictions that limit workers from knowing whether or not they will be hired at all.

4.3 Unemployment Inequality

The model delivers a Pareto *distribution* for job shortage rates across "islands." The distribution of local shortages follows directly from that of local shocks. In the model, the logarithm of the stationary cumulative distribution of unemployment rates should be linearly related to log-unemployment deviations from the logarithm of the mode, i.e.,

$$\ln[\Pr\{\tilde{u}^*(x) \leq u\}] = \gamma_\varphi[\ln(u) - \ln(u^+)]. \tag{27}$$

Following (27), Table 2 uses the empirical cumulative distribution of the lower tail of the unemployment distribution, and its relationship with the unemployment deviations from the mode, to estimate γ_φ across several segments of the US labor market. I separately consider the distribution of unemployment rates across 12 industries, 10 occupations, and

51 states (including the District of Columbia) from the monthly files of the CPS, as well as unemployment differences across large metropolitan areas from the LAUS.[18]

Table 2. Estimates of lower tail unemployment inequality across "islands."

	Cross-sectional unit				Cross-sectional unit			
	I	O	C	S	I	O	C	S
$\ln[\tilde{u}_{i,t} - u_t^+]$	2.04	0.67	4.78	3.31	2.04	0.67	4.78	3.31
	(0.02)	(0.01)	(0.04)	(0.01)	(0.02)	(0.01)	(0.04)	(0.01)
Recession					0.10	0.06	−0.02	−0.12
					(0.01)	(0.01)	(0.02)	(0.01)
R^2	0.77	0.75	0.66	0.74	0.78	0.75	0.66	0.74
N. groups	12	10	36	51	12	10	36	51
N. obs.	3,421	3,453	5,646	16,136	3,421	3,453	5,646	16,136

Note: OLS estimates of equation (27) using monthly variation in unemployment rates across (I)ndustries, (O)ccupations, (S)tates, and large metropolitan areas or (C)ities. Standard errors in parentheses. Recession is a control according to the NBER chronology. Cross-sectional data for seasonally adjusted unemployment rates for (I), (O), and (S) from the Current Population Survey (CPS). Unemployment measures in (C) from the Local Area Unemployment Statistics.

The mode of the distribution of unemployment rates is a consistent estimate of u^+. The observed distribution of unemployment rates, however, is unlikely to be stationary due to business cycle fluctuations. To capture changes in aggregate conditions, I consider a time-varying mode u_t^+. The underlying assumption is that aggregate business cycles proportionally shift the distribution of local unemployment rates. I also control for recessions, as dated by the NBER chronology. (I consider a constant mode u^+ and many alternative estimates of u^+ and γ_φ in the Appendix, but the reported findings here are robust.)

The baseline calibration based on wage dispersion delivers $\gamma_\varphi \equiv 1 - 2\mu_z/\sigma_z^2 = 3.3$, which is identical to the estimate of γ_φ across states (S) in Table 2. An advantage of considering unemployment rates across states is the large number of "islands" relative to industries (I) and occupations (O). These other segments deliver smaller values for γ_φ, which imply heavier tails and higher unemployment inequality. The reduced number of cross-sectional units in

[18] I consider the Current Population Survey (CPS, https://www.bls.gov/cps/) from 1976-01 to 2018-07. I also consider Local Area Unemployment Statistics (LAUS, https://www.bls.gov/lau/) from 1990-01 to 2019-10. The model has distributional implications for vacancy rates, but the normalization of φ^- makes vacancy rates too sensitive to the local shocks, which is an unpleasant feature of the model.

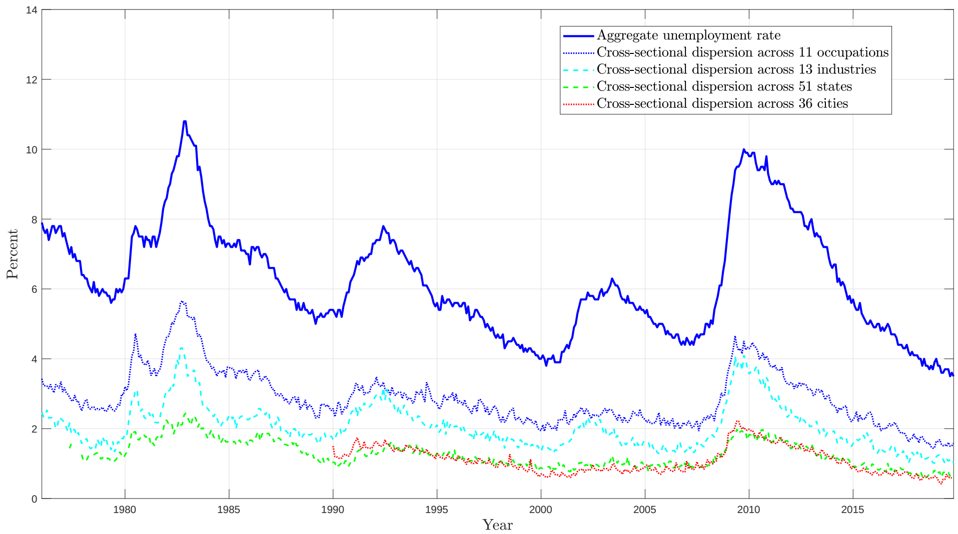

Figure 1: Aggregate unemployment rates and cross-sectional dispersion in unemployment.

(I) and (O) is likely responsible for the higher unemployment inequality. The estimate of γ_φ across large metropolitan areas (C) implies lower unemployment inequality, but these estimates rely on more recent samples and fewer cross-sectional units than in (S). Table 2 also suggests that recessions shift the distribution of the lower tail of unemployment rates, but the shifts do not alter the log-log relationship implied by expression (27).

Is the value of γ_φ reasonable? Under the alternate value of $\mu_z = -0.10$, the Pareto coefficient for the distribution of the unemployment rate is $\gamma_\varphi = 21$, which is counterfactual. While there is no single data analog for an "island," the degree of lower tail inequality in unemployment rates across industries, occupations, states, and cities is roughly in line with the baseline calibration set to match frictional wage dispersion.[19] Moreover, the model also matches the empirical relationship between the cross-sectional dispersion of local unemployment rates and their mean values. In the model, the dispersion in unemployment rates is proportional to the aggregate unemployment rate, $\tilde{u}^*$, as in $\sigma_u = \tilde{u}^*/\sqrt{\gamma_\varphi(2+\gamma_\varphi)}$.

Figure 1 shows that there is a strong positive relationship between the aggregate unemployment rate and the cross-sectional dispersion of local unemployment rates across occupations, industries, states, and large metropolitan areas in the US. This positive relationship

[19] The estimates of the log-log relationship (27) in Table 2 are similar to alternative estimates in the Appendix. The Appendix also shows that a log-log relationship accounts for more of the variability in the empirical cumulative distribution of unemployment rates at the lower end of the distribution than at the upper end.

is important for two reasons. First, the baseline calibration of $\gamma_\varphi = 3.3$ matches the slope of the observed relationship, while the alternate value of $\gamma_\varphi = 21$ implies that these moments are essentially unrelated. Second, there are no significant breaks in the relationship between the first two moments of the distribution of unemployment rates during the Great Recession. The stability of this relationship during the Great Recession, compared to previous downturns, suggests that mismatch was likely as significant then as it was in prior downturns.

In particular, in the 12 industry monthly cross-sectional data compiled by Şahin et al. [55] between December 2000 and June 2011, the negative correlation between the cross-sectional average of unemployment and vacancies from the Job Openings and Labor Turnover Survey (JOLTS) equals the correlation between the cross-sectional dispersion of unemployment and vacancies, as in $\mathbb{C}orr(u_t, v_t) = -0.878$, $\mathbb{C}orr(\sigma_{u,t}, \sigma_{v,t}) = -0.882$, because the first two moments of the distributions of unemployment and vacancies move together, i.e., $\mathbb{C}orr(u_t, \sigma_{u,t}) = 0.996$, $\mathbb{C}orr(v_t, \sigma_{v,t}) = 0.943$. During the Great Recession, local labor markets experienced a widening of the distribution of unemployment rates and a compression of the distribution of vacancies. These movements are consistent with the mismatch model, but a definitive assessment of the importance of mismatch requires much more disaggregated unemployment and vacancy data than is currently available.[20]

5 Some Concluding Remarks

This paper studied an economy where workers and jobs are matched and *mis*matched using more explicit assumptions and aggregation principles than in the matching-function based approach. The paper highlighted, through a simple example, that incorporating explicit informational imperfections, market incompleteness, costly worker mobility, and general competitive equilibrium principles do not necessarily introduce intractable complexities to the macroeconomic study of labor markets. Macroeconomics has evolved from building *ad*

[20] Aggregate vacancy rates and their cross-sectional dispersion across metropolitan areas in the Conference Board Help-Wanted Series are positively associated; see Andolfatto [5]. The correlation between aggregate and the cross-sectional dispersion for unfilled vacancies across 12 regions in the UK from January 1980 to April 2001, when the survey ended (www.nomisweb.co.uk), is 0.87.

hoc models to directly integrating microeconomic principles and disaggregated data. The ability to confront distributional questions that involve cross-sectional data and time series behavior of an economy is one of the many potential benefits from peering into the "black-box" of the aggregate matching function.

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6 Appendix: Omitted Derivations and Proofs

6.1 Contingent (First-Best) Allocations

The paper considers a non-contingent static assignment of jobs. This subsection considers a version of the static assignment problem under contingent allocations to illustrate the outcome if the informational frictions can be overcome. The jobs assigned can now be indexed by ω, so the planner's contingent assignment problem is:

$$\max \int_0^{\zeta(\omega)} z(x) \min\{k(x,\omega), \epsilon(\omega) l(x)^{\phi}\} dx, \text{ s.t. } \int_0^{\zeta(\omega)} k(x,\omega) dx \leq K.$$

The First-Best (FB) optimal job assignment is $k^{FB}(x,\omega) = \epsilon(\omega) l(x)^{\phi}$ for $x < \zeta(\omega)$ and $k^{FB}(x,\omega) = 0$ for $x \geq \zeta(\omega)$, thereby eliminating local imbalances in active and inactive locations. Still, depending on the value of K, the economy might exhibit either vacancies or unemployment, but not both. Let

$$K^{FB}(\omega) \equiv \int_0^{\zeta(\omega)} \epsilon(\omega) l(x)^{\phi} dx,$$

denote the total number of jobs needed by the informed planner. *If $K > K^{FB}(\omega)$ then, there is a surplus of jobs given by $K - K^{FB}(\omega)$. If $K < K^{FB}(\omega)$, then the least productive locations (i.e., those with low productivities $z(x)$) will have a surplus of workers.* If $K = K^{FB}(\omega)$, the planner will have the exact same number of jobs as needed, given the state of the aggregate demand shocks $\zeta(\omega)$ and uncertain factor requirement $\epsilon(\omega)$.

In the First-Best allocation, there is no coexistence of unemployment and vacancies, so it would *not* be possible to construct an aggregate matching function. The First-Best contingent allocation differs from the job assignment in Lagos [36] (L). In Lagos [36], the market equalizes *average* productivities across active locations leading to an *inefficient* outcome. Let r be the opportunity cost of a job. Suppose that local jobs are assigned to satisfy:

$$\{z(x) \min\left[1, \epsilon(\omega) l(x)^{\phi} / k^L(x,\omega)\right] - r\} k^L(x,\omega) = 0,$$

for $x < \zeta(\omega)$ or all active locations.

In Lagos [36], the static job assignment is based on $\min\{1, \epsilon(\omega) l(x)^{\phi} / k^L(x,\omega)\}$, which can be interpreted as the probability of a pairwise match between workers and jobs. If the local shock $z(x)$ in an active location exceeds r, their assigned job capital is given by $z(x)\{\epsilon(\omega) l(x)^{\phi} / k^L(x,\omega)\} = r$. Jobs, in other words, are directed toward the best locations. As $z(x)/r > 1$, these locations have a surplus of jobs, i.e., $k^L(x,\omega) = [z(x)/r] \epsilon(\omega) l(x)^{\phi} > \epsilon(\omega) l(x)^{\phi}$. Let

$$K^L(\omega) \equiv \int_0^{\zeta(\omega)} k^L(x,\omega) dx = \int_0^{\zeta(\omega)} [z(x)/r] \epsilon(\omega) l(x)^{\phi} dx,$$

denote the number of jobs needed by the market assignment. *If K is small, i.e., $K <$*

$K^L(\omega)$, the economy will simultaneously exhibit a surplus of jobs in the most productive locations as $k^L(x,\omega) > \epsilon(\omega)l(x)^\phi$ in the best locations, and a surplus of workers in the least productive locations as $k^L(x,\omega) = 0$ in such locations. The market is inefficient due to a coordination problem, and unemployment and vacancies will coexist in the economy.

Besides coordination problems, if an active location has a productivity level given by $z(x) < r$, this location should have no jobs assigned to them (i.e., $k^L(x,\omega) = 0$) as it does not cover the opportunity cost of capital. Any assignment, even if job capital is abundant, would be unproductive. This case resembles Akerlof [2] and adds another layer to the potential sources of unemployment in Lagos [36].

6.2 Proofs

For convenience, I omit the index x unless it is essential for the proofs. I also suppress the index ω when $\epsilon(\omega)$ is the only random variable under consideration.

Proof of Proposition 1. Assumption 1 implies a probability density for ϵ of the form

$$[1+\pi\epsilon_+]\psi_\epsilon(s) = \begin{cases} \pi \text{ if } 0 \le s < \epsilon_+ \\ (1-\alpha)\epsilon_+^{1-\alpha}/s^{1+(1-\alpha)} \text{ if } s \ge \epsilon_+, \end{cases} \tag{A1}$$

with π and ϵ_+ as parameters to be determined by simple normalizations. This density integrates to one for any positive value of π and ϵ_+.

Mean output satisfies

$$\mathbb{E}[\min\{k, \epsilon l^\phi\}] = l^\phi \mathbb{E}[\epsilon|\epsilon < k/l^\phi] \Pr\{\epsilon < k/l^\phi\} + k \Pr\{\epsilon \ge k/l^\phi\}. \tag{A2}$$

The first term in (A2) satisfies:

$$l^\phi \mathbb{E}[\epsilon|\epsilon < k/l^\phi] \Pr\{\epsilon < k/l^\phi\} = \frac{l^\phi}{1+\pi\epsilon_+}\left(\frac{\pi\epsilon_+}{2} - \frac{1-\alpha}{\alpha}\right)\epsilon_+ + \left(\frac{(1-\alpha)\epsilon_+^{1-\alpha}}{\alpha(1+\pi\epsilon_+)}\right) k^\alpha l^{\phi(1-\alpha)}, \tag{A3}$$

so that if $\pi\epsilon_+/2 = (1-\alpha)/\alpha$, the first term in (A3) drops out leading to $l^\phi\mathbb{E}[\epsilon|\epsilon < k/l^\phi] \Pr\{\epsilon < k/l^\phi\} = \{(1-\alpha)\epsilon_+^{1-\alpha}/\alpha(1+\pi\epsilon_+)\}k^\alpha l^{\phi(1-\alpha)}$. The second term in (A2) satisfies $k \Pr\{\epsilon \ge k/l^\phi\} = k[1 - \Pr\{\epsilon < k/l^\phi\}]$ or

$$k \Pr\{\epsilon \ge k/l^\phi\} = k\left\{\frac{\epsilon_+^{1-\alpha}}{1+\pi\epsilon_+}\right\}\left(\frac{k}{l^\phi}\right)^{\alpha-1}. \tag{A4}$$

Combining (A3) and (A4) yields

$$\mathbb{E}[\min\{k, \epsilon l^\phi\}] = \left\{\frac{\epsilon_+^{1-\alpha}}{\alpha(1+\pi\epsilon_+)}\right\} k^\alpha l^{\phi(1-\alpha)}, \tag{A5}$$

which completes the proof under the normalization $\epsilon_+^{1-\alpha} = \alpha(1+\pi\epsilon_+) = \alpha + 2(1-\alpha)$.

Problem (2) is globally concave. Its first-order condition is

$$\alpha\varphi(k^*/l^\phi)^{\alpha-1} = r, \text{ with } \varphi \equiv qz. \tag{A6}$$

Integrating both sides of its first-order condition and using (1) yields

$$\frac{k^*}{K} = \frac{\varphi^{1/(1-\alpha)}}{\displaystyle\int_0^1 \varphi^{1/(1-\alpha)} dx}, \tag{A7}$$

or simply $k^*/K = (\varphi/Z)^{1/(1-\alpha)}$, with Z defined by (5), which shows that k^* is increasing in φ, and K. Mean local output y is also increasing in φ, and K as y just raises (A7) to α, and φ has a direct effect on y. Substituting (A7) into the maximand yields (4). ■

For the following proofs, I assume that the expectation terms are conditional on $k/l^\phi > \epsilon_+$. This assumption is not problematic as long as the level of local productivities φ is not "too small." The dynamic case will have the relevant productivity shocks regulated by a lower bound φ^-. The technical issue is that if $\varphi \to 0$, $k^* \to 0$ so the unemployment and vacancy rates would tend to 'incorrect' limits in (8) and (10). With a proper lower bound for the local shocks, the following proofs are the only relevant cases to consider. For completeness, I consider the case when the local shock is not bounded from below in an Appendix not for publication.

Proof of Proposition 2. The first event in the unemployment equation (7) occurs with probability $(1-q)$. The second, with probability q, satisfies $l \Pr\{\epsilon \geq k/l^\phi\} - k^{1/\phi}\mathbb{E}[\epsilon^{-1/\phi}|\epsilon \geq k/l^\phi] \Pr\{\epsilon \geq k/l^\phi\}$. (Unemployment in the third event is zero.) From (A4), the first part of this expression is simply $l\alpha(k/l^\phi)^{\alpha-1}$. The second part satisfies

$$\begin{aligned} \frac{k^{1/\phi}(1-\alpha)\epsilon_+^{1-\alpha}}{1+\pi\epsilon_+}\left(\int_{k/l^\phi}^\infty s^{-1/\phi}/s^{1+(1-\alpha)}ds\right) &= k^{1/\phi}\alpha(1-\alpha)\left\{\frac{s^{-1+\alpha-1/\phi}}{(-1+\alpha-1/\phi)}\Bigg|_{k/l^\phi}^\infty\right\} \\ &= k^{1/\phi}\frac{\alpha(1-\alpha)}{(1-\alpha+1/\phi)}\left(\frac{k}{l^\phi}\right)^{-1+\alpha-1/\phi}. \end{aligned} \tag{A8}$$

Given the previous expression, the (mean) local unemployment rate satisfies $\tilde{u} = (1-q) + (1+\phi(1-\alpha))^{-1}q\alpha\left(k/l^\phi\right)^{\alpha-1}$, which equals (8) after using (A6). ■

Proof of Proposition 3. As before, the first event in the vacancies equation (9) occurs with probability $(1-q)$. The second, with probability q, satisfies $k \Pr\{\epsilon < k/l^\phi\} - l^\phi\mathbb{E}[\epsilon|\epsilon < k/l^\phi] \Pr\{\epsilon < k/l^\phi\}$. The first part of this expression uses $\Pr\{\epsilon < k/l^\phi\} = 1 - \alpha(k/l^\phi)^{\alpha-1}$. The second part directly divides (A3) by k to yield $(1-\alpha)(k/l^\phi)^{\alpha-1}$, so one obtains $\tilde{v} = (1-q) + q\{1-(k/l^\phi)^{\alpha-1}\} = 1 - q(k/l^\phi)^{\alpha-1}$, which can be simplified, using (A6), to get (10). ■

Proof of Proposition 4. The mean conditional values of unemployment and vacancy rates are inversely related to z. The relevant variables are $\tilde{u} = 1 - q + z^{-1}(r/[1+\phi(1-\alpha)])$ and $\tilde{v} = 1 - z^{-1}(r/\alpha)$. Hence, $\mathbb{C}ov(\tilde{u},\tilde{v}) = -(r/[1+\phi(1-\alpha)])(r/\alpha)\,\mathbb{C}ov(z^{-1},z^{-1})$. As $\mathbb{V}ar(\tilde{u}) = (r/[1+\phi(1-\alpha)])^2\,\mathbb{V}ar(z^{-1})$ and $\mathbb{V}ar(\tilde{v}) = (r/\alpha)^2\,\mathbb{V}ar(z^{-1})$, $\mathbb{C}orr(\tilde{u},\tilde{v}) = \mathbb{C}ov(\tilde{u},\tilde{v})/\mathbb{V}ar(\tilde{u})^{1/2}\mathbb{V}ar(\tilde{v})^{1/2} = -1$.

Local wages are increasing in z hence covary negatively with unemployment rates,

$$\mathbb{C}ov(\tilde{u}, w) = (r/[1+\phi(1-\alpha)])\,[\phi(1-\alpha)(\alpha/r)^{\alpha/(\alpha-1)}q^{1/1-\alpha}l^{\phi-1}]\mathbb{C}ov(z^{-1}, z^{1/(1-\alpha)}).$$

The covariance term is of the form $\mathbb{C}ov(z^{-1}, z^{1/(1-\alpha)}) = \mathbb{E}[z^{-1}z^{1/(1-\alpha)}] - \mathbb{E}[z^{-1}]\mathbb{E}[z^{1/(1-\alpha)}]$. The first term can be written as $\mathbb{E}[z^{\alpha/(1-\alpha)}]$. As $z^{\alpha/(1-\alpha)}$ is a striclty concave function, Jensen's inequality implies that $\mathbb{E}[z^{\alpha/(1-\alpha)}] < \mathbb{E}[z]^{\alpha/(1-\alpha)}$. In the second term, as z^{-1} and $z^{1/(1-\alpha)}$ are striclty convex functions, Jensen's inequality implies $\mathbb{E}[z^{-1}] > \mathbb{E}[z]^{-1}$ and $\mathbb{E}[z^{1/(1-\alpha)}] > \mathbb{E}[z]^{1/(1-\alpha)}$. Combining inequalities yields

$$\mathbb{E}[z^{-1}]\mathbb{E}[z^{1/(1-\alpha)}] > \mathbb{E}[z]^{-1}\mathbb{E}[z]^{1/(1-\alpha)} = \mathbb{E}[z]^{\alpha/(1-\alpha)} > \mathbb{E}[z^{\alpha/(1-\alpha)}],$$

leading to a negative covariance. ■

Proof of Proposition 5. In the Cobb-Douglas production function (4), $Y(\lambda < 1) = \lambda Y(\lambda = 1)$. Hence, $r(\lambda < 1) = \lambda r(\lambda = 1)$ from the Envelope Theorem. As the opportunity cost of capital r changes with λ, expression (8) yields

$$\begin{aligned} u(\lambda < 1) &= l\left\{1 - \lambda\left[q - \frac{r}{1+\phi(1-\alpha)}\frac{q}{\varphi}\right]\right\} \\ &= l - \lambda\left[l - u(\lambda = 1)\right], \end{aligned} \tag{A9}$$

which gives (11). Changes in the vacancy rates can be derived analogously because

$$v(\lambda < 1) = k^*\left\{1 - \lambda\left[\frac{r}{\alpha}\frac{q}{\varphi}\right]\right\} = k^* - \lambda\left\{k^* - v(\lambda = 1)\right\}. \tag{A10}$$

■

Proof of Proposition 6. In Proposition 5, unemployment shares satisfy

$$\frac{u}{U}(\lambda < 1) = (1-\lambda_U) + \lambda_U\frac{u}{U}(\lambda = 1), \tag{A11}$$

or in the Δ_λ notation as $\Delta_\lambda(u/U) = (1-\lambda_U)[1-u/U]$ with $\lambda_U \equiv \lambda U/(\lambda U + (1-\lambda)L) < \lambda$. Therefore, one can write MM^L as

$$\mathsf{MM}^L(\lambda < 1) \equiv \frac{1}{2}\int_0^1 \left(\frac{u}{U}(\lambda < 1) - 1\right)^2 dx = \frac{1}{2}\int_0^1 \lambda_U^2\left(\frac{u}{U}(\lambda = 1) - 1\right)^2 dx,$$

where the last expression is simply $\lambda_U^2\mathsf{MM}^L(\lambda = 1)$. Hence $\Delta_\lambda\mathsf{MM}^L = (\lambda_U^2 - 1)\mathsf{MM}^L$ as needed.

For MM^J, notice that in Proposition 5, vacancy shares $v(x)/V$ shift as in (A11) but with

$\lambda_V \equiv \lambda V/(\lambda V + (1-\lambda)K)$. Therefore, one can write

$$\begin{aligned}\mathsf{MM}^J(\lambda<1) &\equiv \frac{1}{2}\int_0^1 \lambda_U \left(\frac{u}{U}(\lambda=1) - \frac{v}{V}(\lambda=1)\right) + (\lambda_V - \lambda_U)\left|1 - \frac{v}{V}(\lambda=1)\right| dx,\\ &\leq \lambda_U \mathsf{MM}^J(\lambda=1) + \frac{|\lambda_V-\lambda_U|}{2}\int_0^1 \left|1-\frac{v}{V}(\lambda=1)\right| dx,\\ &\leq \lambda_U \mathsf{MM}^J(\lambda=1) + \frac{|\lambda_V-\lambda_U|}{2}, \end{aligned} \tag{A12}$$

where the second inequality follows when all vacancies are in a single location, which is the highest possible concentration. Hence $\Delta_\lambda \mathsf{MM}^J \leq (\lambda_U - 1)\mathsf{MM}^J + |\lambda_V - \lambda_U|/2$.

For MM^S, notice that

$$\frac{u}{U}(\lambda<1) \geq \frac{u}{U}(\lambda=1) \text{ and } \frac{v}{V}(\lambda<1) \geq \frac{v}{V}(\lambda=1),$$

which implies that

$$\left(\frac{v}{V}(\lambda<1)\right)^a \left(\frac{u}{U}(\lambda<1)\right)^{1-a} \geq \left(\frac{v}{V}(\lambda=1)\right)^a \left(\frac{u}{U}(\lambda=1)\right)^{1-a}.$$

so one obtains $\mathsf{MM}^S(\lambda<1) < \mathsf{MM}^S(\lambda=1)$ upon integration. ■

Proof of Proposition 7. Output for the new matches is $\min\{V, \epsilon U^\phi\}$, where ϵ satisfies Assumption 1. The number of exits from unemployment is

$$H = U - \begin{cases} [V/\epsilon]^{1/\phi} & \text{if } V \leq \epsilon U^\phi \\ 0 & \text{if } V > \epsilon U^\phi, \end{cases} \tag{A13}$$

so one needs to compute $V^{1/\phi}\mathbb{E}[\epsilon^{-1/\phi}|\epsilon \geq V/U^\phi]\Pr\{\epsilon \geq V/U^\phi\}$. This expression is of the same form as (A8), so one gets (15). ■

To verify the log-linear approximation for $h(U,V)$, let $n = V/U^\phi$ and write $h(n) = 1 - cn^b$, with $c \equiv \alpha(1-\alpha)\phi/[1+\phi(1-\alpha)]$ and $b \equiv \alpha - 1$. To log-linearize $h(n)$ around $h_0 = 1 - cn_0^b$, first note $dh(n_0) = -bcn_0^{b-1}dn_0$ so that using $dh(n_0)/h(n_0) \simeq d\ln h(n_0)$ and $dn_0/n_0 \simeq d\ln n_0$ gives

$$[\ln h(n) - \ln h_0] \simeq -b\frac{cn_0^b}{1-cn_0^b}[\ln n - \ln n_0].$$

At $n_0 = 1$, the previous expression yields $\ln h(n) \simeq \ln(1-c) - [bc/(1-c)]\ln n$. As $-[bc/(1-c)] = \alpha(1-\alpha)^2\phi/[1+\phi(1-\alpha)^2]$, it is possible to write $\ln h(U,V) \simeq \ln h_0 + \tilde{\alpha}\ln[V/U^\phi]$, where $\tilde{\alpha} \equiv \alpha(1-\alpha)^2\phi/[1+\phi(1-\alpha)^2] < 1$, as in (16). The total number of matches, $M(U,V) = h(U,V)U$, can be approximated by $\ln M(U,V) \simeq \ln h_0 + \tilde{\alpha}\ln V - \tilde{\alpha}\ln U^\phi + \ln U$, as in the text.

7 Appendix: Dynamic Assignment Problem

I collect here the detailed and technical solution to the dynamic assignment problem (21).

As noted in the text, the spatial allocation of workers and jobs, and the accumulation of aggregate capital can be treated as separate problems. First, let $\mathcal{V}(\varphi_0(x), l_0(x); \{r_t\}_{t\geq 0})$ be the value of local output in location x when the initial state is $(\varphi_0(x), l_0(x))$ and $\{s_t(x), a_t(x), k_t(x)\}_{x\in[0,1],t\geq 0}$ are chosen to solve:

$$\mathcal{V}(\varphi_0(x), l_0(x); \{r_t\}_{t\geq 0}) \equiv \max \mathbb{E}_\varphi \int_0^\infty [y_t(x) - \theta dl_t(x) - r_t k_t(x)] \exp\{-\rho t\} dt, \tag{B1}$$

where r_t is the Lagrange multiplier on (19). Second, the capital accumulation problem is deterministic and $\{I_t\}_{t\geq 0}$ solves:

$$\mathcal{M}(K_0) \equiv \max \int_0^\infty [r_t K_t - I_t] \exp\{-\rho t\} dt, \text{ s.t., (20)}. \tag{B2}$$

Taking the previous two problems together yields (21) or

$$\max \int_0^1 \mathcal{V}(\varphi_0(x), l_0(x); \{r_t\}_{t\geq 0}) \psi_t(x) dx + \mathcal{M}(K_0).$$

Job capital. The job capital accumulation problem (B2) is deterministic. The Hamilton-Jacobi-Bellman (HJB) equation associated with (B2) is $\rho\mathcal{M}(K_t) = \max_{I_t}\{r_t K_t - I_t + \mathcal{M}'(K_t)[I_t - \delta K_t]\}$. The first-order condition for I_t is of the form $\{\mathcal{M}'(K_t) - 1\} I_t^* = 0$, and the envelope condition is $(\rho + \delta)\mathcal{M}'(K_t) = r_t$, as (B2) is linear in K_t, i.e., $\mathcal{M}''(K_t) = 0$. Preferences for consumption are linear hence, in the stationary solution, $r_t = r^* = \rho + \delta$. If K_t and $\psi_t(x)$ are "too far" from their stationary value, one should have $r_t > \rho + \delta$ to prevent negative consumption, but the previous inequality can only hold in finite time. I therefore consider only the case with $r^* = \rho + \delta$.

Some remarks. Before considering the spatial allocations, I next present a few remarks about the job capital accumulation in the dynamic assignment problem. In a world with concave preferences for consumption, there would be an added self-insurance motive and a precautionary (over)accumulation of job capital. As in traditional Bewley-Aiyagari models, the equilibrium rate of return r^* will be lower than the adjusted rate of time preference $\rho + \delta$. The rate of return would also be a function of the distribution of the state of the economy, rather than a constant value, as in here.

It is possible to assume that households diversify the risk associated with search histories by aggregating local consumptions internally. This would allow different locations face different trading opportunities, while still delivering a stationary outcome where the rate of return to capital equals the augmented rate of time preference. The downside of concave utilities, even in this simple case, is that the allocation is no longer separable and the stationary values of K^* and $\psi^*(x)$ need to be solved simultaneously instead of recursively, as done here. I have also ignored home production and leisure. These activities can be implicitly included as one location in the model, or as a separate activity with additional value. Treating home production as a separate activity would make the total labor force L_t endogenous.

Spatial allocation of labor. Consider next the spatial allocation problem given the stationary value r^*. The value function $\mathcal{V}(\varphi(x), l(x); r^*)$ in (B1) does not depend directly on time, so write $\mathcal{V}(\varphi(x), l(x); r^*)$ as $\mathcal{V}(x)$. The HJB equation associated with (B1) is

$$\rho\mathcal{V}(x) = \max\left\{y(x) - \theta[(1-\eta)a(x) - s(x) - \bar{\eta}]l(x) - r^*k(x) + \frac{\mathbb{E}_\varphi[d\mathcal{V}(x)]}{dt}\right\}, \tag{B3}$$

where the last term, given (17) and Ito's Lemma, is

$$\frac{\mathbb{E}_\varphi[d\mathcal{V}(x)]}{dt} = \mathcal{V}_l(x)[(1-\eta)a(x) - s(x) - \bar{\eta}]l(x) + \mu_z\varphi(x)\mathcal{V}_\varphi(x) + \frac{1}{2}\sigma_z^2\varphi(x)^2\mathcal{V}_{\varphi\varphi}(x). \tag{B4}$$

As in the static problem, the marginal product of capital equals r^*. The marginal product of labor coincides with the local wage $w(x)$, as the dynamic assignment is efficient. I treat them interchangeably. Expressions (B3) and (B4) imply that the optimal fraction of searchers and arrivals, $s^*(x)$ and $a^*(x)$, satisfy a series of (variational) inequalities. Worker search $s^*(x)$ satisfies

$$\{\theta - w(x) - \mathcal{V}_l(x)\}\, s^*(x) = 0, \tag{B5}$$

and arrivals $a^*(x)$ satisfy

$$(1-\eta)\{\mathcal{V}_l(x) - \theta\}\, a^*(x) = 0. \tag{B6}$$

Proposition 8 *The value function for a particular location satisfies $\mathcal{V}(\varphi(x), l(x); r^*) = \mathcal{W}(w(x); r^*)l(x)$, where $\mathcal{W}(w(x); r^*)$ is the present discounted value of output per worker. There is a search threshold so that workers leave a location if wages reach w^-; and arrivals are directed to a location with a wage $w^+ > w^-$.*

Proof. The proof uses a series of transformations:

(i) Net out capital choices through an *indirect production function* $g_t(x)$ which gives the value of local output once capital adjusts optimally to labor reallocations.

(ii) In locations without searchers and arrivals, log-wages $\tilde{w}_t(x) \equiv \log(w_t(x))$ evolve as

$$d\tilde{w}_t(x) = \mu_w dt + \sigma_w dB_t - \frac{\alpha}{1-\alpha}\frac{dr_t}{r_t}, \tag{B7}$$

with drift and diffusion terms

$$\mu_w \equiv \frac{\mu_z}{(1-\alpha)} + \alpha\frac{\sigma_w^2}{2} + (1-\phi)\bar{\eta}, \text{ and } \sigma_w^2 \equiv \frac{\sigma_z^2}{(1-\alpha)^2}. \tag{B8}$$

The drift μ_w captures the random flow of workers and the amplification of local shocks through the job assignments, i.e., set $\bar{\eta} = 0$ temporarily to see that $\mu_w > \mu_z$ as long as $\alpha > 0$. Amplification follows because a positive shock makes worker and job inflows more attractive on their own but also due to their complementarity. (This is so because wages are convex in $\varphi(x)$. Convexity also explains the relationship between σ_w^2 and σ_z^2 in (B8) and the presence of σ_w^2 in μ_w.) The dependence of local wages on dr_t/r_t in (B7) is problematic under aggregate uncertainty but in a stationary environment $r_t = r^*$ and wages eventually settle to a stationary density.

(iii) By the competitive nature of search, the value of a location for a particular worker in (B3), should only be a function of the wage. Homogeneity in the wage function makes possible to write (B1) in terms of the present discounted value of output *per worker*, i.e., $\mathcal{W}^*(w(x); r^*)$. That is, in the *inaction* region, (B3) becomes

$$\frac{\sigma_z^2}{2}\varphi(x)^2\mathcal{V}_{\varphi\varphi}(x) + \mu_z\varphi(x)\mathcal{V}_\varphi(x) - \rho\mathcal{V}(x) + g(x) + [\theta - \mathcal{V}_l(x)]\bar{\eta}l(x) = 0, \tag{B9}$$

which is a second-order *partial* differential equation. Homogeneity in the wage function implies $\mathcal{V}(\varphi(x), l(x); r^*) = \mathcal{W}(w(x); r^*)l(x)$ so, $\mathcal{V}_l(x) = \mathcal{W}(x) + \mathcal{W}'(x)(\phi - 1)w(x)$, $\mathcal{V}_\varphi(x) = \mathcal{W}'(x)w(x)l(x)/(1-\alpha)\varphi(x)$, and $\mathcal{V}_{\varphi\varphi}(x) = \{\mathcal{W}''(x)w(x)^2 + \alpha\mathcal{W}'(x)w(x)\}l(x)/(1-\alpha)^2\varphi(x)^2$. Substitution of these expressions into (B9) yields:

$$\frac{\sigma_w^2}{2}\mathcal{W}''(x)w(x)^2 + \mu_w\mathcal{W}'(x)w(x) - \bar{\rho}\mathcal{W}(x) + w(x)/\phi + \theta\bar{\eta} = 0, \tag{B10}$$

with μ_w and σ_w listed in (B8), and with $\bar{\rho} \equiv \rho + \bar{\eta}$.

This value function satisfies a second-order ordinary differential equation $\sigma_w^2\beta(\beta-1)/2 + \mu_w\beta - \bar{\rho} = 0$ whose solution is standard. A homogeneous part is based on $A_1 w(x)^{\beta_1} + A_2 w(x)^{\beta_2}$, with $\beta_{1,2}$ as the roots of $\mathcal{Q}(a) = \sigma_w^2 a(a-1)/2 + \mu_w a - \bar{\rho} = 0$, of the form

$$\beta_{1,2} = (1-\gamma)/2 \pm \sqrt{[(1-\gamma)/2]^2 + 2\bar{\rho}/\sigma_w^2}, \text{ with } \gamma \equiv 2\mu_w/\sigma_w^2. \tag{B11}$$

Under $\bar{\rho} > 0$, the roots are of opposite sign. The value of a location should go to zero as $w(x) \to 0$, so $A_2 = 0$.

Let β_1 denote the positive root. This root is associated with the *option value* of search. For search to have a positive *option value* (i.e., $\beta_1 > 1$) one must have $\mathcal{Q}(1) = \mu_w - \bar{\rho} < 0$, which, from (B8), requires the (technical) assumption that *time must be discounted at a sufficiently high rate to make search valuable*, i.e.,

$$\bar{\rho} > \mu_w \Longrightarrow \rho > \frac{\mu_z}{(1-\alpha)} + \frac{\alpha}{(1-\alpha)^2}\frac{\sigma_z^2}{2} - \phi\bar{\eta}. \tag{B12}$$

For a particular solution consider the adjusted wage process $w(x)/\phi + \theta\bar{\eta}$ when workers are required to remain in a location indefinitively (i.e., when no control is undertaken). Then, the value function $\mathcal{W}^*(x)$ satisfies $\mathcal{W}^*(w(x); r^*) = C_w + w(x)/\phi(\bar{\rho} - \mu_w)$, where $\bar{\rho} \equiv \rho + \bar{\eta} > \rho$ is the *augmented* discount rate and C_w is a constant, $C_w \equiv \theta\bar{\beta}^{-1}\left[1 - \bar{\eta}/\bar{\rho} - \bar{\beta}\right]$, for $\bar{\beta} \equiv \beta_1(\beta_1 - 1)[(\beta_1 - 1) - \phi\beta_1]$, and with the explicit dependence of $w(x)$ on r^* omitted for convenience.

(iv) At the threshold where worker search takes place, expression (B5) corresponds to $\theta = w(x) + \mathcal{W}(x) - \mathcal{W}'(x)(1-\phi)w(x)$, while the smooth-pasting condition is $0 = 1 + \phi\mathcal{W}'(x) - \mathcal{W}''(x)(1-\phi)w(x)$. Combining these expressions yields a *search threshold* w^-

given by

$$w^{-}\left(1+\frac{1}{\bar{\rho}-\mu_w}\right)=\theta\left(\frac{\beta_1}{\beta_1-1}\right)\left(1-\frac{\bar{\eta}}{\bar{\rho}}\right). \tag{B13}$$

At the threshold where directed arrivals take place, (B6) is $\{\mathcal{W}(x)-\mathcal{W}'(x)(1-\phi)w(x)-\theta\}=0$ with a smooth-pasting condition $0=\phi\mathcal{W}'(x)-\mathcal{W}''(x)(1-\phi)w(x)$. Combining expressions as in the other threshold yields an *arrival threshold* w^{+} given by

$$w^{+}\left(\frac{1}{\bar{\rho}-\mu_w}\right)=\theta\left(\frac{\beta_1}{\beta_1-1}\right)\left(1-\frac{\bar{\eta}}{\bar{\rho}}\right). \tag{B14}$$

For both thresholds to be positive, the exogenous flow term $1-\bar{\eta}/\bar{\rho}$ must be positive.

To understand the previous expressions, notice that the right-hand sides of (B13) and (B14) are equal. Both expressions represent the "lifetime" cost of search: the direct cost θ, times the *option value multiple* $\beta_1/(\beta_1-1)>1$, times a random flow adjustment. The direct cost θ and the random exit term $\bar{\eta}/\bar{\rho}$ are primitive parameters. The left-hand side of (B13) is the "lifetime" benefit of staying in a location with a wage w^{-}: the current wage and its expected present value given an adjusted growth at a rate $\mu_w-\bar{\rho}$. Likewise, the left-hand-side of (B14) is the "lifetime" benefit of arriving to a location with a wage w^{+}.

The ratio between the arrival and search thresholds in (B13) and (B14), $w^{+}/w^{-}=1+\bar{\rho}-\mu_w$, measures the opportunity cost of not working for an instant, while searching: the lost time and the missed wage appreciation during that instant. ■

Derivation of the Kolmogorov Forward Equation (KFE) and boundary conditions. Before considering the stationary distribution of wages, it is neccesary to derive the KFE and its boundary conditions. They are related to the behavior of workers at the search and arrival thresholds.

Because $\tilde{w}(x)$ is a sufficient statistic for the local state, $\psi_t(\varphi(x),l(x))=\psi_t(\tilde{w}(x))$. To derive the Kolmogorov Forward equation (KFE) for the state of the local economy (i.e., log-wages), notice that in the absence of *any* labor movement, and when $r_t=r^{*}$, log-wages evolve as a regular Brownian motion with drift μ_w and diffusion σ_w^2 in (B8). Labor movements, however, induce changes to log-wages due to random arrivals, directed arrivals, and worker search. Random arrivals take place at all points in the support $[\tilde{w}^{-},\infty)$ whereas directed arrivals only take place in $\tilde{w}^{+}$ and workers search once the wage reaches $\tilde{w}^{-}$. Therefore, over the support $\tilde{w}\in(\tilde{w}^{-},\tilde{w}^{+})\cup(\tilde{w}^{+},\infty)$, that excludes the search and arrival thresholds, log-wages evolve as a regular Brownian motion (B7), and their density $\psi_t(\tilde{w})$ satisfies a Kolmogorov Forward equation (KFE)

$$\frac{\partial\psi_t(\tilde{w})}{\partial t}=\frac{\sigma_w^2}{2}\psi_t''(\tilde{w})-\mu_w\psi_t'(\tilde{w})-\bar{\eta}\psi_t(\tilde{w}), \tag{B15}$$

with $\psi_t'(\tilde{w})\equiv\partial\psi_t(\tilde{w})/\partial\tilde{w}$ and $\psi_t''(\tilde{w})\equiv\partial_t^2\psi(\tilde{w})/\partial\tilde{w}^2$.[21]

[21] To see the connection between the KFE in the original state $(\varphi_t(x),l_t(x))$ and the KFE using the sufficient statistic $\tilde{w}_t$, let the "original" density be $\tilde{\psi}_t(\varphi_t(x),l_t(x))$. When the state is $(\varphi_t(x),l_t(x))$, the "original" KFE satisfies $\partial\tilde{\psi}_t(x)/\partial t=(\sigma_z^2/2)\partial^2[\varphi_t(x)^2\tilde{\psi}(x)]/\partial\varphi_t(x)^2-\mu_z\partial[\varphi_t(x)\tilde{\psi}_t(x)]/\partial\varphi_t(x)+\partial[\{((1-\eta)a_t^{*}(x)-s_t^{*}(x)-\bar{\eta})l_t(x)\}\tilde{\psi}_t(x)]/\partial l_t(x)$, where the first two terms are due to the diffusion of local shocks

The previous expression must be supplemented with boundary conditions: $\psi_t(\tilde{w})$ must satisfy integrability conditions and continuity at the wage at which workers direct their search, i.e., $\psi_{t-}(\tilde{w}^+) = \psi_{t+}(\tilde{w}^+)$, where $\psi_{t-}(\tilde{w}^+)$ and $\psi_{t+}(\tilde{w}^+)$ denote the left and right limits of $\psi_t(\tilde{w})$ at $\tilde{w}^+$, respectively. The boundary condition at the search threshold implies that $\tilde{w}^-$ behaves as a *reflecting barrier*. Integrating (B15) yields

$$\begin{aligned}\int_{\tilde{w}^-}^{\infty} \frac{\partial \psi_t(\tilde{w})}{\partial t} d\tilde{w} = & \frac{\sigma_w^2}{2}\{[\psi'_{t-}(\tilde{w}^+) - \psi'_{t+}(\tilde{w}^+)] + [\psi'_t(\infty) - \psi'_t(\tilde{w}^-)]\} \\ & - \mu_w\{[\psi_{t-}(\tilde{w}^+) - \psi_t(\tilde{w}^-)] + [\psi_t(\infty) - \psi_{t+}(\tilde{w}^+)]\} \\ & - \bar{\eta}.\end{aligned}$$

Integrability requires that $\psi_t(\infty) = \psi'_t(\infty) = 0$ and continuity requires that $\psi_{t-}(\tilde{w}^+) = \psi_{t+}(\tilde{w}^+)$ leading to

$$\begin{aligned}\int_{\tilde{w}^-}^{\infty} \frac{\partial \psi_t(\tilde{w})}{\partial t} d\tilde{w} = & \frac{\sigma_w^2}{2}[\psi'_{t-}(\tilde{w}^+) - \psi'_{t+}(\tilde{w}^+)] \\ & - \frac{\sigma_w^2}{2}\psi'_t(\tilde{w}^-) + \mu_w \psi_t(\tilde{w}^-) - \bar{\eta}.\end{aligned} \tag{B16}$$

The righ-hand-side terms capture how the density $\psi_t(\tilde{w})$ changes over time in the absence of worker search. The first line is associated with the distributional influence of arrivals at $\tilde{w}^+$ and the second with wage behavior at the boundary $\tilde{w}^-$ and the random flow of $\bar{\eta}$ workers. To understand the boundary behavior, the search and arrival decisions are needed.

Proposition 9 *Search unemployment is determined by the outflow of workers at the search threshold, and it is given by*

$$s_t^*(\tilde{w}^-) = \frac{\sigma_w^2}{2}\frac{\psi_t(\tilde{w}^-)}{1-\phi}, \tag{B17}$$

while arrivals satisfy

$$(1-\eta)a_t^*(\tilde{w}^+) = \frac{\psi_t(\tilde{w}^-)}{(1-\phi)} + \eta_w, \tag{B18}$$

with $\eta_w \equiv 2\bar{\eta}/\sigma_w^2$.

and the third to the optimally chosen drift for the local labor force. Since $\tilde{\psi}_t(\varphi_t(x), l_t(x)) = \psi_t(\tilde{w}_t(x))$, the changes in the wage drift due to changes in the local labor force take place in three different ways. First, there is an inflow of $\bar{\eta}$ in the entire support $[\tilde{w}^-, \infty)$, an inflow of directed arrivals in $\tilde{w}^+$, and an outflow of searchers at $\tilde{w}^-$. Since the first changes take place over the entire support, the change in the drift due to random arrivals is of the form $\partial[\bar{\eta} l_t(x)\tilde{\psi}_t(x)]/\partial l_t(x) = \bar{\eta}\{\psi_t(\tilde{w}_t(x)) + \psi'_t(\tilde{w}_t(x))(\phi - 1)\}$, where the second term arises simply because $l_t(x)\partial\tilde{\psi}_t(x)/\partial l_t(x) = l_t(x)\psi'_t(\tilde{w}_t(x))d\tilde{w}_t(x)/dl_t(x) = \psi'_t(\tilde{w}_t(x))(\phi-1)$. The other two changes in the drift are discussed below because they take place at sets of measure zero, i.e., at the 'exit' and 'entry' points. Supplementing the diffusion terms for the log-wages with the change in drift due to random arrivals yields

$$\frac{\partial \psi_t(\tilde{w})}{\partial t} = \frac{\sigma_w^2}{2}\psi''_t(\tilde{w}) - \left(\frac{\mu_z}{(1-\alpha)} + \frac{\alpha}{(1-\alpha)^2}\frac{\sigma_z^2}{2}\right)\psi'_t(\tilde{w}) - \{\bar{\eta}\psi_t(\tilde{w}) - (1-\phi)\bar{\eta}\psi'_t(\tilde{w})\},$$

which is the KFE (B15) in the text.

Proof. As noted by expression (23), worker search acts as an *instantaneous control* to ensure that no log-wage crosses $\tilde{w}^-$, so that $s_t^*(x) = -(\sigma_w^2/2)\tilde{\psi}_t(\tilde{w}^-)/(\phi-1)$.[22] As noted in the text, heuristically speaking, search prevents a crossing of the search threshold. More specifically, the variation of log-wages in a short time interval is proportional to σ_w^2, and by symmetry half of the density of log-wages near the threshold will make its way to $\tilde{w}^-$. These terms yield $(\sigma_w^2/2)\psi_t(\tilde{w}^-)$ in (B17) (or $\sigma_z^2\psi_t(\tilde{w}^-)/2(1-\alpha)^2$ in (23)). As the control is given in terms of workers leaving, $1/(1-\phi)$ translates worker flows into log-wage changes.

The arrival of $(1-\eta)a_t^*(\tilde{w}^+)$ workers takes place at a log-wage $\tilde{w}^+$. The fact that wages move across $\tilde{w}^+$ and that $\tilde{\psi}_t(\tilde{w})$ is assumed to be continuous means that $\lim_{\tilde{w}\uparrow\tilde{w}^+}\psi_t(\tilde{w}) = \lim_{\tilde{w}\downarrow\tilde{w}^+}\psi_t(\tilde{w})$ so that the left and right derivatives of $\psi_t(\tilde{w})$ at $\tilde{w}^+$ satisfy $\psi'_{t-}(\tilde{w}^+) - \psi'_{t+}(\tilde{w}^+) = (1-\eta)a_t^*(\tilde{w}^+)$. For the spatial assignment of workers to be *feasible*,

$$\int_{\tilde{w}^-}^{\infty}\psi_t(\tilde{w})d\tilde{w} = 1 \text{ for all } t, \text{ or } \int_{\tilde{w}^-}^{\infty}\frac{\partial\psi_t(\tilde{w})}{\partial t}d\tilde{w} = 0,$$

which, from expression (B16), implies $(\sigma_w^2/2)(1-\eta)a_t^*(\tilde{w}^+) - s_t^*(\tilde{w}^-) - \bar{\eta} = 0$ or (B18). In other words, for the spatial assignment of workers to be *feasible*, the directed arrivals in (B18) ensure that $\psi_t(\tilde{w})$ behaves like a probability density at all t. ■

Once search behavior is taken into account, the boundary terms of the KFE at $\tilde{w}^-$ must satisfy $-(\sigma_w^2/2)\psi_t'(\tilde{w}^-) + \mu_w\psi_t(\tilde{w}^-) - (\sigma_w^2/2)(\phi-1)^{-1}\psi_t(\tilde{w}^-) = 0$. Therefore, for $\gamma \equiv (2\mu_w/\sigma_w^2)$, the boundary condition at $\tilde{w}^-$ is

$$\psi_t'(\tilde{w}^-) = (\gamma + (1-\phi)^{-1})\psi_t(\tilde{w}^-). \tag{B19}$$

Proposition 10 *The stationary log-wage density $\psi^*(\tilde{w})$ can be approximated by (22).*

Proof. The stationary value of the distribution of log-wages over $\tilde{w} \in (\tilde{w}^-, \tilde{w}^+)\cup(\tilde{w}^+,\infty)$ is given by $\psi''(\tilde{w}) - \gamma\psi'(\tilde{w}) - \eta_w\psi(\tilde{w}) = 0$, where $\gamma \equiv 2\mu_w/\sigma_w^2$ and $\eta_w \equiv 2\bar{\eta}/\sigma_w^2$. A general solution for the previous homogeneous second-order ordinary differential equation is of the form

$$\psi^*(\tilde{w}) = A_1\exp\{\gamma_1\tilde{w}\} + A_2\exp\{\gamma_2\tilde{w}\}, \tag{B20}$$

where the constants $A_{1,2}$ need to be determined, and the exponents are the roots of $\mathcal{Q}(a) = a^2 - \gamma a - \eta_w = 0$, i.e.,

$$\gamma_{1,2} = \gamma/2 \pm \sqrt{(\gamma/2)^2 + \eta_w}.$$

For the roots to be real, $(\gamma/2)^2 > -\eta_w$, which holds immediately if $\bar{\eta} > 0$. Since $\mathcal{Q}(0) = -\eta_w = \gamma_1\gamma_2$, if $\bar{\eta} < 0$, and if both roots are real, the roots will be positive and no solution of the form (B20) converges to zero as $\tilde{w} \to \infty$. *For the existence of a stationary density it is then neccesary to assume that $\bar{\eta} > 0$.*

[22] This term can be obtained in multiple ways. Alvarez and Shimer ([4], equation 16) contain a derivation based on the approximation of a discrete grid. An alternative derivation based on the hypothetical movement of wages below the threshold, and the amount of control needed to bring them back to the set $[\tilde{w}^-,\infty)$ is available upon request.

If $\bar{\eta} > 0$, the roots are not only real but they are of opposite sign, $\gamma_1 > 0 > \gamma_2$. The numerical value of negative root is important to guarantee the existence of mean values associated with $\tilde{\psi}^*(\tilde{w})$. If $\mathcal{Q}(-1) = 1 + \gamma - \eta_w < 0$, then $\gamma_2 < -1$. This requires $\bar{\eta} > \sigma_w^2/2 + \mu_w$, which assumes a 'large' fraction of random flows relative to the log-wage process or a sufficiently negative drift in wages. Moreover, as $\gamma_1 + \gamma_2 = \gamma$, $\gamma_1 \gtreqless -\gamma_2$ iff $\gamma \gtreqless 0$, with the roots being symmetric (i.e., $\gamma_1 = -\gamma_2$) if $\gamma = 0$ and asymmetric if $\gamma \neq 0$.

The stationary density $\psi^*(\tilde{w})$ can be solved as a pair of second-order ordinary differential equations, one on $(\tilde{w}^-, \tilde{w}^+)$, and the other on $(\tilde{w}^+, \infty)$, with boundary conditions that ensure that the density is continuous at the threshold $\tilde{w}^+$. Consider first the support $(\tilde{w}^-, \tilde{w}^+)$. Using the threshold condition for $\tilde{w}^-$ given by (B19) in (B20) implies that

$$\psi^*(\tilde{w}) = A_2 \exp\{\gamma_2 \tilde{w}\} \left\{1 - \frac{[\gamma_2 - (\gamma + (1-\phi)^{-1})]}{[\gamma_1 - (\gamma + (1-\phi)^{-1})]} \exp\{(\gamma_1 - \gamma_2)[\tilde{w} - \tilde{w}^-]\}\right\}, \tag{B21}$$

for $\tilde{w} \in [\tilde{w}^-, \tilde{w}^+)$. Consider next $(\tilde{w}^+, \infty)$. For the boundary condition $\psi^*(\infty) = 0$ to hold in (B20), only the negative root must be active so $A_1' = 0$, and

$$\psi^*(\tilde{w}) = A_2' \exp\{\gamma_2 \tilde{w}\}, \text{ for } \tilde{w} \in (\tilde{w}^+, \infty). \tag{B22}$$

The densities (B21) and (B22) must agree on $\tilde{w}^+$ so over $[\tilde{w}^-, \tilde{w}^+]$ one has $\tilde{\psi}^*(\tilde{w}) = A_2' \exp\{\gamma_2 \tilde{w}^+\}[\exp\{\gamma_1[\tilde{w} - \tilde{w}^+]\} + \bar{\gamma}(\exp\{\gamma_2[\tilde{w} - \tilde{w}^+] - \exp\{\gamma_1[\tilde{w} - \tilde{w}^+]\})]$ with

$$\bar{\gamma} \equiv \frac{[\gamma_1 - (\gamma + (1-\phi)^{-1})]e^{\gamma_2[\tilde{w}^+ - \tilde{w}^-]}}{[\gamma_1 - (\gamma + (1-\phi)^{-1})]\, e^{\gamma_2[\tilde{w}^+ - \tilde{w}^-]} - [\gamma_2 - (\gamma + (1-\phi)^{-1})]\, e^{\gamma_1[\tilde{w}^+ - \tilde{w}^-]}},$$

while $\tilde{\psi}^*(\tilde{w}) = A_2' \exp\{\gamma_2 \tilde{w}^+\} \exp\{\gamma_2[\tilde{w} - \tilde{w}^+]\}$ over $[\tilde{w}^+, \infty)$. As an *approximation*, one can write

$$\psi^*(\tilde{w}) \simeq A_2' \exp\{\gamma_2 \tilde{w}^+\} \begin{cases} \exp\{\gamma_1[\tilde{w} - \tilde{w}^+]\} \text{ for } \tilde{w} \in [\tilde{w}^-, \tilde{w}^+] \\ \exp\{\gamma_2[\tilde{w} - \tilde{w}^+]\} \text{ for } \tilde{w} \in [\tilde{w}^+, \infty), \end{cases} \tag{B23}$$

which can be used to obtain (22). As the density must integrate to one, the constant term C_ψ is such that one obtains $A_2' \exp\{\gamma_2 \tilde{w}^+\}[\gamma_1^{-1}(1 - \exp\{\gamma_1[\tilde{w}^- - \tilde{w}^+]) + \bar{\gamma}\gamma_2^{-1}(1 - \exp\{\gamma_2[\tilde{w}^- - \tilde{w}^+]) - \gamma_1^{-1}((1 - \exp\{\gamma_1[\tilde{w}^- - \tilde{w}^+]\})) - \gamma_2^{-1}] = 1$. ■

Some remarks on the wage density. The approximation in (B23) is proportional to $\bar{\gamma}(\exp\{\gamma_2[\tilde{w} - \tilde{w}^+] - \exp\{\gamma_1[\tilde{w} - \tilde{w}^+]\})$, which has been dropped from $\psi^*(\tilde{w})$ over $[\tilde{w}^-, \tilde{w}^+]$. Since the density of wages is $\psi^*(\log w)/w$, applying $\exp\{\gamma_1[\tilde{w} - \tilde{w}^+]\} \equiv \exp\{\gamma_1 \log[w/w^+]\}$ makes the double Pareto nature of the density of wages evident. In terms of wages,

$$\psi^*(\log w)/w \simeq C_\psi \begin{cases} \dfrac{w^{\gamma_1 - 1}}{(w^+)^{\gamma_1}} \text{ for } w^- \leq w \leq w^+ \\[2ex] \dfrac{w^{\gamma_2 - 1}}{(w^+)^{\gamma 2}} \text{ for } w > w^+. \end{cases} \tag{B24}$$

Also, notice that the stationary values of search unemployment and arrivals are directly obtainable from $\psi^*(\tilde{w})$. Search under $\mu_w = 0$, for example, satisfies

$$s^*(\tilde{w}^-) \simeq \frac{\sigma_w \sqrt{\bar{\eta}/2}}{(1-\phi)\{1+2[1+\rho+\bar{\eta}]^{\sqrt{2\bar{\eta}/\sigma_w^2}}\}}, \tag{B25}$$

which is increasing in wage dispersion σ_w due to a higher likelihood of reaching the search threshold. When $\mu_w = 0$, arrivals $a^*(\tilde{w}^+)$ are

$$a^*(\tilde{w}^+) \simeq \frac{2\psi^*(\tilde{w}^+)\sqrt{\bar{\eta}/2}}{(1-\eta)(\sigma_w/2)}. \tag{B26}$$

These are closed-form solutions to the stationary wage density and the frictional worker flows.

Local labor force and job assignments. One needs to still determine the the stationary density of the labor force, as a function of the local state $\tilde{w}$. Let such density be denoted by $l^*(\tilde{w})$. I assume that $l^*(\tilde{w})$ is a piecewise continuous function of the state (i.e., log-wages). Consider a path for log-wages with a starting value of $\tilde{w}_0$. The labor force (18) evolves as a stochastic differential equation which, when integrated, yields a realization

$$l(\tilde{w}_t) = l(\tilde{w}_0) \exp\left\{-\bar{\eta}t - (\eta-1)a^*(\tilde{w}^+)\int_0^t \mathbb{I}_{\{\tilde{w}^+\}} d\tau - s^*(\tilde{w}^-)\int_0^t \mathbb{I}_{\{\tilde{w}^-\}} d\tau\right\}, \tag{B27}$$

where $\mathbb{I}_{\{\tilde{w}^{-,+(\theta)}\}}$ denote the indicator functions that equal one when log-wages reach the thresholds. This expression can be written as

$$l_t(\tilde{w}) = l(\tilde{w}_0) \exp\left\{-\bar{\eta}t - (\eta-1)a^*(\tilde{w}^+)\mathbb{L}_t^+ - s^*(\tilde{w}^-)\mathbb{L}_t^-\right\}, \tag{B28}$$

where $\mathbb{L}_t^{-,+}$ denote the *local times* of the log-wage process at $\tilde{w}^-$ and $\tilde{w}^+$. The *local times* at $\tilde{w}^-$ and $\tilde{w}^+$ are defined by $\mathbb{L}_t^{-,+} \equiv \int_0^t \mathbb{I}_{\{\tilde{w}^{-,+}\}} d\tau$ so (B27) and (B28) are identical.[23]

Proposition 11 *The stationary density of the local labor force can be approximated by*

$$l^*(\tilde{w}) \simeq LC_L \left\{\exp\{-\gamma_l \tilde{w}\} + \exp\{\gamma_l \tilde{w}\}\mathbb{I}_{\{[\tilde{w}^-,\tilde{w}^+)\}}\right\}, \tag{B29}$$

where C_L is a constant, $\gamma_l = \sqrt{2\bar{\eta}} > 0$, and $\mathbb{I}_{\{[\tilde{w}^-,\tilde{w}^+)\}}$ is an indicator function.

Proof. The stationary density of the labor force $l^*(\tilde{w})$ satisfies an ordinary second-order differential equation with a jump discontinuity at the arrival threshold $\tilde{w}^+$ and an *elastic barrier* that "kills" workers at a rate $s^*(\tilde{w}^-)$ when log-wages reach $\tilde{w}^-$. The density can be computed using (B27) and the functional equation

$$l^*(\tilde{w}) \equiv \mathbb{E}^{\tilde{w}} \int_0^\infty l(\tilde{w}_t)dt.$$

[23] Local times measure the time log-wages spend in the states $\tilde{w}^-$ and $\tilde{w}^+$, as in $\mathbb{L}_t^- \equiv \int_0^t \mathbb{I}_{\{\tilde{w}^-\}} d\tau$ and $\mathbb{L}_t^+ \equiv \int_0^t \mathbb{I}_{\{\tilde{w}^+\}} d\tau$, where $\mathbb{I}_{\{A\}}$is an indicator function; see Karatzas and Shreve [33].

By the Feynman-Kac formula (Karatzas and Shreve [33], Section 4.4), $l^*(\tilde{w})$ satisfies an ordinary second-order homogeneous ordinary differential equation $(1/2)l''(\tilde{w}) = \bar{\eta}l(\tilde{w})$ over $\tilde{w} \in (\tilde{w}^-, \tilde{w}^+) \cup (\tilde{w}^+, \infty)$, where $l''(\tilde{w}) \equiv \partial^2 l^*(\tilde{w})/\partial\tilde{w}^2$. The general solution $l^*(\tilde{w})$ is of the form

$$l^*(\tilde{w}) = A_1 \exp\{\gamma_l \tilde{w}\} + A_2 \exp\{-\gamma_l \tilde{w}\}, \tag{B30}$$

for $\gamma_l = \sqrt{2\bar{\eta}} > 0$, and with the constants $A_{1,2}$ to be determined by the boundary conditions. *A finite value of the aggregate labor input L^* requires that γ_2 and γ_l are sufficiently negative, i.e., $-\gamma_2 + \gamma_l > 1$.*

Treat $l^*(\tilde{w})$ as a pair of differential equations, one on $(\tilde{w}^-, \tilde{w}^+)$, and the other on $(\tilde{w}^+, \infty)$. Boundary conditions ensure that $l^*(\tilde{w})$ is integrable, jump discontinuous at the arrival threshold, and such that workers leave at an exponential rate at the search threshold. That is, in the limit, $l^*(\infty) = 0$ to ensure integrability. The boundary condition at $\tilde{w}^+$ reflects the proportional entry of directed arrivals, as in $l'_-(\tilde{w}^+) - l'_+(\tilde{w}^+) = (1-\eta)a^*(\tilde{w}^+)l(\tilde{w}^+)$. The left and right derivatives at $\tilde{w}^+$ are *not* equal due to the entry of directed arrivals so $l^*(\tilde{w})$ experiences a jump discontinuity at $\tilde{w}^+$. The boundary at $\tilde{w}^-$ is treated as an *elastic barrier*, as in $l'(\tilde{w}^-) = -s^*(\tilde{w}^-)l(\tilde{w}^-)$. This condition implies that workers are "killed" at an exponential rate $s^*(\tilde{w}^-)$ whenever the wage reaches the threshold $\tilde{w}^-$.

With the previous boundary conditions, the construction of the solution is standard. The elastic barrier gives a linear relationship between A_1 and A_2 over $[\tilde{w}^-, \tilde{w}^+)$: $A_1 = A_2[(\gamma_l - s^*(\tilde{w}^-))/(\gamma_l + s^*(\tilde{w}^-))] \exp\{-2\gamma_l\tilde{w}^-\}$. Over $(\tilde{w}^+, \infty)$, $A'_1 = 0$ since only the negative root remains to ensure integrability. The entry condition at $\tilde{w}^+$ gives a linear relationship between A_2 and A'_2 so one can write

$$l^*(\tilde{w}) = A'_2 \left[\bar{\gamma}_{2,l} \exp\{\gamma_l \tilde{w}\} + \bar{\gamma}_{1,l} \exp\{-\gamma_l \tilde{w}\}\right],$$

over $[\tilde{w}^-, \tilde{w}^+)$ with

$$\bar{\gamma}_{1,l} \equiv \frac{\left[\left(\frac{(1-\eta)a^*(\tilde{w}^+)}{\gamma_l}\right) - 1\right]}{\left(\frac{\gamma_l - s^*(\tilde{w}^-)}{\gamma_l + s^*(\tilde{w}^-)}\right) e^{2\gamma_l[\tilde{w}^+ - \tilde{w}^-]} - 1}, \text{ and } \bar{\gamma}_{2,l} \equiv \bar{\gamma}_{1,l}\left(\frac{\gamma_l - s^*(\tilde{w}^-)}{\gamma_l + s^*(\tilde{w}^-)}\right) e^{-2\gamma_l \tilde{w}^-},$$

and $l^*(\tilde{w}) = A'_2 \exp\{-\gamma_l\tilde{w}\}$ over $[\tilde{w}^+, \infty)$. The constant $A'_2 = LC_L$ is determined since the density $l^*(\tilde{w})$ integrates to 1 (i.e., $A'_2\{(\bar{\gamma}_{2,l}/\gamma_l)[\exp\{\gamma_l\tilde{w}^+\} - \exp\{\gamma_l\tilde{w}^-\}] - (\bar{\gamma}_{1,l}/\gamma_l)[\exp\{\gamma_l\tilde{w}^+\} - \exp\{\gamma_l\tilde{w}^-\}] + [\exp\{-\gamma_l\tilde{w}^+\}]/\gamma_l\} = 1$). As an approximation, small values of search imply that $\bar{\gamma}_{2,l} \simeq \bar{\gamma}_{1,l}$. If one further approximates $\bar{\gamma}_{1,l}$ to one, the previous solution yields (B29). ■

The stationary density $l^*(\tilde{w})$ is jump-discontinuous and composed of two exponential functions. The first function, defined over $[\tilde{w}^-, \tilde{w}^+)$, is of the form $\exp\{-\gamma_l\tilde{w}\} + \exp\{\gamma_l\tilde{w}\}$. As $\tilde{w} \uparrow \tilde{w}^+$, $l^*(\tilde{w})$ jumps down. This downward jump at $\tilde{w}^+$ accounts for the arrival of workers and ensures that the stationary density of wages is continuous. (The approximation in (B29) is about the size of that jump.) The second function is defined over $[\tilde{w}^+, \infty)$ where $l^*(\tilde{w})$ decreases at a rate $\gamma_l = \sqrt{2\bar{\eta}}$ determined by the exogenous worker separation rate.

Since $k_t^*(\tilde{w})$ is proportional to $w_t l_t(\tilde{w})$ in (B31), the stationary assignment of jobs satisfies

$$k^*(\tilde{w}) = \frac{\alpha \exp\{\tilde{w}\} l^*(\tilde{w})}{\phi(1-\alpha)(\rho+\delta)}, \tag{B31}$$

where $\exp\{\tilde{w}\}$ is the local wage and $l^*(\tilde{w})$ is the stationary density (B29). Local output is $g_t(\tilde{w}) = w_t l_t(\tilde{w})/\phi$ so aggregate output and capital equal

$$Y^* = \frac{L^*}{\phi}, \text{ and } K^* = \frac{\alpha L^*}{\phi(1-\alpha)(\rho+\delta)}, \tag{B32}$$

where L^* is the *aggregate labor input*, defined by

$$L^* \equiv \int_{\tilde{w}^-}^{\infty} \exp\{\tilde{w}\} l^*(\tilde{w}) \psi^*(\tilde{w}) d\tilde{w}. \tag{B33}$$

The previous expressions provide a closed-form characterization of the *ex ante* spatial allocation of workers and jobs, the stationary wage distribution, and all aggregate variables.

Local imbalances. Local shocks behave as a geometric Brownian motion of the form $d\varphi_t(x)/\varphi_t(x) = \mu_z dt + \sigma_z dB_t$, with probabilistic initial conditions $\varphi_0(x)$ and a reflecting barrier $\varphi^- \equiv (\rho+\delta)[(w^-/(\rho+\delta)) l^*(\tilde{w}^-)^{1-\phi}]^{1-\alpha}\{\alpha^{-\alpha}[\phi(1-\alpha)]^{-(1-\alpha)}\}$. This barrier is associated with the search threshold and ensures that local shocks will not cross φ^-. Consider $\tilde{\varphi}_t(x) \equiv \log(\varphi_t(x)/\varphi^-)$ such that $d\tilde{\varphi}_t(x) = \tilde{\mu}_z dt + \sigma_z dB_t$, with $\tilde{\mu}_z \equiv \mu_z - \sigma_z^2/2$, and a reflecing barrier at zero. The stationary distribution of the log-shocks is standard and given by

$$\Pr\{\tilde{\varphi}_t(x) \le \tilde{\varphi}\} = \mathcal{N}\left(\frac{\tilde{\varphi} - \tilde{\mu}_z t}{\sigma_z t^{1/2}}\right) - \exp\{2\tilde{\mu}_z \tilde{\varphi}/\sigma_z^2\} \mathcal{N}\left(\frac{-\tilde{\varphi} - \tilde{\mu}_z t}{\sigma_z t^{1/2}}\right), \text{ for all } t \ge 0,$$

where $\mathcal{N}$ is the standard normal distribution; see, e.g., Harrison ([23], p. 15). In the limit, as $t \to \infty$, $\Pr\{\tilde{\varphi}_t(x) \le \tilde{\varphi}\} \to 1 - \exp\{2\tilde{\mu}_z \tilde{\varphi}/\sigma_z^2\}$ for $\tilde{\mu}_z < 0$. The log-shock's limiting density, $-\exp\{2\tilde{\mu}_z/\sigma_z^2\}\{2\tilde{\mu}_z \tilde{\varphi}/\sigma_z^2\}$, is exponential and can be written as $\gamma_\varphi \exp\{-\gamma_\varphi \tilde{\varphi}\}$ for $\gamma_\varphi \equiv -2\tilde{\mu}_z/\sigma_z^2 > 0$ or $\gamma_\varphi \equiv 1 - 2\mu_z/\sigma_z^2$. The stationary distribution of the local shocks is therefore Pareto and given by $\Pr\{\varphi_t(x) \le \varphi\} \to 1 - [\varphi^-/\varphi]^{\gamma_\varphi}$.

The stationary distributions of local imbalances follow from the previous derivations and the static imbalances. Job-shortages are distributed as an inverse Pareto distribution $[u/u^+]^{\gamma_\varphi}$ defined over a support $[0, u^+]$ with upper bound $u^+ \equiv (\rho+\delta)/\varphi^-(1+\phi(1-\alpha))$. (The distribution is also known as the *Power Function Distribution*.) The inverse Pareto characterizes the lower tail of the unemployment distribution. Job shortage rates in the typical location are $\gamma_\varphi u^+/(1+\gamma_\varphi)$ and the cross-sectional variance in unemployment rates is

$$\sigma_u^2 = \frac{(u^+)^2 \gamma_\varphi}{(1+\gamma_\varphi)^2(2+\gamma_\varphi)} = \frac{(\tilde{u}^*)^2}{\gamma_\varphi(2+\gamma_\varphi)}.$$

A feature of the distribution of unemployment is that the *cross-sectional dispersion* of unemployment rates, defined by σ_u, is positively related to the mean unemployment rate $\tilde{u}^*$, i.e., both moments proportional and are functions of u^+, for instance. Although there are no aggregate shocks introducing business cycle dynamics, the positive relationship between these moments is consistent with the time series evidence presented in the quantitative section. Moreover, the shape parameter $\gamma_\varphi \equiv 1 - 2\mu_z/\sigma_z^2$ determines the heaviness of the inverse Pareto tail. A smaller value of γ_φ yields a heavier tail and therefore higher unemployment inequality across locations. I confronted the tail behavior in the main text, and provide some additional empirical analyses below.

Vacancy rates also have a stationary distribution of the Pareto type, i.e., $1-v$ is inverse Pareto $[(1-v)/(1-v^-)]^{\gamma_\varphi}$ defined over $[1-v^-, 1]$ with $1-v^- \equiv (\rho+\delta)/\alpha\varphi^-$. In the calibration exercise, I assume $v^- = (\rho+\delta)/\alpha\varphi^- = 1$ leading to a support $[0,1]$. The relevant moments for the vacancy rate can be calculated as in the case of job shortages. As noted in the text, and for the case of unemployment, the *cross-sectional dispersion* of job shortages and vacancies is positively related to the *mean* job shortages and vacancies across locations.

8 Appendix: Calibration and Quantitative Analyses [Not needed for publication]

This Appendix collects details about the calibration of the parameters used in the baseline model. I also provide additional quantitative analyses to complement those in the text.

Aggregate parameters. As noted in the text, I consider a depreciation rate of $\delta = 0.012$ and a discount rate of $\rho = 0.012$, as it is standard at quarterly frequencies from Cooley ([12], p. 22). I calibrate ϕ using the value for the labor share commonly used in equilibrium search models, i.e., $\phi = 0.64$; see, e.g., Alvarez and Shimer ([4], p. 102). This value assumes that capital adjusts in response to labor flows, as in the indirect production function $g_t(x)$ considered here.

The relevant share of capital in national income is often computed as one minus the labor share. Not all physical capital, however, is reproducible or relevant for the notion of job capital used here. In the US, nonresidential equipment is on average about 15 percent of the Net Stock of Private Fixed Assets according to the BEA (Table 2.1). Since equipment is more in line with the notion of job capital used here, I assume that the job capital share is $\alpha = 0.15 \times (1 - 0.64) = 0.06$. The job shortage rate in the typical location depends on α and ϕ, and it is moderately sensitive to α, as I show below in Table C2.

The value of η plays a secondary role overall so long $\eta > 0$. The reason is that the relevant flow is given by the total number of arrivals $(1 - \eta)a_t(x)$ and not by the separate values for the fraction of direct arrivals η and the number of arrivals. In other words, since $\bar{\eta} \equiv \bar{s} - \eta\bar{a}$, the *gross* flow rates $\bar{a}$ are $\bar{s}$ undetermined and so is η. Because only *net* flows are relevant, provided that $\eta > 0$, I assume an exogenous *net* worker flow rate of $\bar{\eta} = 0.02$ which is the average quit rate that matches wage persistence under random search in Alvarez and Shimer ([4], p. 101). Their value for exogenous worker flows under directed search is only marginally different.

Demand shortages. I assume that the survival function that determines demand shortages is

$$q(\tilde{w}_0(x)) = \min\{\exp\{-\chi[(\tilde{w}^+ - \tilde{w}_0(x))/(\tilde{w}^+ - \tilde{w}^-)]\}, 1\},$$

for $\chi > 0$. Thus, demand uncertainty is relevant only in low wage locations. At the search threshold, $1 - q(\tilde{w}^-) = 1 - \exp\{-\chi\}$ whereas $1 - q(\tilde{w}_0(x)) = 0$ for any wage above the arrival threshold $\tilde{w}_0(x) \geq \tilde{w}^+$. Demand uncertainty, i.e., $1 - q(\tilde{w}_0(x))$ thus declines exponentially at the rate χ.

To calibrate χ, I use measures of *unused capacity* in the product market for manufacturing goods, as in Michaillat and Saez [45]. The Federal Reserve Board (FRB) publishes monthly measures of capacity utilization (https://www.federalreserve.gov/datadownload/). Data is available since 1948 for a fewer sectors, but I use data for 12 manufacturing sectors consistently available since January 1972. I measure the importance of demand shortages on aggregate unemployment using a simple (i.e., naïve) OLS regression between the cross-sectional standard deviation of capacity utilization on aggregate monthly unemployment.

Table C1 reports an OLS coefficient of the cross-sectional standard deviation of capacity utilization on aggregate monthly unemployment of 0.42 (s.e., 0.046). The OLS coefficient of the monthly interquantile range (IQR) of capacity utilization on aggregate unemployment is 0.30 (s.e., 0.020). On average, the standard deviation of capacity utilization in the sample is 6.54 percent and the average IQR is 7.70 percent. Both measures of dispersion suggest that demand shortages associated with cross-sectional differences in capacity utilization amount to about 2 percent unemployment (i.e., $0.0654 \times 0.42 = 0.027$, for the standard deviation, or $0.077 \times 0.30 = 0.023$, for the IQR). Using $\chi = 0.0023$ and the stationary density of wages $\psi^*(\tilde{w}_0(x))$ in (24) yields this naïve estimate.

Table C1. Predicted aggregate unemployment due to demand shortages.

Dep. Var: Monthly aggregate unemployment rate						Predicted unempl.	
	Mean	OLS		OLS			
STDev	6.54	0.29	0.42			1.89	2.74
	(1.71)	(0.038)	(0.046)				
IQR	7.70			0.19	0.30	1.46	2.31
	(3.90)			(0.019)	(0.020)		
Controls		No	Yes	No	Yes		
Months		683	683	683	683		

Note: OLS regression between monthly aggregate unemployment rates and cross-sectional dispersion measures of capacity utilization in manufacturing. Controls include and indicator for recessions, as well as linear and quadratic time trends. The predicted values represent the naïve estimate of demand-driven unemployment rates. The sectors in durable manufacturing are: Electrical equipment, appliance, and component; motor vehicles and parts; aerospace and miscellaneous transportation equipment; furniture and related product; miscellaneous. The sectors in non-durable manufacturing are: Food, beverage, and tobacco; textiles and products; apparel and leather goods; paper; printing and related support activities; petroleum and coal products; and chemical.

The calibration of χ relies on the *cross-sectional dispersion* of utilization measures, rather than on mean values of capacity utilization, which is the main measure in the empirical analysis in Michaillat and Saez [45]. Capacity utilization averages 0.79 in the data, which would imply very high demand-driven unemployment rates if taken at face value. I rely on cross-sectional dispersion measures because in an exponential distribution, χ controls the mean and the standard deviation. A separate difficulty with the use of mean values is that in the national accounts there is no room for imbalances in the output market, as only actual transactions are measured. This can be interpreted as assuming that the output markets clear *ex post*, as I assumed in the model developed here.

Wage inequality. I calibrate the local productivity shocks to match the stationary distribution of frictional wage inequality. Alvarez and Shimer ([4], Table 1) measured the dispersion of *sectoral* level log-wages at a 5 digit industry level and report $\hat{\sigma}_w = 0.037$. I updated their auxiliary statistical model of industry wages. Given an industry fixed-effect δ_i and an across-industries average log-wage $\tilde{w}_t$, I estimate: $\tilde{w}_{i,t} - \tilde{w}_t = \delta_i + \beta_w(\tilde{w}_{i,t-1} - \tilde{w}_{t-1}) + \sigma_w \varepsilon_{i,t}$. Time series estimates for 5-digit level NAICS industries with data from the Current Employment Statistics (CES, https://www.bls.gov/ces/) from January 1990 until February 2019 yield $\hat{\sigma}_w = 0.05$, with $\hat{\sigma}_w = 0.04$ if the Great Recession is dropped, consistent with their slightly smaller estimates of σ_w.

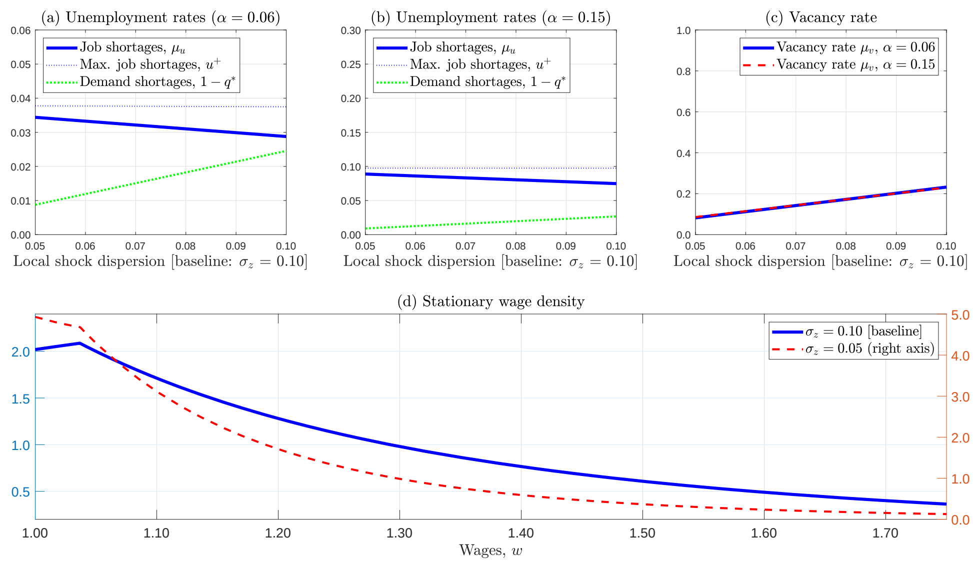

Figure 2: Sensitivity analyses to changes in local shock dispersion and capital share.

As I noted in the text, industrial sectors alone are too broad to be consistent with the notion of a "island" as sectors do not discriminate across geographic and occupational categories. To calibrate the dispersion of wages I consider more disaggregated information. In particular, the residual dispersion of *individual* log-wages is larger than at the sectoral level. Heathcote et al. ([25], Figure 18), for instance, considered an individual log-wage process

$$\tilde{w}_{i,t} = \tilde{w}_{i,t}^{perm} + \tilde{w}_{i,t}^{trans}, \text{ where } \tilde{w}_{i,t}^{perm} = \tilde{w}_{i,t-1}^{perm} + \sigma_w^{perm} \varepsilon_{i,t}^{perm},$$

and report a dispersion of the permanent component of $\hat{\sigma}_w^{perm} \approx 0.14$. I consider $\sigma_z = 0.10$ so that $\sigma_w = \sigma_z/(1-\alpha) = 0.106$, which is an intermediate value between sectoral and individual data.

Given σ_z, I calibrate μ_z to match the estimated heavy upper tail of wages in Toda ([58], p. 368). I focus on the upper Pareto tail exponent, as I treated the lower tail as an approximation. In addition, the upper tail is a more relevant metric for wage inequality. Toda [58] examined, conditional on education and experience, cross-sectional and panel wage data in the US from the Current Population Survey (CPS) and the Panel Study of Income Dynamics (PSID). He showed, and checked by goodness-of-fit tests, that wages are well approximated by a double Pareto distribution. I associate his estimates to the distribution of frictional wage inequality. His estimated upper tail exponent is -2.34. Given σ_z and $\bar{\eta}$, a value $\mu_z = -0.0115$ yields $\gamma_2 = -2.34$. For illustration, the Gini coefficient in a Pareto distribution is $(2|\gamma_2| - 1)^{-1}$ so, ignoring the lower tail in wages, the baseline parameterization yields a Gini coefficient for wages of 0.27. The resulting growth rate of log-wages is $\mu_w = -0.0047$ and the mode/min ratio for wages is $w^+/w^- = 1+\rho+\bar{\eta}-\mu_w = 1.0167$. Toda's ([58], p. 368) lower tail exponent is 1.15 which is slightly lower than the resulting value here, $\gamma_1 = 1.51$.

The stationary density of local productivities is governed by a Pareto exponent $\gamma_\varphi \equiv 1 - 2\mu_z/\sigma_z^2 = 3.3$. The upper tail of the density of the labor force is governed by an exponent $\gamma_l \equiv \sqrt{2\bar{\eta}} = 0.2$. The previous densities are stationary while the US economy features sustained growth. This means that the negative drift terms μ_z and μ_w should be interpreted as productivity and wage growth *relative* to aggregate growth not modeled here.

Normalization. I normalize $L = 0.00021$ to ensure that $\varphi^- = (\rho+\delta)q^*/\alpha$ in (26). Higher values of L yield higher values of φ^- and higher vacancy rates, whereas lower values of L yield lower vacancy rates. For instance, expression (26) would suggest negative vacancies. This expression, however, only applies to values of ϵ above ϵ_+. Smaller values of L or assuming $\varphi^- < (\rho + \delta)q^*/\alpha$ delivers small values of the vacancy rate by using the 'flat' segment of the uncertain factor requirement shock with values of ϵ below the threshold ϵ_+. An Appendix not for publication considers the flat segments for completeness, but to keep the message of the model consistent, I do not consider values of ϵ below ϵ_+.

Table C2. Selected aggregate labor market outcomes.

	Aggregate unemployment rates			Aggregate	Frictional	
	Demand shortages	Job shortages	Search unemployment	Vacancy rate	wage inequality	
	$1 - q^*$		$s^*(\tilde{w}^-)$	$\tilde{v}^*$	γ_1	γ_2
Baseline calibration	2.45	2.88	0.41	23.2	1.51	−2.34
$\mu_z = -0.05$	0.21	3.44	1.39	8.16	0.41	−8.48
$\mu_z = -0.10$	0.02	3.63	2.63	2.97	0.20	−17.6
$\sigma_z = 0.05$	0.88	3.44	0.25	8.21	2.39	−5.89
$\sigma_z = 0.11$	2.78	2.79	0.44	25.6	1.40	−2.08
$\alpha = 0.15$	2.67	7.48	0.46	23.0	1.36	−2.12
$\chi = 0.01$	10.0	2.64	0.41	29.6	1.51	−2.34

Note: The table reports the stationary values of key aggregate labor market variables. All rates are reported in percentages. The coefficients $\gamma_{1,2}$ represent the Pareto exponents of the stationary wage distribution. The parameters used in the baseline calibration are $\alpha = 0.06$, $\chi = 0.0023$, $\mu_z = -0.0115$, and $\sigma_z = 0.10$.

Sensitivity analysis. Table C2 presents the baseline calibration and several alternate values for the main parameters, all under the normalization $\varphi^- = (\rho + \delta)q^*/\alpha$, achieved by changing the value of L. As noted in the text, a more negative drift increases the importance of search unemployment at the expense of demand shortages, and it reduces the mean value of the aggregate vacancy rate. While demand shortages are exogenous, demand-driven unemployment changes by the associated shifting of the stationary distribution of the local state $\psi^*(\tilde{w}(x))$. The downside of lower values of μ_z is a reduced frictional wage dispersion. Table C2 also considers alternate values of α and χ. A higher value for α implies higher job shortages, which is consistent with the closed-form solution in the text, while changes in χ mostly influence demand without altering the other unemployment types. Across the alternate parameterization, the value of the job shortages tends to remain stable except when α changes, as α has a direct effect on $\tilde{u}^*$, as seen in the text.

Panels (a)-(c) in Figure 2 examine the sensitivity to changes in the local shock dispersion. Panel (a) shows that demand shortages in the representative location increase due to a shift in the stationary density $\psi^*(\tilde{w}_0)$. Job shortages μ_u, including the highest job shortage rate

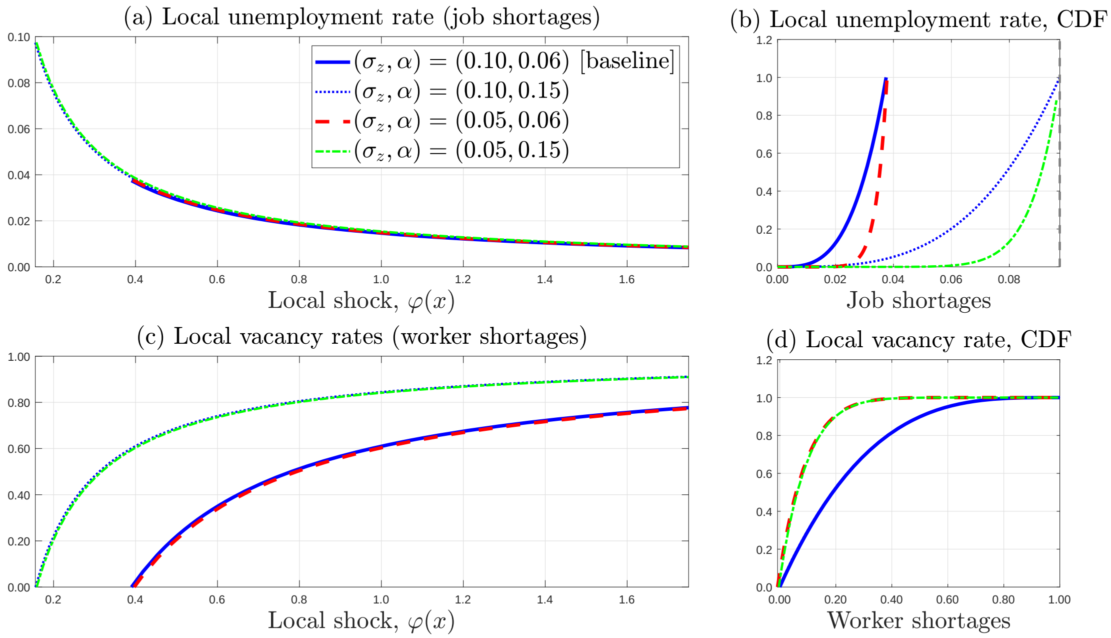

Figure 3: Stationary distributions of unemployment and vacancies.

u^+, also increase. These changes are also evident in Panel (b) which considers $\alpha = 0.15$. An increase in local shock dispersion translates into higher local unemployment rates and higher dispersion in local unemployment rates. Panel (c) confirms that vacancy rates are insensitive to α, due to the normalization used here. Finally, panel (d) shows the double Pareto nature of frictional wages.

Lower tail unemployment inequality. Figure 3 plots the stationary distributions of local unemployment rates (job shortages) and vacancy rates (worker shortages). The left panel, (a) and (c), shows the functional relationship between local shocks $\varphi(\tilde{w})$ and the unemployment and vacancy rates. The right panel, (b) and (d), depicts the cumulative distribution functions of unemployment and vacancies across locations. There are substantial differences in the cross-sectional dispersions of unemployment and vacancies with unproductive locations having higher unemployment and lower vacancy rates in equilibrium. The predicted Pareto tails of unemployment are also evident in panel (b).

Table C3 presents alternative estimates of the lower Pareto tail of the unemployment distribution γ_φ. Panel A considers a time-invariant mode in (27). The point of the time invariant estimates is to show that the tail behavior reported in text is robust to considering a single unemployment mode, rather than a mode that varies over time. A time varying mode u_t^+ is meant to capture shifts in the distribution of unemployment rates across locations. The fact that the estimates for γ_φ in Table C3 are similar to those in Table 2 imply that the shape of the distribution of unemployment rates is not too drastically perturbed by business cycle fluctuations. I used the mode of the unemployment rates to estimate u_t^+, as the mode is a consistent estimate. Panel B looks at deviations from the *median* unemployment rate rather than the mode. The tail behavior, again, is consistent with the estimates reported in Table 2 suggesting that the lower tail behavior is not heavily dependent on the cut-off point used to measure u_t^+.

The previous estimates and the text focused on lower tail inequality in unemployment rates. Table C3, Panel C, also reports estimates for the upper tail coefficients of the distribution of unemployment rates across the same cross-sectional units as in Table 2. Two observations are important. First, the order of estimates coincides with those of the lower tail. That is, estimates based on (O) are lower than those in (I), (S), and (C). This ranking, as noted in text, is likely due to the different time periods under consideration, and the differences in the cross-sectional coverage.

In addition, while the estimates of the upper tail inequality are more stable across "islands," and suggest higher unemployment inequality than in the lower part of the distribution, the R^2 in Panel C are lower than in the lower tail. The lower R^2 imply that a log-log relationship such as (27) is unable to account for a large fraction of the variability in the upper end of the distribution of unemployment rates.

Table C3. Empirical estimates of the lower tail of unemployment distribution.

	Cross-sectional unit				Cross-sectional unit			
A. Time invariant u^+	I	O	C	S	I	O	C	S
$\ln[\tilde{u}_{i,t} - u^+]$	3.02	1.21	5.28	3.86	3.02	1.21	5.28	3.86
	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)
Recession					0.10	0.01	−0.02	−0.01
					(0.01)	(0.01)	(0.02)	(0.01)
R^2	0.91	0.98	0.94	0.95	0.91	0.98	0.94	0.95
B. Median estimate u_t^+								
$\ln[\tilde{u}_{i,t} - u_t^+]$	1.79	0.59	5.30	3.80	1.79	0.59	5.30	3.80
	(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)
Recession					−0.01	−0.01	0.06	−0.06
					(0.02)	(0.02)	(0.01)	(0.01)
R^2	0.74	0.76	0.85	0.87	0.74	0.76	0.85	0.87
C. Upper tail								
$\ln[\tilde{u}_t^+ - u_{i,t}]$	1.03	0.67	1.46	1.03	1.03	0.67	1.47	1.03
	(0.02)	(0.01)	(0.02)	(0.01)	(0.02)	(0.01)	(0.02)	(0.01)
Recession					0.03	0.04	−0.05	−0.01
					(0.01)	(0.01)	(0.01)	(0.01)
R^2	0.47	0.71	0.38	0.45	0.47	0.72	0.38	0.45

Note: For definitions see Table 2 in the text.

Finally, the evolution of unemployment shares during recessions is important for the measurement of mismatch unemployment in the mismatch indices discussed in the text. I examined the relationship between local unemployment rates and their mean values across the cross-section, as a function of the NBER recession indicators. Regardless of the definition of "islands," along the lines previolusy discussed, there is no significant difference in the share of the unemployment rate attributed to particular sub-market and the NBER recession indicators. The findings are available upon request.

Unemployment moments. Table C4 report estimates of the time series relationship between the first two moments of the distribution of unemployment rates. These moments are positively associated for the cross-sectional units considered in Tables 2 and C3, and their positive association is not a consequence of the business cycle. To the extent that cyclical conditions are controlled by the NBER Recession indicator, episodes of high aggregate unemployment rates are associated with high cross-sectional dispersion regardless of the stage of the business cycle. More importantly, the relationship between these moments roughly aligns with the distributional implications of the model, as discussed in the text.

Table C4. Relationship between mean and cross-sectional dispersion in unemployment rates.

	Cross-sectional unit				Cross-sectional unit			
	I	O	C	S	I	O	C	S
$\tilde{u}_t$	0.41	0.45	0.20	0.25	0.40	0.45	0.20	0.25
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Recession					0.15	0.23	0.10	-0.07
					(0.03)	(0.06)	(0.03)	(0.03)
R^2	0.83	0.67	0.76	0.72	0.85	0.68	0.76	0.72
N. groups	12	10	36	51	12	10	36	51
N. obs.	500	500	358	500	500	500	358	500

Note: OLS estimates of the relationship between the first two moments of the distribution of unemployment rates across (I)ndustries, (O)ccupations, (S)tates, and large metropolitan areas or (C)ities. Standard errors in parenthesis. Recession is a control according to the NBER chronology.

Unemployment Dynamics. I next consider the speed at which labor markets reach their stationary equilibrium following a simple aggregate "shock." I start with the economy in the stationary equilibrium in which the distribution of the augmented local shocks is $\Pr\{\varphi^*(x) \leq \varphi\} = 1 - [\varphi^-/\varphi]^{\gamma_\varphi}$. I then assume that local productivities decline unexpectedly to φ^-. I focus on the deterministic transitional dynamics of the aggregate job shortage rate $\tilde{u}_t$ as it returns back towards $\tilde{u}^*$.

Figure 4(a) depicts the adjustment path for the augmented productivity in the typical location, $\varphi_t \equiv \mathbb{E}_{\tilde{w}_t(x)}[\varphi_t(x)]$, as deviations from the stationary equilibrium. Figure 4(b) reports the aggregate job shortage rate $\tilde{u}_t$, taking demand conditions as fixed at q^*. (This is done to focus on the local imbalances only.) There are several notable features in the adjustment paths of φ_t and $\tilde{u}_t$. First, as expected, the aggregate "shock" is recessionary as productivities reach their lowest possible value. Second, the aggregate "shock" is large. Relative to its stationary value, a decline in local productivities to φ^- lowers the productivity of the typical location by about 25 percent. Third, the adjustment path for φ_t and $\tilde{u}_t$ is fairly persistent. After the initial shock, it takes about 20 quarters to reach $|\varphi_t - \varphi^*| = 0.9$, and over 80 quarters to return to $\varphi_t = \varphi^*$. The persistent path is also evident in the job shortage rate $\tilde{u}_t$, that also converges slowly to $\tilde{u}^*$. Finally, while φ_t has a wide range, there is virtually *no* amplification in the job shortage rates, as $\tilde{u}_t$ varies by very small amounts, Panel (b). At its peak, for example, the job shortage rate is at most 3.38 percent. The lack of amplification is expected. Local productivities matter for job shortages depending on $\alpha/[1 + \phi(1 - \alpha)]$, which is fixed by technological conditions. In the baseline calibration, amplification is minimal as $\alpha/[1 + \phi(1 - \alpha)] = 0.04$. Finally, panel (c) reports the aggregate vacancy rate $\tilde{v}_t$. Vacancies are procyclical and trace a Beveridge curve, but they exhibit a

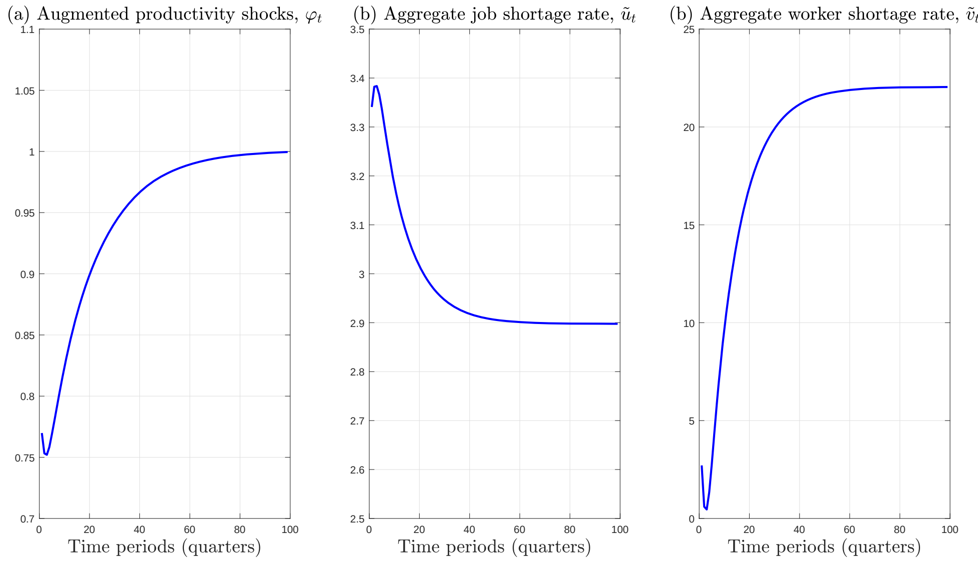

Figure 4: Adjustment to negative aggregate productivity shocks.

large range of variation in the baseline calibration.

9 Appendix: Complementary Results [Not needed for publication]

This subsection collects findings that are not of central importance to the main findings but serve as complements or as verification of claims in the text.

Microfounded aggregate demand uncertainty. The paper considers aggregate demand as uncertain in the extensive margin. This subsection considers a more explicit model of demand uncertainty. The objective of this extension is to show how to obtain an objective function for the assignment problem with the same functional form as (2) but with a more *micro*-founded structure than the original problem studied in the text.

Consumers are of two types, active and inactive. Let a and i index their types so that their consumptions are $c^a(x) > 0$ and $c^i(x) = 0$. The fraction of active consumers in the aggregate is $\xi(\omega)$ and the fraction of inactive consumers is $1 - \zeta(\omega)$. Active and inactive consumers need to be assigned across locations such that the total fraction of active consumes equals $\zeta(\omega)$. Let $p(x)$ denote the fraction of active consumers assigned to x. The consumer assignment problem is feasible if the total assigned mass of active consumers equals $\zeta(\omega)$, with nonnegativity constraints on $\pi(x)$, i.e., $0 \leq \pi(x) \leq 1$.

Individual demands are *strongly complementary*, as in $\min\{c^a(x), c^a(x)\} = c^a(x)$, with all other possible pairs giving zero consumption. The probability that an active consumer meets another active consumer is proportional to the fraction of active consumers in the location, as in $\pi(x)^2$. Likewise, meetings between active and inactive consumers have a likelihood of $2\pi(x)(1 - \pi(x))$ and meetings between inactive consumers one with $(1 - \pi(x))^2$. The planner faces an assignment problem:

$$\max \int_0^1 \pi^2(x) c^a(x) dx, \text{ s.t., } \int_0^1 \pi(x) dx = \zeta(\omega).$$

Due to the convexity of the objective function, it is optimal to have $\pi^*(x) = 1$ in a total mass of $\zeta(\omega)$ locations and $\pi^*(x) = 0$ in a mass $1 - \zeta(\omega)$. The *maximized* value of the consumer assignment problem once active consumers are sorted is

$$\int_0^{\zeta(\omega)} c^a(x) dx,$$

which is of the same form as (2).

Aggregate demand uncertainty is captured by the size of the set of active locations, $\zeta(\omega)$. As there are no additional interactions between the location of active consumers and the spatial allocation of capital, the more *micro*-founded model sketched here does not offer new insights and it will instead divert the attention from the ultimate source of imbalances. It is also possible to consider more general demand schedules in which demand responds to local prices. For instance, an inverse demand schedule of the form

$$p(x, \omega) = \begin{cases} y(x, \omega)^{-\chi} \text{ if } x < \zeta(\omega) \\ 0 \text{ if } x \geq \zeta(\omega), \end{cases} \tag{D1}$$

allows for price-sensitivity since χ denotes the price-elasticity of demand. Firms face a local revenue, $p(x, \omega)y(x, \omega)$, with an uncertain extensive margin that would still give rise to the same type of demand-driven unemployment as discussed in the baseline model. In other words, the baseline model assumes an insensitive demand function with $\chi = 0$ in (D1), but demand sensitivities can be added leaving the results virtually unchanged as long there is uncertainty on the extensive margin of demand.

Remarks about concave utilities and home production. The baseline model assumed linear preferences. It is possible to consider a concave utility for consumption assuming that households diversify the risk associated with local imbalances and search histories by aggregating local consumption 'internally.' Let $U(C_t)$ denote the utility associated with this aggregate consumption,

$$C_t = -I_t + \int_0^1 [y_t(x) - \theta dl_t(x)]\psi_t(x)dx,$$

so that the objective of the dynamic assignment problem is

$$\max \int_0^\infty \left\{ U\left(C_t\right) - r_t \int_0^1 k_t(x)\psi_t(x)dx + r_t K_t \right\} \exp\{-\rho t\}, \text{ s.t. } (20).$$

The static and dynamic problems cannot be treated as independent. The Hamilton-Jacobi-Bellman (HJB) equation associated with the previous problem is

$$\begin{aligned}\rho \mathcal{W}(\psi_t, K_t) = \max \Bigg\{ U &\left(-I_t + \int_0^1 [y_t(x) - \theta dl_t(x)]\psi_t(x)dx \right) \\ &- r_t \int_0^1 k_t(x)\psi_t(x)dx + r_t K_t + \mathcal{W}_K(\psi_t, K_t)[I_t - \delta K_t] \Bigg\}. \\ + \int_0^1 \Bigg[&\mathcal{W}_{l(x)}(\psi_t, K_t)[(1-\eta)a(x) - s(x) - \bar{\eta}]l(x) + \mu_z \varphi(x) \mathcal{W}_{\varphi(x)}(\psi_t, K_t) \\ &+ \frac{1}{2}\sigma_z^2 \varphi(x)^2 \mathcal{W}_{\varphi(x)\varphi(x)}(\psi_t, K_t) \Bigg] \psi_t(x)dx.\end{aligned}$$

The first-order condition for I_t is of the form $\{\mathcal{W}_K(\psi_t, K_t) - U'(C_t)\}I_t^* = 0$. Under positive investment, this means that the marginal value of job capital equals the marginal utility of consumption. For the optimal distribution of labor $l_t^*(x)$, that needs to be determined simultaneously, the first-order condition for $k_t(x)$ is $\alpha U'\left(C_t\right)\varphi_t(x)[k_t^*(x)/l_t^*(x)^\phi]^{\alpha-1} = r_t$, which upon aggregation can be written as

$$\alpha K_t^{\alpha-1} \left[\int_0^1 l_t^*(x)^\phi \varphi_t(x)^{1/(1-\alpha)} \psi_t(x)dx \right]^{1-\alpha} = \frac{r_t}{U'(C_t)}.$$

As this expression shows, the rental rate for capital is now a function of the entire distribution of the labor force across locations, and the distribution of the state of the local economy. It is still possible to derive the stationary value for aggregate stock of capital, but that derivation is not independent of the derivation of the distribution of the local state. For instance, since the envelope condition is $(\rho+\delta)\mathcal{W}_K(\psi_t, K_t) = r_t + \mathcal{W}_{KK}(\psi_t, K_t)dK_t/dt$, combining previous expressions yields a standard Euler equation

$$\left(-\frac{U''(C_t)}{U'(C_t)}\right)\frac{dC_t}{dt} = \alpha K_t^{\alpha-1}\left[\int_0^1 l_t^*(x)^{\phi}\varphi_t(x)^{1/(1-\alpha)}\psi_t(x)dx\right]^{1-\alpha} - \rho - \delta,$$

that determines the stationary value of K^* as an implicit function of the distribution of the labor force and the local state. The first-order conditions for the spatial allocation of labor can be derived following the same steps as before with the amendment that the indirect production function $g_t(x)$ would now be a function of $r_t/U'(C_t)$ which is the relevant market price for job capital. The stationary equilibrium would not be solved recursively, but one must find joint conditions on $\psi_t(x)$ and K_t consistent with the Euler equation and the KFE.

I have also abstained form considering home production or leisure. It is possible to consdier a home sector as an additional location leading to a simple relabeling of the problem. It is also possible to consider the home sector as an alternative activity which requires H_t time units. Given a time constraint, such as $1 = H_t + L_t$, leisure valuations would make the total size of the labor force endogenous. This margin seems important to consider movements into and out of the labor force, but as with workser rest, treating home production would require a dedicated paper.

Remarks about the shock bound in the static model. For completeness, consider the conditional expectation in the case when the random variable ϵ is below the threshold ϵ_+ or when the local shock is not bounded from below so that k/l^{ϕ} is "too low." The first part in the proof of Proposition 2 remains unchanged, but the second event in (7) becomes

$$\frac{k^{1/\phi}}{1+\pi\epsilon_+}\left(\pi\int_{k/l^{\phi}}^{\epsilon_+} s^{-1/\phi}ds + (1-\alpha)\epsilon_+^{1-\alpha}\int_{\epsilon_+}^{\infty} s^{-1/\phi}/s^{1+(1-\alpha)}ds\right),$$

instead of (A8). In terms of the unemployment rate, the previous expression can be written as

$$\frac{(k/l^{\phi})^{1/\phi}}{1+\pi\epsilon_+}\left[\frac{\pi\left(k/l^{\phi}\right)^{1-1/\phi}}{1/\phi-1} - \frac{\pi\epsilon_+^{1-1/\phi}}{1/\phi-1} + \frac{(1-\alpha)\epsilon_+^{-1/\phi}}{(1-\alpha+1/\phi)}\right].$$

As $k/l^{\phi} = (\alpha\varphi/r)^{1/(1-\alpha)}$, from (A6), one can write the previous expression in terms of the same terms as in (8) with a limiting behavior that differs from (8) as $\varphi \to 0$. Instead of exhibiting a strictly declining function of φ, local unemployment rates have an inverse U-shape relationship with the local shock. For instance, the flat part of $\psi_{\epsilon}(s)$ in (A1) ensures that local unemploment rates tend to zero as φ tends to zero. Similarly, for vacancies, being on the flat part of $\psi_{\epsilon}(s)$ implies that $k\Pr\{\epsilon < k/l^{\phi}\} - l^{\phi}\mathbb{E}[\epsilon|\epsilon < k/l^{\phi}]\Pr\{\epsilon < k/l^{\phi}\}$ is given by

$$k\frac{\pi}{1+\pi\epsilon_+}\left(\frac{k}{l^{\phi}}\right) - l^{\phi}\left[\frac{1}{2}\left(\frac{k}{l^{\phi}}\right)^2\frac{\pi}{1+\pi\epsilon_+}\right],$$

so that, in general,

$$\frac{v}{k^*} = (1-q) + q \begin{cases} \frac{\pi}{2(1+\pi\epsilon_+)} \left(\frac{\alpha\varphi}{r}\right)^{1/(1-\alpha)} \text{ if } \varphi < r\epsilon_+^{1-\alpha}/\alpha \\ 1 - \frac{r}{\alpha\varphi} \text{ if } \varphi \geq r\epsilon_+^{1-\alpha}/\alpha \end{cases}.$$

Vacancy rates are strictly increasing in the local shocks φ regardless of its value.

Remarks about demand uncertainty. Drop the location indicator x for notational convenience. Conditional on z, the variance of the distribution of local output is $\mathbb{V}ar\{y(\omega)|z\} = \mathbb{V}ar\{\mathbb{E}_\omega[y(\omega)|\zeta(\omega), z]\} + \mathbb{E}_\omega[\mathbb{V}ar\{y(\omega)|\zeta(\omega), z\}]$. The second term is driven by $\epsilon(\omega)$. The first term is the variance of the distribution of local output due to demand uncertainty, i.e., $q(1-q)[z(k^*)^\alpha l^{\phi(1-\alpha)}]^2$, which is a function of q. The variance of $y(\omega)$ when demand uncertainty is the only source of randomness is:

$$\begin{aligned} \sigma^2_{\zeta(\omega)}\{y(\omega)\} &\equiv \mathbb{E}_\omega\{[y(\omega)^2|\zeta(\omega)]|z)\} - \mathbb{E}_\omega[y(\omega)]^2, \\ &= q)[zk^{*\alpha} l^{\phi(1-\alpha)}]^2 - y^2, \\ &= (q^{-1}-1)y^2. \end{aligned}$$

By the law of total variance, the variance of $y(\omega)$ when $\zeta(\omega)$ and z are both a source of randomness is $\sigma^2\{y(\omega)\} = \mathbb{E}_z[\sigma^2\{\mathbb{E}_\omega[y(\omega)|\zeta(\omega)]|z\}] + \sigma^2_z\{y\} = (q^{-1}-1)\mathbb{E}[z^2] + \sigma^2\{z\}\,(y/z)^2$, which varies with q due to the first term. The variance is hence increasing in the degree of demand uncertainty.

The variance of $u(\omega)$ when demand uncertainty is the only source of randomness is $\sigma^2_{\zeta(\omega)}\{u(\omega)\} \equiv \mathbb{E}_\omega\{[u(\omega)^2|\zeta(\omega)]|z\} - \mathbb{E}_\omega[u(\omega)]^2$. Let $\tilde{u} \equiv r/(1+\phi(1-\alpha))z$. For the first term, notice that $\mathbb{E}_\omega\{[u(\omega)^2|\zeta(\omega)]|z)\} = \{(1-q+q\left[u^k/q\right]^2\}l^2$, so

$$\sigma^2_{\zeta(\omega)}\{u(\omega)\} = \left[q(1-q) - (1-q)2\tilde{u} + (1-q)\tilde{u}^2/q\right] l^2.$$

This yields

$$\frac{d\sigma^2_{\zeta(\omega)}\{u(\omega)\}}{dq} = 1 - 2q + 2\tilde{u} - \frac{\tilde{u}^2}{q^2}.$$

A sufficient condition for the previous derivative to be negative is $q > 1/2 + \tilde{u}$. The variance of $v(\omega)$ when demand uncertainty is the only source of randomness is

$$\sigma^2_{\zeta(\omega)}\{v(\omega)\} \equiv \mathbb{E}_\omega\{[v(\omega)^2|\zeta(\omega)]|z\} - \mathbb{E}_\omega[v(\omega)]^2.$$

The first term equals $(1-q)k^{*2} + q\left[k^* - rk^*/(\alpha q z)\right]^2$ whereas the second is $(k^* - rk^*/\alpha z)^2$. Hence, $\sigma^2_{\zeta(\omega)}\{v(\omega)\} = \{(1-q+q\,(1-r/\alpha z q)^2 - (1-r/\alpha z)^2\}k^{*2}$, which can be written as

$$\sigma^2_{\zeta(\omega)}\{v(\omega)\} = (q^{-1}-1)\left(rk^*/\alpha z\right)^2,$$

which declines as $q(x)$ increases, as noted in the text.

Remarks about approximation of aggregate matching function. The static model was

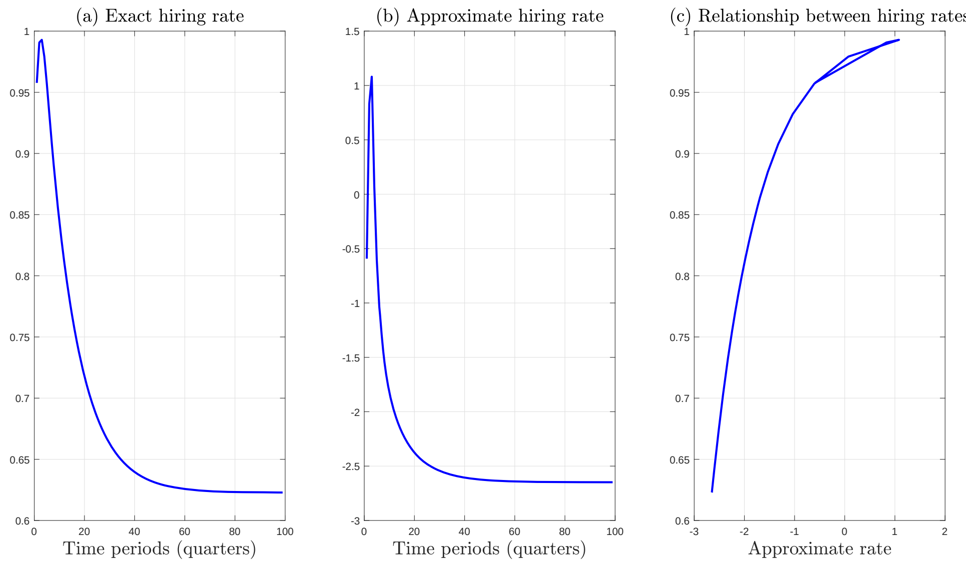

Figure 5: Exact and approximate hiring rates.

used to construct an aggregate matching function. The approximated aggregate matching function is Cobb-Douglas. Figure 5 plots the exact and the approximate exit rates from unemployment. By the exact exit rate, the figure considers

$$h(U,V) = 1 - \frac{\alpha(1-\alpha)\phi}{1+\phi(1-\alpha)}\left(\frac{V}{U^{\phi}}\right)^{\alpha-1}.$$

Both time series are computed in response to the simple aggregate "shock" considered in the text. That is, to an unexpected decline in local productivities to φ^{-}. The deterministic transitional dynamics of the aggregate job shortage rate $\tilde{u}_t$ as it returns back towards $\tilde{u}^*$ were depicted in Figure 4.

Figure 5 shows that the non-linear adjustment patterns in aggregate unemployment and vacancy rates translate into a non-linear adjustment for hires. As the range of variation in aggregate variables is small since there is no amplification (Figure 4), the range of the hiring rate is also small. The approximate hiring rate exhibits a higher range, since the approximation has not been scaled. Panel (c) shows the nonlinear nature of the relationship between market tightness and hires, and the 'errors' associated with the log-approximation.